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SOME RESULTS ON PROJECTIVE CURVATURE TENSOR IN SASAKIAN MANIFOLDS

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Abstract. In the present paper, we study certain curvature conditions satisfying by the projective curvature tensor in Sasakian manifolds with respect to the generalized-Tanaka-Webster connection. Finally, we give an example of a 3-dimensional Sasakian manifold with respect to the generalized-Tanaka-Webster connection.

1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [15]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure.

The Tanaka-Webster connection [14, 17] is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-submanifold. S. Tanno [16] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

A Riemannian manifold is said to be Ricci semisymmetric if the curvature tensor R satisfies

$$R \cdot S = 0, \tag{1}$$

where $R(X, Y)$ is considered as a field of linear operators acting on S and S is the Ricci tensor of type $(0, 2)$. The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset.

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In 1977, T. Takahashi [13] introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. U. C. De [3] and D. G. Prakasha [8] have studied ϕ-symmetric Kenmotsu manifolds and have obtained some interesting results. In 2010, S. S. Shukla and M. K. Shukla [11] have studied ϕ-symmetric para-Sasakian manifolds.

The paper is organized as follows: In Section 2, we give a brief introduction of Sasakian manifolds. In Section 3, we deduce the relation between the curvature tensors with respect to the g-Tanaka-Webster connection and the Levi-Civita connection. In Section 4, we show that the Ricci symmetries in Sasakian manifolds with respect to the g-Tanaka-Webster connection and the Levi-Civita connection are equivalent if and only if the manifold is an Einstein manifold. Section 5 deals with the study of partially Ricci-pseudosymmetric Sasakian manifolds with respect to the g-Tanaka-Webster connection. Section 6 is devoted to study projectively flat, pseudoprojectively flat and ϕ-projectively semisymmetric Sasakian manifolds with respect to the g-Tanaka-Webster connection. In Section 7, we show that a Sasakian manifold is locally projectively ϕ-symmetric with respect to the g-Tanaka-Webster connection if and only if it is so with respect to the Levi-Civita connection. Finally, we construct an example of 3-dimensional Sasakian manifold with the generalized-Tanaka-Webster connection.

2. Preliminaries

Let M be an almost contact metric manifold of dimension $(2n+1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g) admitting a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then [1]

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ g(X, \xi) = \eta(X), \tag{3}$$

$$g(X, \phi Y) = -g(\phi X, Y) \tag{4}$$

for all vector fields X, Y on M.

A normal contact metric manifold is known as a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if ([9, 10])

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{5}$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection of the Riemannian metric. From the above equations it follows that

$$\nabla_X \xi = -\phi X, \tag{6}$$

$$(\nabla_X \eta)Y = g(X, \phi Y). \tag{7}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Sasakian manifold M with respect to the Levi-Civita connection satisfy

([2,12])

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$

$$S(X,\xi) = 2n\eta(X),\ Q\xi = 2n\xi, \tag{9}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2n\eta(X)\eta(Y), \tag{10}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{11}$$

for any vector fields X, Y and Z on M.

Sasakian manifolds have been studied by many authors in several ways to a different extent such as U. C. De and G. Ghosh [4], M. K. Dwivedi and J. S. Kim [5], I. Hasegawa and I. Mihai [6], V. A. Khan and M. A. Khan [7] and many others.

Definition. A Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{12}$$

where a and b are scalar functions on M.

A relation between the g-Tanaka-Webster connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in a Sasakian manifold M is given by [4]

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,\phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y \tag{13}$$

for all vector fields X, Y on M.

3. Curvature tensor and Ricci tensor in Sasakian manifolds with respect to the g-Tanaka-Webster connection

Let $\bar{R}$, $\bar{S}$, $\bar{r}$ and $\bar{Q}$ be the curvature tensor, the Ricci tensor, the scalar curvature and the Ricci operator with respect to the g-Tanaka-Webster connection $\bar{\nabla}$ in a Sasakian manifold M, respectively. Then we have [4]

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi - g(Y,\phi Z)\phi X \\ & + g(X,\phi Z)\phi Y + 2g(Y,\phi X)\phi Z - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y,\end{aligned} \tag{14}$$

$$\bar{R}(X,Y)\xi = \bar{R}(\xi,Y)Z = \eta(\bar{R}(X,Y)Z) = 0, \tag{15}$$

$$\bar{S}(Y,Z) = S(Y,Z) - 2g(Y,Z) - 2(n-1)\eta(Y)\eta(Z), \tag{16}$$

$$\bar{S}(X,\xi) = 0, \tag{17}$$

$$\bar{Q}Y = QY - 2Y - 2(n-1)\eta(Y)\xi,\ \bar{Q}\xi = 0, \tag{18}$$

$$\bar{r} = r - 6n \tag{19}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Now, let R and $\bar{R}$ be the curvature tensors of type $(0,4)$ of the connections ∇ and $\bar{\nabla}$, respectively on M given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \text{ and } \bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W).$$

