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ROBUST SPECIAL ANOSOV ENDOMORPHISMS

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Abstract. In this paper we introduce the notion of *"robust special Anosov endomorphisms"*, and show that Anosov endomorphisms of tori which are not neither an Anosov diffeomorphism nor an expanding map, are not robust special.

1. Introduction

Let M be a compact, connected, boundaryless finite dimensional C^∞ manifold. The big difference between the set of all C^1 Anosov diffeomorphisms and C^1 expanding maps on M and the set of all non-invertible C^1 Anosov endomorphisms on M is that the maps in the first set are C^1-structurally stable but the maps in the second are not. Anosov [1] proved that every Anosov diffeomorphism is C^1-structurally stable, and Shub [10] showed the same result for expanding differentiable maps. In [10], Shub also remarked that the techniques which prove expanding maps are structurally stable should also prove Anosov endomorphisms are structurally stable. However, Mané and Pugh [6] and Przytycki [9] proved that Anosov endomorphisms which are not diffeomorphisms nor expanding do not be C^1-structurally stable.

Przytycki [9] showed that the set of all C^1 Anosov endomorphisms is an open subset of $C^1(M, M)$, the set of all C^1 maps from M to itself. We consider C^1 special Anosov endomorphisms. The natural question is that: Is the set of all C^1 special Anosov endomorphisms an open subset of $C^1(M, M)$?

In this paper we show that the answer in general is negative. The question is equivalent to: Is specialness robust for C^1 special Anosov endomorphisms?

A property $\mathcal{P}$ for a map f is called *robust* if there is a neighborhood $\mathcal{U}$ of f, in the desired topology, such that the property $\mathcal{P}$ holds for all $g \in \mathcal{U}$. In this paper, we investigate the following question:

Question. Let f be a special Anosov endomorphism. Can we perturb f such that the new perturbed map is not special any more?

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In this paper, we show that the answer for the above question is negative in general. For this we first introduce the concept of a robust special Anosov endomorphism.

Definition. Let f be a C^1 special Anosov endomorphism on a closed manifold. We say that f is a C^1 *robust special Anosov endomorphism* if there exists a neighborhood U_f in C^1 topology, such that for any $g \in U_f$, g is also a special Anosov endomorphism. (For the definition of special Anosov endomorphisms and details see next sections.)

Example 1.1. C^1 Anosov diffeomorphisms are invertible and then special. It is well known that if f is a C^1 Anosov diffeomorphism on a closed manifold, then f is C^1-structurally stable, and hence it is C^1 robust special.

Firstly, we show that any TA-covering map on an infra-nil-manifold is topologically mixing and therefore topologically transitive.

Proposition 1.2. *Let N/Γ be an infra-nil-manifold. If $f : N/\Gamma \to N/\Gamma$ is a TA-covering map, then f is topologically mixing and therefore topologically transitive.*

In [8], we proved the following theorem which for the convenience of reader, we will bring a sketch of the proof in Section 4:

Theorem 1.3 ([8, Theorem 1.10])**.** *Let $f : N/\Gamma \to N/\Gamma$ be a covering map of a nil-manifold and denote as $A : N/\Gamma \to N/\Gamma$ the nil-endomorphism homotopic to f. If f is a special TA-map, then A is a hyperbolic nil-endomorphism and f is topologically conjugate to A.*

Secondly, as a result of Proposition 1.2 and Theorem 1.3, because a torus is a nil-manifold (and so an infra-nil-manifold), we prove the following theorem.

Theorem 1.4. *Suppose $n \geq 2$. Then for generic $f \in A^*(\mathbb{T}^n)$, there exists a residual set $R \subset \mathbb{T}^n$ such that for all $x \in R$, x has infinitely many unstable directions, where*

$$A(\mathbb{T}^n) = \{f : \mathbb{T}^n \to \mathbb{T}^n : f \text{ is a } C^1 \text{Anosov endomorphism}\},$$
$$A^*(\mathbb{T}^n) = A(\mathbb{T}^n) \setminus (\{Anosov\ diffeomorphisms\} \cup \{expanding\ maps\}).$$

Finally, we conclude the main result:

Theorem 1.5 (Main Theorem)**.** *Every $f \in A^*(\mathbb{T}^n)$ is not a robust special Anosov endomorphism.*

This theorem means if $f \in A^*(\mathbb{T}^n)$, we can perturb f such that the result map is not special any more or the set of all C^1 special Anosov endomorphisms is not an open subset of $C^1(M, M)$, in general.

2. Preliminaries

Let X be a compact metric space with metric d. For a continuous surjection $f : X \to X$, we let

$$X_f = \{\tilde{x} = (x_i) : x_i \in X \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\},$$
$$\sigma_f((x_i)) = (f(x_i)).$$