Therefore from (14), it follows that

$$\begin{aligned} \bar{R}(X,Y,Z,U) = {} & R(X,Y,Z,U) + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\eta(U) \\ & - g(Y,\phi Z)g(\phi X,U) + g(X,\phi Z)g(\phi Y,U) \\ & + 2g(Y,\phi X)g(\phi Z,U) - \eta(Y)\eta(Z)g(X,U) \\ & + \eta(X)\eta(Z)g(Y,U). \end{aligned} \tag{20}$$

Interchanging X and Y in (20), we have

$$\begin{aligned} \bar{R}(Y,X,Z,U) = {} & R(Y,X,Z,U) + [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U) \\ & - g(X,\phi Z)g(\phi Y,U) + g(Y,\phi Z)g(\phi X,U) \\ & + 2g(X,\phi Y)g(\phi Z,U) - \eta(X)\eta(Z)g(Y,U) \\ & + \eta(Y)\eta(Z)g(X,U). \end{aligned} \tag{21}$$

On adding (20) and (21) and using the fact that $R(X,Y,Z,U)+R(Y,X,Z,U) = 0$, we get

$$\bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0. \tag{22}$$

Now interchanging U and Z in (20), we have

$$\begin{aligned} \bar{R}(X,Y,U,Z) = {} & R(X,Y,U,Z) + [g(X,U)\eta(Y) - g(Y,U)\eta(X)]\eta(Z) \\ & - g(Y,\phi U)g(\phi X,Z) + g(X,\phi U)g(\phi Y,Z) \\ & + 2g(Y,\phi X)g(\phi U,Z) - \eta(Y)\eta(U)g(X,Z) \\ & + \eta(X)\eta(U)g(Y,Z). \end{aligned} \tag{23}$$

By adding (20) and (23) and using the fact that $R(X,Y,Z,U)+R(X,Y,U,Z) = 0$, we get

$$\bar{R}(X,Y,Z,U) + \bar{R}(X,Y,U,Z) = 0. \tag{24}$$

Interchanging pair of slots in (20), we have

$$\begin{aligned} \bar{R}(Z,U,X,Y) = {} & R(Z,U,X,Y) + [g(Z,X)\eta(U) - g(U,X)\eta(Z)]\eta(Y) \\ & - g(U,\phi X)g(\phi Z,Y) + g(Z,\phi X)g(\phi U,Y) \\ & + 2g(U,\phi Z)g(\phi X,Y) - \eta(U)\eta(X)g(Z,Y) \\ & + \eta(Z)\eta(X)g(U,Y). \end{aligned} \tag{25}$$

By subtracting (25) from (20) and using fact that $R(X,Y,Z,U)-R(Z,U,X,Y) = 0$, we get

$$\bar{R}(X,Y,Z,U) - \bar{R}(Z,U,X,Y) = 0. \tag{26}$$

Interchanging X and Y in (23), we have

$$\bar{R}(Y,X,U,Z) = R(Y,X,U,Z) + [g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \tag{27}$$

$$- g(X, \phi U)g(\phi Y, Z) + g(Y, \phi U)g(\phi X, Z) + 2g(X, \phi Y)g(\phi U, Z) - \eta(X)\eta(U)g(Y, Z) + \eta(Y)\eta(U)g(X, Z).$$

By subtracting (27) from (23) and using the fact that

$$R(X, Y, Z, U) - R(Y, X, U, Z) = 0,$$

we get

$$\bar{R}(X, Y, Z, U) - \bar{R}(Y, X, U, Z) = 0. \tag{28}$$

Thus in view of (22), (24), (26) and (28), we can state the following theorem:

Theorem 3.1. *In a Sasakian manifold with respect to the g-Tanaka-Webster connection, we have*
(i) $\bar{R}(X, Y, Z, U) + \bar{R}(Y, X, Z, U) = 0,$
(ii) $\bar{R}(X, Y, Z, U) + \bar{R}(X, Y, U, Z) = 0,$
(iii) $\bar{R}(X, Y, Z, U) - \bar{R}(Z, U, X, Y) = 0,$
(iv) $\bar{R}(X, Y, Z, U) - \bar{R}(Y, X, U, Z) = 0$
for any vector fields $X, Y, Z, U \in \chi(M)$.