The map $\sigma_f : X_f \to X_f$ is called the *shift map* determined by f. We call (X_f, σ_f) the *inverse limit* of (X, f). A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $e > 0$ (called an *expansive constant*) such that if x and y are any two distinct points of X, then $d(f^i(x), f^i(y)) > e$ for some integer i. A continuous surjection $f : X \to X$ is called *c-expansive* if there is a constant $e > 0$ such that for $\tilde{x}, \tilde{y} \in X_f$ if $d(x_i, y_i) \leq e$ for all $i \in \mathbb{Z}$, then $\tilde{x} = \tilde{y}$. In particular, if there is a constant $e > 0$ such that for $x, y \in X$ if $d(f^i(x), f^i(y)) \leq e$ for all $i \in \mathbb{N}$, then $x = y$, we say that f is *positively expansive*. A sequence of points $\{x_i : a < i < b\}$ of X is called a δ-pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. Given $\epsilon > 0$ a δ-pseudo orbit of $\{x_i\}$ is called to be ϵ-traced by a point $x \in X$ if $d(f^i(x), x_i) < \epsilon$ for every $i \in (a, b-1)$. Here the symbols a and b are taken as $-\infty \leq a < b \leq \infty$ if f is bijective and as $-1 \leq a < b \leq \infty$ if f is not bijective. f has the *pseudo orbit tracing property* (abbrev. POTP) if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f can be ϵ-traced by some point of X. We say that a homeomorphism $f : X \to X$ is a topological Anosov map (abbrev. TA-map) if f is expansive and has POTP. Analogously, we say that a continuous surjection $f : X \to X$ is a topological Anosov map if f is c-expansive and has POTP, and say that f is a topological expanding map if f is positively expansive and open. We can check that every topological expanding map is a TA-map (see [2, Remark 2.3.10]).

We bring here the definitions of nil-manifolds and infra-nil-manifolds from Karel Dekimpe in [4] and [5].

Let N be a Lie group and $\text{Aut}(N)$ be the set of all automorphisms of N. Assume that $\overline{A} \in \text{Aut}(N)$ is an automorphism of N, such that there exists a discrete and cocompact subgroup Γ of N, with $\overline{A}(\Gamma) = \Gamma$. Then the space of left cosets N/Γ is a closed manifold, and $\overline{A}$ induces a diffeomorphism $A : N/\Gamma \to N/\Gamma$, $g\Gamma \mapsto \overline{A}(g)\Gamma$. If we want this diffeomorphism to be Anosov, $\overline{A}$ must be hyperbolic. It is known that this can happen only when N is nilpotent. So we restrict ourselves to that case, where the resulting manifold N/Γ is said to be a *nil-manifold*. Such a diffeomorphism A induced by an automorphism $\overline{A}$ is called a nil-automorphism and is said to be a hyperbolic nil-automorphism, when $\overline{A}$ is hyperbolic.

All tori, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are examples of nil-manifolds.

Now we give an extended definition of nil-manifolds. Let N be a connected and simply connected nilpotent Lie group and $\text{Aut}(N)$ be the group of continuous automorphisms of N. Then $Aff(N) = N \rtimes \text{Aut}(N)$ acts on N in the following way:

$$\forall (n, \gamma) \in Aff(N), \forall x \in N : (n, \gamma).x = n\gamma(x).$$

So an element of $Aff(N)$ consists of a translational part $n \in N$ and a linear part $\gamma \in \mathrm{Aut}(N)$ (as a set $Aff(N)$ is just $N \times \mathrm{Aut}(N)$) and $Aff(N)$ acts on N by first applying the linear part and then multiplying on the left by the translational part). In this way, $Aff(N)$ can also be seen as a subgroup of $Diff(N)$.

Now, let C be a compact subgroup of $\mathrm{Aut}(N)$ and consider any torsion free discrete subgroup Γ of $N \rtimes C$, such that the orbit space N/Γ is compact. Note that Γ acts on N as being also a subgroup of $Aff(N)$. The action of Γ on N will be free and properly discontinuous, so N/Γ is a manifold, which is called an *infra-nil-manifold.*

Klein bottle is an example of infra-nil-manifolds.

In what follows, we will identify N with the subgroup $N \times \{id\}$ of $N \rtimes \mathrm{Aut}(N) = Aff(N)$, F with the subgroup $\{id\} \times F$ and $\mathrm{Aut}(N)$ with the subgroup $\{id\} \times \mathrm{Aut}(N)$.

It follows from Theorem 1 of L. Auslander in [3], that $\Gamma \cap N$ is a uniform lattice of N and that $\Gamma/(\Gamma \cap N)$ is a finite group. This shows that the fundamental group of an infra-nil-manifold N/Γ is virtually nilpotent (i.e., has a nilpotent normal subgroup of finite index). In fact $\Gamma \cap N$ is a maximal nilpotent subgroup of Γ and it is the only normal subgroup of Γ with this property. (This also follows from [3]).

If we denote by $p : N \rtimes C \to C$ the natural projection on the second factor, then $p(\Gamma) = \Gamma \cap N$ is a uniform lattice of N and that $\Gamma/(\Gamma \cap N)$. Let F denote this finite group $p(\Gamma)$, then we will refer to F as being the *holonomy group of* Γ (or of the infra-nil-manifold N/Γ). It follows that $\Gamma \subseteq N \rtimes F$. In case $F = \{id\}$, so $\Gamma \subseteq N$, the manifold N/Γ is a nil-manifold. Hence, any infra-nil-manifold N/Γ is finitely covered by a nil-manifold $N/(\Gamma \cap N)$. This also explains the prefix "infra".

Fix an infra-nil-manifold N/Γ, so N is a connected and simply connected nilpotent Lie group and Γ is a torsion free, uniform discrete subgroup of $N \rtimes F$, where F is a finite subgroup of $\mathrm{Aut}(N)$. We will assume that F is the holonomy group of Γ (so for any $\mu \in F$, there exists an $n \in N$ such that $(n, \mu) \in \Gamma$).

We can say that an element of Γ is of the form $n\mu$ for some $n \in N$ and some $\mu \in F$. Also, any element of $Aff(N)$ can uniquely be written as a product $n\psi$, where $n \in N$ and $\psi \in \mathrm{Aut}(N)$. The product in $Aff(N)$ is then given as

$$\forall n_1, n_2 \in N, \forall \psi_1, \psi_2 \in \mathrm{Aut}(N) : n_1\psi_1 n_2\psi_2 = n_1\psi_1(n_2)\psi_1\psi_2.$$

Now we can define infra-nil-endomorphisms as follows:

Let N be a connected, simply connected nilpotent Lie group and $F \subseteq \mathrm{Aut}(N)$ a finite group. Assume that Γ is a torsion free, discrete and uniform subgroup of $N \rtimes F$. Let $\overline{\mathcal{A}} : N \rtimes F \to N \rtimes F$ be an automorphism, such that $\overline{\mathcal{A}}(F) = F$ and $\overline{\mathcal{A}}(\Gamma) \subseteq \Gamma$, then, the map

$$A : N/\Gamma \to N/\Gamma,\ \Gamma \cdot n \mapsto \Gamma \cdot \overline{\mathcal{A}}(n)$$

is the *infra-nil-endomorphism* induced by $\overline{\mathcal{A}}$. In case $\overline{\mathcal{A}}(\Gamma) = \Gamma$, we call A an *infra-nil-automorphism.*

In the definition above, $\Gamma \cdot n$ denotes the orbit of n under the action of Γ. The computation above shows that A is well defined. Note that infra-nil-automorphisms are diffeomorphisms, while in general an infra-nil-endomorphism is a self-covering map.

The following theorem shows that the only maps of an infra-nil-manifold, that lift to an automorphism of the corresponding nilpotent Lie group are exactly the infra-nil-endomorphisms defined above.

Theorem 2.1 ([5, Theorem 3.4])**.** *Let N be a connected and simply connected nilpotent Lie group, $F \subseteq \mathrm{Aut}(N)$ a finite group and Γ a torsion free discrete and uniform subgroup of $N \rtimes F$ and assume that the holonomy group of Γ is F. If $\overline{A} : N \to N$ is an automorphism for which the map*

$$A : N/\Gamma \to N/\Gamma,\ \Gamma \cdot n \mapsto \Gamma \cdot \overline{A}(n)$$

is well defined (meaning that $\Gamma \cdot \overline{A}(n) = \Gamma \cdot \overline{A}(\gamma \cdot n)$ for all $\gamma \in \Gamma$), then

$$\overline{\mathcal{A}} : N \rtimes F \to N \rtimes F : x \mapsto \phi x \phi^{-1}\ (\textit{conjugation in } Aff(N))$$

is an automorphism of $N \rtimes F$, with $\overline{\mathcal{A}}(F) = F$ and $\overline{\mathcal{A}}(\Gamma) \subseteq \Gamma$. Hence, A is an infra-nil-endomorphism.

Let X and Y be metric spaces. A continuous surjection $f : X \to Y$ is called a *covering map* if for $y \in Y$ there exists an open neighborhood V_y of y in Y such that

$$f^{-1}(V_y) = \bigcup_i U_i \quad (i \neq i' \Rightarrow U_i \cap U_i' = \emptyset),$$

where each of U_i is open in X and $f_{|U_i} : U_i \to V_y$ is a homeomorphism. A covering map $f : X \to Y$ is especially called a *self-covering map* if $X = Y$. We say that a continuous surjection $f : X \to Y$ is a local homeomorphism if for $x \in X$ there is an open neighborhood U_x of x in X such that $f(U_x)$ is open in Y and $f_{|U_x} : U_x \to f(U_x)$ is a homeomorphism. It is clear that every covering map is a local homeomorphism. Conversely, if X is compact, then a local homeomorphism $f : X \to Y$ is a covering map (see [2, Theorem 2.1.1]).

Let X be a compact metric set and $f : X \to X$ a continuous surjection. A point $x \in X$ is said to be a *nonwandering point* if for any neighborhood U of x there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set $\Omega(f)$ of all nonwandering points is called the *nonwandering set.* Clearly $\Omega(f)$ is closed in X and invariant under f.

f is said to be *topologically transitive* (here X may be not necessarily compact) if there is $x_0 \in X$ such that the orbit $O^+(x_0) = \{f^i(x_0) : i \in \mathbb{Z}^{\geq 0}\}$ is dense in X. It is easy to check that if X is compact, a continuous surjection

$f : X \to X$ is topologically transitive if and only if for any U, V nonempty open sets there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

A continuous surjection $f : X \to X$ of a metric space is *topologically mixing* if for nonempty open sets U, V there exists $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. Topological mixing implies topological transitivity.

A map $f \in C^1(M, M)$ is said to be C^1-*structurally stable* if there is an open neighborhood $\mathcal{N}(f)$ of f in $C^1(M, M)$ such that $g \in \mathcal{N}(f)$ implies that f and g are topologically conjugate.