4. Ricci symmetry in a Sasakian manifold with respect to the connections $\bar{\nabla}$ and ∇

Assuming that the manifold is Ricci symmetric with respect to the g-Tanaka-Webster connection $\bar{\nabla}$, we have

$$(\bar{R}(X, Y) \cdot \bar{S})(U, V) = -\bar{S}(\bar{R}(X, Y)U, V) - \bar{S}(U, \bar{R}(X, Y)V) \tag{29}$$

for all $X, Y, U, V \in \chi(M)$. In view of (14) and (16), (29) takes the form

$$\begin{aligned}(\bar{R}(X, Y) \cdot \bar{S})(U, V) = & - \bar{S}(R(X, Y)U, V) + g(Y, \phi U)\bar{S}(\phi X, V) \\ & - g(X, \phi U)\bar{S}(\phi Y, V) - 2g(Y, \phi X)\bar{S}(\phi U, V) \\ & + \eta(Y)\eta(U)\bar{S}(X, V) - \eta(X)\eta(U)\bar{S}(Y, V) \\ & - \bar{S}(U, R(X, Y)V) + g(Y, \phi V)\bar{S}(\phi X, U) \\ & - g(X, \phi V)\bar{S}(\phi Y, U) - 2g(Y, \phi X)\bar{S}(\phi V, U) \\ & + \eta(Y)\eta(V)\bar{S}(X, U) - \eta(X)\eta(V)\bar{S}(Y, U).\end{aligned} \tag{30}$$

By using (4) and (16) in (30), we have

$$\begin{aligned}(\bar{R}(X, Y) \cdot \bar{S})(U, V) = & (R(X, Y) \cdot S)(U, V) + 2[R(X, Y, U, V) + R(X, Y, V, U)] \\ & + 2(n-1)[\eta(R(X, Y)U)\eta(V) + \eta(R(X, Y)V)\eta(U)] \\ & + S(\phi X, V)g(Y, \phi U) - g(X, \phi U)S(\phi Y, V) \\ & - 2g(Y, \phi X)S(\phi U, V) + \eta(Y)\eta(U)S(X, V) \\ & - 2\eta(Y)\eta(U)g(X, V) - \eta(X)\eta(U)S(Y, V) \\ & + 2\eta(X)\eta(U)g(Y, V) + g(Y, \phi V)S(\phi X, U)\end{aligned} \tag{31}$$

$$- g(X, \phi V)S(\phi Y, U) - 2g(Y, \phi X)S(\phi V, U) + \eta(Y)\eta(V)S(X, U) - 2\eta(Y)\eta(V)g(X, U) - \eta(X)\eta(V)S(Y, U) + 2\eta(X)\eta(V)g(Y, U)$$

which by using (11) and the fact that $R(X, Y, U, V) + R(X, Y, V, U) = 0$ turns to

$$\begin{aligned}(\bar{R}(X, Y) \cdot \bar{S})(U, V) = {} & (R(X, Y) \cdot S)(U, V) + S(\phi X, V)g(Y, \phi U) \\ & - g(X, \phi U)S(\phi Y, V) + g(Y, \phi V)S(\phi X, U) \\ & - g(X, \phi V)S(\phi Y, U) + \eta(Y)\eta(U)S(X, V) \\ & - \eta(X)\eta(U)S(Y, V) + \eta(Y)\eta(V)S(X, U) \\ & - \eta(X)\eta(V)S(Y, U) + 2n[g(Y, U)\eta(X)\eta(V) \\ & - g(X, U)\eta(Y)\eta(V) + g(Y, V)\eta(X)\eta(U) \\ & - g(X, V)\eta(Y)\eta(U)].\end{aligned} \tag{32}$$

Suppose that $(\bar{R}(X, Y) \cdot \bar{S})(U, V) = (R(X, Y) \cdot S)(U, V)$, then from (32), it follows that

$$\begin{aligned}& g(Y, \phi U)S(\phi X, V) - g(X, \phi U)S(\phi Y, V) + g(Y, \phi V)S(\phi X, U) \\ & - g(X, \phi V)S(\phi Y, U) + \eta(Y)\eta(U)S(X, V) - \eta(X)\eta(U)S(Y, V) \\ & + \eta(Y)\eta(V)S(X, U) - \eta(X)\eta(V)S(Y, U) + 2n[g(Y, U)\eta(X)\eta(V) \\ & - g(X, U)\eta(Y)\eta(V) + g(Y, V)\eta(X)\eta(U) - g(X, V)\eta(Y)\eta(U)] = 0.\end{aligned} \tag{33}$$

Taking $X = U = \xi$ in (33) and then using (2), (3) and (9), we obtain

$$S(Y, V) = 2ng(Y, V).$$

If $S(Y, V) = 2ng(Y, V)$, then from (32), it follows that

$$(\bar{R}(X, Y) \cdot \bar{S})(U, V) = (R(X, Y) \cdot S)(U, V).$$

Thus we have the following theorem:

Theorem 4.1. *Ricci symmetries in a Sasakian manifold with respect to the g-Tanaka-Webster connection and the Levi-Civita connection are equivalent if and only if the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

5. Partially Ricci-pseudosymmetric Sasakian manifolds with respect to the g-Tanaka-Webster connection

Definition. A Sasakian manifold M is said to be partially Ricci-pseudosymmetric if and only if the relation

$$R \cdot S = f(p)Q(g, S) \tag{34}$$

holds on the set $A = [x \in M : Q(g,S) \neq 0 \text{ at } x]$, where $f \in C^\infty(A)$ for $p \in (A)$, $R \cdot S$ and $Q(g,S)$ are respectively defined by

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V) \tag{35}$$

and

$$Q(g,S) = ((X \wedge_g Y) \cdot S)(U,V), \text{ where } (X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y \tag{36}$$

for all X, Y, U and $V \in \chi(M)$.

Let a $(2n+1)$-dimensional Sasakian manifold with respect to the g-Tanaka-Webster connection be partially Ricci-pseudosymmetric. Then we have

$$(\bar{R}(X,Y) \cdot \bar{S})(U,V) = f(p)\bar{Q}(g,\bar{S})(X,Y;U,V). \tag{37}$$

It is equivalent to

$$(\bar{R}(X,Y) \cdot \bar{S})(U,V) = f(p)[((X \wedge_g Y) \cdot \bar{S})(U,V)]$$

for all $X, Y, U, V \in \chi(M)$. From the above relation it follows that

$$\begin{aligned} &\bar{S}[\bar{R}(X,Y)U,V] + \bar{S}[U,\bar{R}(X,Y)V] \\ &= f(p)[\bar{S}((X \wedge_g Y)U,V) + \bar{S}(U,(X \wedge_g Y)V)] \end{aligned} \tag{38}$$

which by taking $X = U = \xi$ and then using (15) and (17) reduces to

$$f(p)\bar{S}(Y,V) = 0. \tag{39}$$

In view of (16), we have from (39)

$$f(p)[S(Y,V) - 2g(Y,V) - 2(n-1)\eta(Y)\eta(V)] = 0. \tag{40}$$

Then we have, either $f(p) = 0$ or, the manifold is an η-Einstein manifold of the form

$$S(Y,V) = 2g(Y,V) + 2(n-1)\eta(Y)\eta(V). \tag{41}$$

Thus we can state the following theorem:

Theorem 5.1. *A partially Ricci-pseudosymmetric Sasakian manifold with respect to the g-Tanaka-Webster connection is an η-Einstein manifold with respect to the Levi-Civita connection, provided $f(p) \neq 0$.*

6. Projectively flat, pseudo projectively flat and ϕ-projectively semisymmetric Sasakian manifolds with respect to the g-Tanaka-Webster connection

Let M be a $(2n+1)$-dimensional Riemannain manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor P with respect to the Levi-Civita connection ∇ is defined by [18]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{42}$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor and S is the Ricci tensor with respect to the Levi-Civita connection.

Now the projective curvature tensor with respect to the g-Tanaka-Webster connection $\bar{\nabla}$ is given by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \tag{43}$$

for all $X, Y, Z \in \chi(M)$, where $\bar{R}$ is the Riemannian curvature tensor and $\bar{S}$ is the Ricci tensor with respect to the g-Tanaka-Webster connection.

Definition. A Sasakian manifold is said to be

(i) projectively flat with respect to the g-Tanaka-Webster connection, if

$$\bar{P}(X,Y)Z = 0, \qquad X, Y, Z \in \chi(M), \tag{44}$$

(ii) pseudoprojectively flat with respect to the g-Tanaka-Webster connection, if

$$g(\bar{P}(\phi X,Y)Z, \phi W) = 0, \qquad X, Y, Z, W \in \chi(M), \tag{45}$$

(iii) ϕ-projectively semisymmetric with respect to the g-Tanaka-Webster connection, if

$$(\bar{P}(X,Y)\cdot\phi)Z = 0, \qquad X, Y \in \chi(M). \tag{46}$$

First, we consider that the manifold M with respect to the g-Tanaka-Webster connection is projectively flat. Then from (43) and (44) it follows that

$$\bar{R}(X,Y)Z = \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{47}$$

Taking inner product of (47) with ξ and using (3), we have

$$g(\bar{R}(X,Y)Z,\xi) = \frac{1}{2n}[\bar{S}(Y,Z)\eta(X) - \bar{S}(X,Z)\eta(Y)] \tag{48}$$

which in view of (15) turns to

$$\bar{S}(Y,Z)\eta(X) - \bar{S}(X,Z)\eta(Y) = 0. \tag{49}$$

By taking $Y = \xi$ in (49) and then using (2) and (15), we get

$$\bar{S}(X,Z) = 0. \tag{50}$$

In view of (16), (50) yields

$$S(X,Z) = 2g(X,Z) + 2(n-1)\eta(X)\eta(Z). \tag{51}$$

Thus we can state the following theorem:

Theorem 6.1. *A projectively flat Sasakian manifold with respect to the g-Tanaka-Webster connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Secondly, we consider that the manifold M with respect to the g-Tanaka-Webster connection is pseudoprojectively flat. Then from (43) and (45) it follows that

$$g[\bar{R}(\phi X, Y)Z, \phi W] = \frac{1}{2n}[\bar{S}(Y, Z)g(\phi X, \phi W) - \bar{S}(\phi X, Z)g(Y, \phi W)]. \tag{52}$$

By using (14) and (16) in the above equation, we have

$$\begin{aligned} g[R(\phi X, Y)Z, \phi W] = &\; g(Y, \phi Z)g(\phi^2 X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \\ &- 2g(Y, \phi^2 X)g(\phi Z, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W) \\ &+ \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) \\ &- 2(n-1)\eta(Y)\eta(Z))g(\phi X, \phi W) \\ &- (S(\phi X, Z) - 2g(\phi X, Z))g(Y, \phi W)]. \end{aligned} \tag{53}$$

Using the relation $g(\phi^2 X, \phi W) = -g(X, \phi W)$, (53) becomes

$$\begin{aligned} g[R(\phi X, Y)Z, \phi W] = &- g(Y, \phi Z)g(X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \\ &+ 2g(\phi X, \phi Y)g(\phi Z, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W) \\ &+ \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) \\ &- 2(n-1)\eta(Y)\eta(Z))g(\phi X, \phi W) \\ &- (S(\phi X, Z) - 2g(\phi X, Z))g(Y, \phi W)]. \end{aligned} \tag{54}$$

Let $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of the vector fields in M. Then $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (54) and sum up with respect to i, then we have

$$\begin{aligned} &\sum_{i=1}^{2n} g[R(\phi e_i, Y)Z, \phi e_i] \\ &= - g(Y, \phi Z)\sum_{i=1}^{2n} g(e_i, \phi e_i) - \sum_{i=1}^{2n} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\ &\quad + 2\sum_{i=1}^{2n} g(\phi e_i, \phi Y)g(\phi Z, \phi e_i) + \eta(Y)\eta(Z)\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) \\ &\quad + \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) - 2(n-1)\eta(Y)\eta(Z))\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) \\ &\quad - \sum_{i=1}^{2n}(S(\phi e_i, Z) - 2g(\phi e_i, Z))g(Y, \phi e_i)]. \end{aligned} \tag{55}$$

It can be easily verified that

$$\sum_{i=1}^{2n} g[R(\phi e_i, Y)Z, \phi e_i] = S(Y, Z) - g(Y, Z) + \eta(Y)\eta(Z), \tag{56}$$

$$\sum_{i=1}^{2n} g(e_i, \phi e_i) = 0, \tag{57}$$

$$\sum_{i=1}^{2n} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z), \tag{58}$$

$$\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{59}$$

$$\sum_{i=1}^{2n} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z) - \eta(Y)\eta(Z), \tag{60}$$

$$\sum_{i=1}^{2n} S(\phi e_i, Z)g(Y, \phi e_i) = S(Y, Z) - 2n\eta(Y)\eta(Z). \tag{61}$$

By virtue of (56)-(61), the equation (55) takes the form

$$\begin{aligned} S(Y, Z) &= 2g(\phi Y, \phi Z) + 2n\eta(Y)\eta(Z) \\ &\quad + \frac{2n-1}{2n}[S(Y, Z) - 2g(Y, Z) - 2(n-1)\eta(Y)\eta(Z)] \end{aligned} \tag{62}$$

which after simplification gives

$$S(Y, Z) = 2g(Y, Z) + 2(n-1)\eta(Y)\eta(Z). \tag{63}$$

Thus we can state the following theorem:

Theorem 6.2. *A pseudoprojectively flat Sasakian manifold with respect to the g-Tanaka-Webster connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Now, we consider that the manifold M with respect to the g-Tanaka-Webster connection is ϕ-projectively semisymmetric. Then from (46), we have

$$(\bar{P}(X, Y) \cdot \phi)Z = \bar{P}(X, Y)\phi Z - \phi\bar{P}(X, Y)Z = 0. \tag{64}$$

Taking $X = \xi$ in (64), we have

$$(\bar{P}(\xi, Y) \cdot \phi)Z = \bar{P}(\xi, Y)\phi Z - \phi\bar{P}(\xi, Y)Z = 0. \tag{65}$$