Let M be a closed smooth manifold and let $C^1(M, M)$ be the set of all C^1 maps of M endowed with the C^1 topology. A map $f \in C^1(M, M)$ is called an *Anosov endomorphism* if f is a C^1 regular map and if there exist $C > 0$ and $0 < \lambda < 1$ such that for every $\tilde{x} = (x_i) \in M_f = \{\tilde{x} = (x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1},\ i \in \mathbb{Z}\}$ there is a splitting

$$T_{x_i}M = E^s_{x_i} \oplus E^u_{x_i}, \quad i \in \mathbb{Z}$$

(we show this by $T_{\tilde{x}}M = \bigcup_i (E^s_{x_i} \oplus E^u_{x_i})$) so that for all $i \in \mathbb{Z}$:

(1) $D_{x_i}f(E^\sigma_{x_i}) = E^\sigma_{x_{i+1}}$ where $\sigma = s, u$,

(2) for all $n \geq 0$

$$\| D_{x_i}f^n(v) \| \leq C\lambda^n \| v \| \text{ if } v \in E^s_{x_i},$$
$$\| D_{x_i}f^n(v) \| \geq C^{-1}\lambda^{-n} \| v \| \text{ if } v \in E^u_{x_i}.$$

If, in particular, $T_{\tilde{x}}M = \bigcup_i E^u_{x_i}$ for all $\tilde{x} = (x_i) \in M_f$, then f is said to be *expanding differentiable map*, and if an Anosov endomorphism f is injective, then f is called an *Anosov diffeomorphism.* We can check that every Anosov endomorphism is a TA-map, and that every expanding differentiable map is a topological expanding map (see [2, Theorem 1.2.1]).

We define *special TA-maps* as follows. Let $f : X \to X$ be a continuous surjection of a compact metric space. Define the *stable* and *unstable* sets

$$W^s(x) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\},$$
$$W^u(\tilde{x}) = \{y_0 \in X : \exists \tilde{y} = (y_i) \in X_f \text{ s.t. } \lim_{i\to\infty} d(x_{-i}, y_{-i}) = 0\}$$

for $x \in X$ and $\tilde{x} \in X_f$. A TA-map $f : X \to X$ is special if f satisfies the property that $W^u(\tilde{x}) = W^u(\tilde{y})$ for every $\tilde{x}, \tilde{y} \in X_f$ with $x_0 = y_0$. In other words, every x has only one unstable direction. Every hyperbolic nil-endomorphism is a special TA-covering map (See [11, Remark 3.13]). By this and Theorem 1.3 we have the following corollary:

Corollary 2.2. *A TA-covering map of a nil-manifold is special if and only if it is conjugate to a hyperbolic nil-endomorphism.*

3. Proof of Proposition 1.2

To prove Proposition 1.2, we need the following lemmas and theorems:

Lemma 3.1 ([11, Lemma 1.5]). *Let $f : N/\Gamma \to N/\Gamma$ be a self-covering map and let $\overline{f} : N \to N$ be a lift of f by the natural projection $\pi : N \to N/\Gamma$. If f is a TA-covering map, then $\overline{f}$ has exactly one fixed point.*

Lemma 3.2 ([11, Lemma 5.4]). *If $f : N/\Gamma \to N/\Gamma$ is a TA-covering map, then $\Omega(f) = N/\Gamma$.*

For continuing, we need the following theorem whose proof can be found in Theorem 3.4.4 in [2].

Theorem 3.3 (Topological decomposition theorem). *Let $f : X \to X$ be a continuous surjection of a compact metric space. If $f : X \to X$ is a TA-map, then the following properties hold*:

(1) (*Spectral decomposition theorem due to Smale*) *The nonwandering set, $\Omega(f)$, contains a finite sequence B_i $(1 \leq i \leq l)$ of f-invariant closed subsets such that*
  (i) $\Omega(f) = \bigcup_{i=1}^{l} B_i$ *(disjoint union),*
  (ii) $f_{|B_i} : B_i \to B_i$ *is topologically transitive.*
  Such the subsets B_i are called basic sets.
(2) (*Decomposition theorem due to Bowen*) *For a basic set B there exist $a > 0$ and a finite sequence C_i $(0 \leq i \leq a-1)$ of closed subsets such that*
  (i) $C_i \cap C_j = \emptyset$ $(i \neq j)$, $f(C_i) = C_{i+1}$ *and* $f^a(C_i) = C_i$,
  (ii) $B = \bigcup_{i=0}^{a-1} C_i$,
  (iii) $f^a_{|C_i} : C_i \to C_i$ *is topologically mixing,*
  Such the subsets C_i are called elementary sets.

Proposition 3.4. *If $f : N/\Gamma \to N/\Gamma$ is a TA-covering map, then N/Γ is indeed an elementary set.*

Proof. By Lemma 3.1, let $\overline{f} : N \to N$ be the lift of f such that $\overline{f}(e) = e$. By the commuting diagram:

$$\begin{array}{ccc} N & \xrightarrow{\overline{f}} & N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ N/\Gamma & \xrightarrow[f]{} & N/\Gamma \end{array}$$

we have,

$$f([e]) = f(\pi(e)) = \pi(\overline{f}(e)) = \pi(e) = [e].$$

Therefore, $[e]$ is a fixed point of f. By Lemma 3.2, $\Omega(f) = N/\Gamma$. As N is connected and π is a continuous surjection then N/Γ is connected. In the proof of part (1) of spectral decomposition theorem, they prove that basic sets are close and open. Hence by connectedness of $\Omega(f) = N/\Gamma$, it consists of only one basic set, say B. On the other hand, by part (2) of spectral decomposition theorem, $N/\Gamma = B$ is the union of elementary sets. There is an elementary set, say C, such that $[e] \in C$. Since elementary sets are disjoint, by condition $f(C_i) = C_{i+1}$, $N/\Gamma = B$ consists of only one elementary set. □

Proof. (Proof of Proposition 1.2) By Theorem 3.4, N/Γ is an elementary set and the "a" in item (i) of part (2) of Theorem 3.3 must be equal to 1 and by item (iii) of part (2), f is topologically mixing on N/Γ. Of course, by this result f is topologically transitive on N/Γ. □

4. Proof of Theorem 1.3

Here we give a brief outline proof of Theorem 1.3:

Proof. Suppose that $f : N/\Gamma \to N/\Gamma$ is a special TA-covering map of an infra-nil-manifold. Sumi [11] proved that if f is injective (or expanding), then f conjugates to a hyperbolic infra-nil-automorphism (or an expanding infra-nil-endomorphism). By Dekimpe [5], the main results of [11] are incorrect for infranil-manifolds. But if we consider nil-manifolds instead of infra-nil-manifolds, we can repair main results. So, we consider them for nil-manifolds. So consider the case f is not injective nor expanding, and $A : N/\Gamma \to N/\Gamma$ is the unique nil-endomorphism homotopic to f (see [8], for details), and let $\overline{f}, \overline{A} : N \to N$ be the automorphisms which are lifts of f and A, respectively, by the natural projection π. Sumi [11] proved that there is a unique continuous surjective map $\overline{h} : N \to N$ such that

$$\overline{A} \circ \overline{h} = \overline{h} \circ \overline{f}.$$

So $\overline{h}$ is a so-called *semi-conjugacy* between $\overline{f}$ and $\overline{A}$. We should find a conjugacy between f and A.

For continuous maps f and g of N we define $D(f, g) = \sup\{D(f(x), g(x)) : x \in N\}$ where D denotes a left invariant, Γ-invariant Riemannian distance for N. Notice that $D(f, g)$ is not necessary finite. Since $D_e\overline{A}$ is hyperbolic, the Lie algebra $Lie(N)$ of N splits into the direct sum $Lie(N) = E_e^s \oplus E_e^u$ of subspaces E_e^s and E_e^u such that $D_e\overline{A}(E_e^s) = E_e^s$, $D_e\overline{A}(E_e^u) = E_e^u$ and there are $c > 1, 0 < \lambda < 1$ so that for all $n \geq 0$

$$||D_e\overline{A}^n(v)|| \leq c\lambda^n||v|| \quad (v \in E_e^s),$$
$$||D_e\overline{A}^{-n}(v)|| \leq c\lambda^n||v|| \quad (v \in E_e^u),$$

where $||\cdot||$ is the Riemannian metric. Let $\overline{L}^\sigma(e) = \exp(E_e^\sigma)$ $(\sigma = s, u)$ and let $\overline{L}^\sigma(x) = x \cdot \overline{L}^\sigma(e)(\sigma = s, u)$ for $x \in N$. Since left translations are isometries under the metric D, it follows that for all $x \in N$

$$\overline{L}^s(x) = \{y \in N : D(\overline{A}^i(x), \overline{A}^i(y)) \to 0\ (i \to \infty)\},$$
$$\overline{L}^u(x) = \{y \in N : D(\overline{A}^i(x), \overline{A}^i(y)) \to 0\ (i \to -\infty)\}.$$

Let $x \in N$, we define the stable set and unstable sets of x for f and A as follow (for more details see [2]):

$$\overline{W}^s(x) = \{y \in N : \lim_{i\to\infty} D(\overline{f}^i(x), \overline{f}^i(y)) = 0\},$$

$$\overline{W}^u(x, \mathbf{e}) = \{y \in N : \lim_{i\to-\infty} D(\overline{f}^i(x), \overline{f}^i(y)) = 0\},$$

where $\mathbf{e} = (\ldots, e, e, e, \ldots)$.

Lemma 4.1 ([11, Lemma 2.4])**.** *For the semi-conjugacy $\overline{h}$, we have the following properties*:

1. *There exists $K > 0$ such that $D(\overline{h} \circ \gamma(x), \gamma \circ \overline{h}(x)) < K$ for $x \in \mathbb{N}$ and $\gamma \in \Gamma$.*
2. *For any $\lambda > 0$, there exists $L \in \mathbb{N}$ such that $D(\overline{h} \circ \gamma(x), \gamma \circ \overline{h}(x)) < \lambda$ for $x \in N$ and $\gamma \in \overline{A}_*^L(\Gamma)$.*
3. *For $x \in N$ and $\gamma \in \bigcap_{i=0}^{\infty} \overline{A}_*^i(\Gamma)$, we have $\overline{h} \circ \gamma(x) = \gamma \circ \overline{h}(x)$.*
4. *For $x \in N$ and $\gamma \in \Gamma$, we have $\overline{h} \circ \gamma(x) \in \overline{L}^s(\gamma \circ \overline{h}(x))$.*

Remark 4.2. Since $\overline{h}$ is D-uniformly continuous then $\overline{h}(\overline{W}^s(x)) = \overline{L}^s(\overline{h}(x))$ and $\overline{h}(\overline{W}^u(x; \mathbf{e})) = \overline{L}^u(\overline{h}(x))$.

Lemma 4.3. *The following statements hold*:

1. $\gamma(\overline{W}^s(x)) = \overline{W}^s(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
2. $\gamma(\overline{W}^u(x; \boldsymbol{e})) = \overline{W}^u(\gamma(x); \boldsymbol{e})$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
3. $\gamma(\overline{L}^s(x)) = \overline{L}^s(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
4. $\gamma(\overline{L}^u(x)) = \overline{L}^u(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
5. *If* $x \in \overline{W}^u(e; \boldsymbol{e})$, *then* $\overline{W}^u(x; \boldsymbol{e}) = \overline{W}^u(e; \boldsymbol{e})$,
6. *If* $x \in \overline{L}^u(e)$, *then* $\overline{L}^u(x) = \overline{L}^u(e)$.