From (43), we find

$$\bar{P}(\xi, Y)\phi Z = -\frac{1}{2n}\bar{S}(Y, \phi Z)\xi, \qquad \phi\bar{P}(\xi, Y)Z = 0. \tag{66}$$

By combining (65) and (66), we obtain $\bar{S}(Y, \phi Z) = 0$, which in view of (16) gives

$$S(Y, \phi Z) = 2g(Y, \phi Z). \tag{67}$$

Now by replacing Y by ϕY then using (3) and (10), (67) takes the form

$$S(Y, Z) = 2g(Y, Z) + 2(n-1)\eta(Y)\eta(Z). \tag{68}$$

Thus we can state the following theorem:

Theorem 6.3. *A ϕ-projectively semisymmetric Sasakian manifold with respect to the g-Tanaka-Webster connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

Remark 6.4. A Sasakian manifold with respect to the g-Tanaka-Webster connection is always ξ-projectively flat.

7. Locally projectively ϕ-symmetric Sasakian manifolds with respect to the g-Tanaka-Webster connection

Definition. A Sasakian manifold with respect to the g-Tanaka-Webster connection is said to be locally projectively ϕ-symmetric if

$$\phi^2((\bar{\nabla}_W \bar{P})(X, Y)Z) = 0 \tag{69}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Using (13), we can write

$$\begin{aligned} &(\bar{\nabla}_W \bar{P})(X, Y)Z + \bar{P}(\bar{\nabla}_W X, Y)Z + \bar{P}(X, \bar{\nabla}_W Y)Z + \bar{P}(X, Y)\bar{\nabla}_W Z \\ &= (\nabla_W \bar{P})(X, Y)Z + \bar{P}(\nabla_W X, Y)Z + \bar{P}(X, \nabla_W Y)Z + \bar{P}(X, Y)\nabla_W Z \\ &\quad + g(W, \phi\bar{P}(X, Y)Z)\xi + \eta(\bar{P}(X, Y)Z)\phi W - \eta(W)\phi\bar{P}(X, Y)Z \end{aligned} \tag{70}$$

which in view of (13) can be written as

$$\begin{aligned} &(\bar{\nabla}_W \bar{P})(X, Y)Z \\ &= (\nabla_W \bar{P})(X, Y)Z + g(W, \phi\bar{P}(X, Y)Z)\xi + \eta(\bar{P}(X, Y)Z)\phi W \\ &\quad - \eta(W)\phi\bar{P}(X, Y)Z - g(W, \phi X)\bar{P}(\xi, Y)Z - \eta(X)\bar{P}(\phi W, Y)Z \\ &\quad + \eta(W)\bar{P}(\phi X, Y)Z - g(W, \phi Y)\bar{P}(X, \xi)Z - \eta(Y)\bar{P}(X, \phi W)Z \\ &\quad + \eta(W)\bar{P}(X, \phi Y)Z - \eta(Z)\bar{P}(X, Y)\phi W + \eta(W)\bar{P}(X, Y)\phi Z. \end{aligned} \tag{71}$$

By Differentiating (43), (14) and (16) with respect to W, we have

$$\begin{aligned} &(\nabla_W \bar{P})(X, Y)Z \\ &= (\nabla_W \bar{R})(X, Y)Z - \frac{1}{2n}[(\nabla_W \bar{S})(Y, Z)X - (\nabla_W \bar{S})(X, Z)Y], \end{aligned} \tag{72}$$

$$\begin{aligned} &(\nabla_W \bar{R})(X, Y)Z \\ &= (\nabla_W R)(X, Y)Z + g(X, Z)g(W, \phi Y)\xi - g(Y, Z)g(W, \phi X)\xi \end{aligned} \tag{73}$$

$$
\begin{aligned}
&- g(X,Z)\eta(Y)\phi W + g(Y,Z)\eta(X)\phi W - g(W,Z)\eta(Y)\phi X \\
&+ \eta(Z)g(Y,W)\phi X + g(Y,\phi Z)g(X,W)\xi - g(Y,\phi Z)\eta(X)W \\
&+ g(W,Z)\eta(X)\phi Y - \eta(Z)g(X,W)\phi Y + g(X,\phi Z)g(Y,W)\xi \\
&- g(X,\phi Z)\eta(Y)W + 2g(X,W)\eta(Y)\phi Z - 2\eta(X)g(Y,W)\phi Z \\
&+ 2g(Y,\phi X)g(W,Z)\xi - 2g(Y,\phi X)\eta(Z)W - g(W,\phi Y)\eta(Z)X \\
&- g(W,\phi Z)\eta(Y)X + g(W,\phi X)\eta(Z)Y + g(W,\phi Z)\eta(X)Y,
\end{aligned}
$$

$$
\begin{aligned}
(74)\qquad & (\nabla_W \bar{S})(Y,Z)X \\
&= (\nabla_W S)(Y,Z)X - 2(n-1)[g(W,\phi Y)\eta(Z)X + g(W,\phi Z)\eta(Y)X]
\end{aligned}
$$

respectively.