Proof. For proof, see [8], Lemmas 3.13, 3.14 and 4.2. □

Lemma 4.4 ([2, Lemma 8.6.2])**.** *For $\epsilon > 0$ there is $\delta > 0$ such that if $D(x, y) < \delta$, $x, y \in N$, then $\overline{W}^s(x) \subset U_\epsilon(\overline{W}^s(y))$ and $\overline{W}^u(x; \boldsymbol{e}) \subset U_\epsilon(\overline{W}^u(y; \boldsymbol{e}))$. Where for a set S, $U_\epsilon(S) = \{y \in N : D(y, S) < \epsilon\}$.*

Remark 4.5. There is a $\delta_K > 0$ such that $D(\overline{h}(x), x) < \delta_K$ for $x \in N$, we have (see [2] page 270 (8.5))

$$\overline{W}^s(x) \subset U_{\delta_K}(\overline{L}^s(\overline{h}(x))) \quad \text{and} \quad \overline{W}^u(x; \mathbf{e}) \subset U_{\delta_K}(\overline{L}^u(\overline{h}(x))).$$

For simplicity, let $\Gamma_{\overline{f}} = \big(\overline{W}^u(e; \mathbf{e}) \rtimes \{id_{\mathrm{Aut}(N)}\}\big) \cap \Gamma$ and $\Gamma_{\overline{A}} = \big(\overline{L}^u(e) \rtimes \{id_{\mathrm{Aut}(N)}\}\big) \cap \Gamma$.

Lemma 4.6. *The following statements hold*:

1. $\Gamma_{\overline{A}}$ *and* $\Gamma_{\overline{f}}$ *are subgroups of* Γ.
2. $\Gamma_{\overline{f}} \subset \Gamma_{\overline{A}}$.
3. $\overline{h}(\gamma(v)) = \gamma(\overline{h}(v))$ *for each* $\gamma \in \Gamma_{\overline{f}}$ *and* $v \in \overline{W}^u(e; \boldsymbol{e})$.

(4) *If* $\overline{W}^u(\gamma_1(e);\boldsymbol{e}) = \overline{W}^u(\gamma_2(e);\boldsymbol{e})$ *for some* $\gamma_1, \gamma_2 \in \Gamma$*, then we have*

$$\gamma_1(\overline{h}(\gamma_1^{-1}(x)) = \gamma_2(\overline{h}(\gamma_2^{-1}(x)) \text{ for } x \in \overline{W}^u(\gamma_1(e);\boldsymbol{e}).$$

Proof. (1) Let $x, y \in \overline{L}^u(e)$, since $A^i(e) = e$ for all i, then by definition,

$$\begin{aligned} &\lim_{i\to-\infty} D(A^i(x), e) = \lim_{i\to-\infty} D(A^i(x), A^i(e)) = 0,\\ &\lim_{i\to-\infty} D(A^i(y), e) = \lim_{i\to-\infty} D(A^i(y), A^i(e)) = 0. \end{aligned} \tag{4.1}$$

As D is left invariant we have

$$\begin{aligned} 0 \le \lim_{i\to-\infty} D(A^i(xy^{-1}), A^i(e)) &= \lim_{i\to-\infty} D(A^i(x)A^i(y^{-1}), e)\\ &= \lim_{i\to-\infty} D(A^i(x)A^{-i}(y), A^i(x)A^{-i}(x))\\ (D \text{ is left invariant}) \quad &= \lim_{i\to-\infty} D(A^{-i}(y), A^{-i}(x))\\ &\le \lim_{i\to-\infty} D(A^{-i}(y), e) + D(e, A^{-i}(x))\\ (\text{equation } (4.1)) \quad &= \lim_{i\to-\infty} D(A^i(y), e) + D(A^i(x), e) = 0. \end{aligned}$$

Thus $xy^{-1} \in \overline{L}^u(e)$ and $\overline{L}^u(e)$ is a subgroup of N. So $\left(\overline{L}^u(e) \rtimes \{id_{\mathrm{Aut}(N)}\}\right) \cap \Gamma$ is a subgroup of Γ.

For the second part, let $\gamma_1, \gamma_2 \in \Gamma_{\overline{f}}$. Since Γ is a group we have $\gamma_1\gamma_2^{-1} \in \Gamma$. Now consider that $\gamma_1, \gamma_2 \in \left(\overline{W}^u(e;\mathbf{e}) \rtimes \{id_{\mathrm{Aut}(N)}\}\right)$. There exist $x_1, x_2 \in \overline{W}^u(e;\mathbf{e})$ such that $\gamma_1 = (x_1, id_N)$ and $\gamma_2 = (x_2, id_N)$. Therefore,

$$\begin{aligned} x_1\overline{W}^u(e;\mathbf{e}) &= \gamma_1(\overline{W}^u(e;\mathbf{e}))\\ (\text{Lemma 4.3, part (2)}) \quad &= \overline{W}^u(\gamma_1(e);\mathbf{e})\\ &= \overline{W}^u(x_1;\mathbf{e})\\ (\text{Lemma 4.3, part (5)}) \quad &= \overline{W}^u(e;\mathbf{e}). \end{aligned}$$

Similarly, $x_2\overline{W}^u(e;\mathbf{e}) = \overline{W}^u(e;\mathbf{e})$. So we have $x_1\overline{W}^u(e;\mathbf{e}) = x_2\overline{W}^u(e;\mathbf{e})$ and then $x_1x_2^{-1} \in \overline{W}^u(e;\mathbf{e})$. Finally,

$$\begin{aligned} \gamma_1\gamma_2^{-1} &= (x_1, id_N)(x_2, id_N)^{-1}\\ &= (x_1, id_N)(x_2^{-1}, id_N)\\ &= (x_1x_2^{-1}, id_N) \in \overline{W}^u(e;\mathbf{e}) \rtimes \{id_{\mathrm{Aut}(N)}\}, \end{aligned}$$

and we have the result.