Now combining the equations (72)-(74), we obtain

$$
\begin{aligned}
(75)\qquad & (\nabla_W \bar{P})(X,Y)Z \\
&= (\nabla_W R)(X,Y)Z + g(X,Z)g(W,\phi Y)\xi - g(Y,Z)g(W,\phi X)\xi \\
&- g(X,Z)\eta(Y)\phi W + g(Y,Z)\eta(X)\phi W - g(W,Z)\eta(Y)\phi X \\
&+ \eta(Z)g(Y,W)\phi X + g(Y,\phi Z)g(X,W)\xi - g(Y,\phi Z)\eta(X)W \\
&+ g(W,Z)\eta(X)\phi Y - \eta(Z)g(X,W)\phi Y + g(X,\phi Z)g(Y,W)\xi \\
&- g(X,\phi Z)\eta(Y)W + 2g(X,W)\eta(Y)\phi Z - 2\eta(X)g(Y,W)\phi Z \\
&+ 2g(Y,\phi X)g(W,Z)\xi - 2g(Y,\phi X)\eta(Z)W - g(W,\phi Y)\eta(Z)X \\
&- g(W,\phi Z)\eta(Y)X + g(W,\phi X)\eta(Z)Y + g(W,\phi Z)\eta(X)Y \\
&- \frac{1}{2n}[(\nabla_W S)(Y,Z)X - 2(n-1)g(W,\phi Y)\eta(Z)X \\
&- 2(n-1)g(W,\phi Z)\eta(Y)X - (\nabla_W S)(X,Z)Y \\
&+ 2(n-1)g(W,\phi X)\eta(Z)Y + 2(n-1)g(W,\phi Z)\eta(X)Y].
\end{aligned}
$$

Now differentiating (42) with respect to W, we have

$$
\begin{aligned}
(76)\qquad & (\nabla_W P)(X,Y)Z \\
&= (\nabla_W R)(X,Y)Z - \frac{1}{2n}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y].
\end{aligned}
$$

By the use of (76), (75) takes the form

$$
\begin{aligned}
(77)\qquad & (\nabla_W \bar{P})(X,Y)Z \\
&= (\nabla_W P)(X,Y)Z + g(X,Z)g(W,\phi Y)\xi - g(Y,Z)g(W,\phi X)\xi \\
&- g(X,Z)\eta(Y)\phi W + g(Y,Z)\eta(X)\phi W - g(W,Z)\eta(Y)\phi X \\
&+ \eta(Z)g(Y,W)\phi X + g(Y,\phi Z)g(X,W)\xi - g(Y,\phi Z)\eta(X)W \\
&+ g(W,Z)\eta(X)\phi Y - \eta(Z)g(X,W)\phi Y + g(X,\phi Z)g(Y,W)\xi \\
&- g(X,\phi Z)\eta(Y)W + 2g(X,W)\eta(Y)\phi Z - 2\eta(X)g(Y,W)\phi Z
\end{aligned}
$$

$$+ 2g(Y, \phi X)g(W, Z)\xi - 2g(Y, \phi X)\eta(Z)W - \frac{1}{n}[g(W, \phi Y)\eta(Z)X + g(W, \phi Z)\eta(Y)X - g(W, \phi X)\eta(Z)Y - g(W, \phi Z)\eta(X)Y].$$

Operating ϕ^2 to the both sides of (71), we have

$$\begin{aligned} &\phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z \\ &= \phi^2(\nabla_W \bar{P})(X, Y)Z + \frac{1}{2n}[\bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y)]\phi W \\ &\quad + \eta(W)\phi\bar{P}(X, Y)Z - \eta(X)\phi^2\bar{P}(\phi W, Y)Z + \eta(W)\phi^2\bar{P}(\phi X, Y)Z \\ &\quad - \eta(Y)\phi^2\bar{P}(X, \phi W)Z + \eta(W)\phi^2\bar{P}(X, \phi Y)Z - \eta(Z)\phi^2\bar{P}(X, Y)\phi W \\ &\quad + \eta(W)\phi^2\bar{P}(X, Y)\phi Z \end{aligned} \tag{78}$$