(2) Take $(x, \alpha) = \gamma \in \Gamma_{\overline{f}}$ $(x \in N, \alpha \in C)$, such that $\gamma \notin \Gamma_{\overline{A}}$. So, $x \notin \overline{L}^u(e)$ and for each $n \in \mathbb{Z}$, $n \neq 0$, $x^n \notin \overline{L}^u(e)$. By part (1), Remark 4.5 and the fact that $\overline{h}(e) = e$ for all $n \in \mathbb{Z}$, we have $x^n \in \overline{W}^u(e;\mathbf{e}) \subset U_{\delta_K}(\overline{L}^u(e))$, which is impossible.

(3) Let $\gamma = (x, id_N)$ for some $x \in N$ and $v \in \overline{W}^u(e; \mathbf{e})$. We have

$$\begin{aligned} & \gamma(v) \in \gamma(\overline{W}^u(e;\mathbf{e})) \\ \text{(Lemma 4.3, part (2))} \quad & = \overline{W}^u(\gamma(e);\mathbf{e}) \\ & = \overline{W}^u(x;\mathbf{e}) \\ \text{(Lemma 4.3, part (5))} \quad & = \overline{W}^u(e;\mathbf{e}), \end{aligned}$$

so,

$$\begin{aligned} & \overline{h}(\gamma(v)) \in \overline{h}(\overline{W}^u(e;\mathbf{e})) \\ \text{(Remark 4.2)} \quad & = \overline{L}^u(e). \end{aligned}$$

By part (2), $\gamma \in \Gamma_{\overline{A}}$. Thus

$$\begin{aligned} & \gamma(\overline{h}(v)) \in \gamma(\overline{h}(\overline{W}^u(e;\mathbf{e})) \\ \text{(Remark 4.2)} \quad & = \gamma(\overline{L}^u(e)) \\ \text{(Lemma 4.3, part (4))} \quad & = (\overline{L}^u(\gamma(e))) \\ & = \overline{L}^u(x) \\ \text{(Lemma 4.3, part (6))} \quad & = \overline{L}^u(e). \end{aligned}$$

Again by Lemma 4.3 and last part of the above relation, $\overline{L}^u(\gamma(\overline{h}(v))) = \overline{L}^u(e)$, and

$$\overline{h}(\gamma(v)) \in \overline{L}^u(e) = \overline{L}^u(\gamma(\overline{h}(v))).$$

On the other hand, by part (4) of Lemma 4.1, $\overline{h}(\gamma(v)) \in \overline{L}^s(\gamma(\overline{h}(v)))$. Since $\overline{L}^u(\gamma(\overline{h}(v))) \cap \overline{L}^s(\gamma(\overline{h}(v))) = \{\gamma(\overline{h}(v))\}$ (see [11, Lemma 2.1]), then $\overline{h}(\gamma(v)) = \gamma(\overline{h}(v))$.

(4) We have $x \in \overline{W}^u(\gamma_1(e);\mathbf{e}) = \gamma_1(\overline{W}^u(e;\mathbf{e}))$. Thus, $\gamma_1^{-1}(x) \in \overline{W}^u(e;\mathbf{e})$. Similarly, $\gamma_2^{-1}(x) \in \overline{W}^u(e;\mathbf{e})$. Now, by part (3),

$$\begin{aligned} \gamma_1\big(\overline{h}(\gamma_1^{-1}(x)\big) &= \overline{h}\big(\gamma_1(\gamma_1^{-1}(x))\big) \\ &= \overline{h}(x) \\ &= \overline{h}\big(\gamma_2(\gamma_2^{-1}(x))\big) \\ &= \gamma_2\big(\overline{h}(\gamma_2^{-1}(x)\big). \end{aligned}$$

□

According to part (4) of Lemma 4.6, we can define a map

$$\overline{h}' : \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e}) \to \bigcup_{\gamma\in\Gamma} \overline{L}^u(\gamma(e))$$

by

$$\overline{h}'(x) = \gamma(\overline{h}(\gamma^{-1}(x))) \quad x \in \overline{W}^u(\gamma(e);\mathbf{e})\ (\gamma \in \Gamma).$$

Next lemma shows some properties of $\overline{h}'$:

Lemma 4.7. *The following statements hold*:

(1) $\overline{A} \circ \overline{h}' = \overline{h}' \circ \overline{f}$ *on* $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})$,
(2) $D(\overline{h}', id_{|\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})}) < \infty$,
(3) $\overline{h}'(\gamma(e)) = \gamma(e)$ *for* $\gamma \in \Gamma$,
(4) *if* $x \in \overline{W}^u(\gamma(e);\boldsymbol{e})(\gamma \in \Gamma)$, *then* $\overline{h}'(x) \in \overline{L}^u(\gamma(e))$ *and* $\overline{h}'(x) \in \overline{L}^s(\overline{h}(x))$,
(5) *if* $y \in \overline{W}^s(x)$ *for* $x, y \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})$, *then* $\overline{h}'(y) \in \overline{L}^s(\overline{h}'(x))$.

Proof. For proof, see [8], Lemma 4.4. □

Lemma 4.8. $\overline{h}'$ *is D-uniformly continuous.*

Proof. For proof, see [8], Lemma 4.7. □

Lemma 4.9. $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})$ *is dense in* N.

Proof. For proof, see [8], Lemma 4.13. □

By Lemma 4.8, $\overline{h}'$ is extended to a continuous map $\tilde{h} : N \to N$. From Lemma 4.7(1), (2) and (3), and uniqueness of $\overline{h}$, we have $\overline{h} = \tilde{h}$ and $\overline{h}(\gamma(e)) = \gamma(e)$ for all $\gamma \in \Gamma$.

Lemma 4.10. *For all* $\gamma \in \Gamma$ *and* $x \in N$, $\overline{h}(\gamma(x)) = \gamma(\overline{h}(x))$.

Proof. According to Lemma 4.1(4), we have

$$\overline{h}(\gamma(x)) \in \overline{L}^s(\gamma(\overline{h}(x))). \tag{4.2}$$

Suppose that $x \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e})$. Then there is $\gamma_x \in \Gamma$ such that $x \in \overline{W}^u(\gamma_x(e);\mathbf{e})$. For each $\gamma \in \Gamma$ we have

$$\gamma(x) \in \overline{h}\big(\overline{W}^u(\gamma_x(e);\mathbf{e})\big) = \overline{W}^u\big(\gamma(\gamma_x(e));\mathbf{e}\big).$$

Thus

$$\begin{aligned} \overline{h}(\gamma(x)) &\in \overline{h}\Big(\overline{W}^u\big(\gamma(\gamma_x(e));\mathbf{e}\big)\Big) \\ \text{(by Remark 4.2)} \quad &= \overline{L}^u\big(\overline{h}(\gamma(\gamma_x(e)))\big) \\ &= \overline{L}^u\big(\gamma(\gamma_x(e))\big). \end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned} \gamma(\overline{h}(x)) &\in \gamma\big(\overline{h}(\overline{W}^u(\gamma_x(e);\mathbf{e}))\big) \\ \text{(by Remark 4.2)} \quad &= \gamma\big(\overline{L}^u(\overline{h}(\gamma_x(e)))\big) \\ &= \gamma\big(\overline{L}^u(\gamma_x(e))\big) \\ \text{(by Lemma 4.3, part (4))} \quad &= \overline{L}^u(\gamma(\gamma_x(e))). \end{aligned} \tag{4.4}$$

By (4.4), we have $\overline{L}^u\big(\gamma(\gamma_x(e))\big) = \overline{L}^u\big(\gamma(\overline{h}(x))\big)$. Therefore, by (4.3) we have

$$\overline{h}(\gamma(x)) \in \overline{L}^u\big(\gamma(\overline{h}(x))\big). \tag{4.5}$$

By (4.2) and (4.5) we have

$$\overline{h}(\gamma(x)) \in \overline{L}^u\big(\gamma(\overline{h}(x))\big) \cap \overline{L}^s\big(\gamma(\overline{h}(x))\big) = \{\gamma(\overline{h}(x))\}. \tag{4.6}$$

Thus for each $x \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e})$ we have $\overline{h}(\gamma(x)) = \gamma(\overline{h}(x))$. Since $\overline{h}$ is continuous and $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e})$ is dense in N, we have the desired result. □

Hence, $\overline{h}$ induces a homeomorphism $h : N/\Gamma \to N/\Gamma$ such that $h \circ \pi = \pi \circ \overline{h}$, i.e., the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{\ \overline{h}\ } & N \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ N/\Gamma & \xrightarrow[\ h\ ]{} & N/\Gamma \end{array}$$

Now, it is easy to see that h is the desired conjugacy between f and A. □

5. Proof of Theorems 1.4 and 1.5

From now on suppose that $f \in C^1(\mathbb{T}^n, \mathbb{T}^n)$. To prove Theorem 1.4, we need the following theorems:

Zhang [12], by use of the fact that C^1 Anosov endomorphisms, which are not injective nor expanding, are not C^1-structurally stable, showed that:

Theorem 5.1 ([12], Main Theorem)**.** *Suppose $n \geq 2$. Then for generic $f \in A^*(\mathbb{T}^n)$, f is not topologically conjugate to any hyperbolic toral endomorphism.*

Micena and Tahzibi [7], proved that:

Theorem 5.2 ([7], Theorem 1.4)**.** *Let M be a closed manifold and $f : M \to M$ topologically transitive endomorphism. Then*

(1) *either f is a special Anosov endomorphism*
(2) *or there exists a residual subset $R \subset M$ such that for every $x \in R$, x has infinitely many unstable directions.*

Proof. (Proof of Theorem 1.4) We know that $\mathbb{T}^n$ is a case of nil-manifold. Clearly, every C^1 Anosov endomorphism on a closed manifold is a TA-covering map.

According to Theorem 5.1, for generic $f \in A^*(\mathbb{T}^n)$, f is not topologically conjugate to any hyperbolic toral endomorphism. Hence by Theorem 1.3 for $\mathbb{T}^n$ $(n \geq 2)$, f is not special.

On the other hand Proposition 1.2 says that f is topologically transitive. Finally, by Theorem 5.2 and the fact that f is not special, there exists a residual subset $R \subset \mathbb{T}^n$ such that for every $x \in R$, x has infinitely many unstable directions with respect to f. □

End of the proof of Theorem 1.5 (Main Theorem). Let $\Lambda \subset A^*(\mathbb{T}^n)$ be the generic set gotten in Theorem 1.4. According to Baire category theorem, Λ is dense in $A^*(\mathbb{T}^n)$. Thus if $f \in A^*(\mathbb{T}^n)$, then for every neighborhood U of f there is at least one $f \in \Lambda$ which is not special.

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