which by using (77) becomes

$$\begin{aligned} &\phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z \\ &= \phi^2(\nabla_W P)(X, Y)Z + g(X, Z)\eta(Y)\phi W \\ &\quad - g(Y, Z)\eta(X)\phi W + g(W, Z)\eta(Y)\phi X - \eta(Z)g(Y, W)\phi X \\ &\quad - g(Y, \phi Z)\eta(X)\phi^2 W - g(W, Z)\eta(X)\phi Y + \eta(Z)g(X, W)\phi Y \\ &\quad - g(X, \phi Z)\eta(Y)\phi^2 W - 2g(X, W)\eta(Y)\phi Z + 2\eta(X)g(Y, W)\phi Z \\ &\quad - 2g(Y, \phi X)\eta(Z)\phi^2 W - \frac{1}{n}[g(W, \phi Y)\eta(Z)\phi^2 X \\ &\quad + g(W, \phi Z)\eta(Y)\phi^2 X - g(W, \phi X)\eta(Z)\phi^2 Y - g(W, \phi Z)\eta(X)\phi^2 Y] \\ &\quad + \frac{1}{2n}[\bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y)]\phi W + \eta(W)\phi\bar{P}(X, Y)Z \\ &\quad - \eta(X)\phi^2\bar{P}(\phi W, Y)Z + \eta(W)\phi^2\bar{P}(\phi X, Y)Z - \eta(Y)\phi^2\bar{P}(X, \phi W)Z \\ &\quad + \eta(W)\phi^2\bar{P}(X, \phi Y)Z - \eta(Z)\phi^2\bar{P}(X, Y)\phi W + \eta(W)\phi^2\bar{P}(X, Y)\phi Z. \end{aligned} \tag{79}$$

By taking X, Y, Z and W orthogonal to ξ, (79) reduces to

$$\phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z = \phi^2(\nabla_W P)(X, Y)Z. \tag{80}$$

Thus we can state the following theorem:

Theorem 7.1. *A Sasakian manifold is locally projectively ϕ-symmetric with respect to the g-Tanaka-Webster connection if and only if it is so with respect to the Levi-Civita connection.*

Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M and hence form a basis of T_pM. Let g be the Riemannian metric defined by

$$g(e_1,e_1)= g(e_2,e_2)= g(e_3,e_3)=1,\ g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

Let η be the 1-form on M defined by $\eta(X)=g(X,e_3)=g(X,\xi)$ for all $X\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi e_1=-e_2,\ \phi e_2=e_1,\ \phi e_3=0.$$

The linearity property of ϕ and g yields

$$\eta(\xi)=g(\xi,\xi)=1,\ \phi^2X=-X+\eta(X)\xi,\ \eta(\phi X)=0,$$
$$g(X,\xi)=\eta(X),\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for all $X,Y\in\chi(M)$.

Now, by direct computations we obtain

$$[e_1,e_2]=-2e_3,\qquad [e_1,e_3]=0,\qquad [e_2,e_3]=0.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])$$
$$+g(Y,[Z,X])+g(Z,[X,Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1}e_1=0,\ \nabla_{e_2}e_1=e_3,\ \nabla_{e_3}e_1=e_2,\ \nabla_{e_1}e_2=-e_3,\ \nabla_{e_2}e_2=0,$$
$$\nabla_{e_3}e_2=-e_1,\ \nabla_{e_1}e_3=e_2,\ \nabla_{e_2}e_3=-e_1,\ \nabla_{e_3}e_3=0. \tag{81}$$

Let $X=\sum_{i=1}^{3}X^ie_i=X^1e_1+X^2e_2+X^3e_3\in\chi(M)$.

Also, one can easily verify that

$$\nabla_X\xi=-\phi X\qquad\text{and}\qquad(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X.$$

Thus the manifold M is a Sasakian manifold. Using (13) in (81), we obtain

$$\bar\nabla_{e_1}e_1=0,\ \bar\nabla_{e_2}e_1=0,\ \bar\nabla_{e_3}e_1=2e_2,\ \bar\nabla_{e_1}e_2=0,\ \nabla_{e_2}e_2=0,$$
$$\bar\nabla_{e_3}e_2=-2e_1,\ \bar\nabla_{e_1}e_3=0,\ \bar\nabla_{e_2}e_3=0,\ \bar\nabla_{e_3}e_3=0. \tag{82}$$

It is known that

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z. \tag{83}$$

By using the above results, we can easily obtain the components of the curvature tensors as follows:

$$R(e_1,e_2)e_1=3e_2,\ R(e_1,e_2)e_2=-3e_1,\ R(e_1,e_2)e_3=0,$$
$$R(e_2,e_3)e_1=0,\ R(e_2,e_3)e_2=-e_3,\ R(e_2,e_3)e_3=e_2,$$
$$R(e_1,e_3)e_1=-e_3,\ R(e_1,e_3)e_2=0,\ R(e_1,e_3)e_3=e_1, \tag{84}$$

and

$$\bar R(e_1,e_2)e_1=4e_2,\ \bar R(e_1,e_2)e_2=-4e_1,\ \bar R(e_1,e_2)e_3=0,$$
$$\bar R(e_2,e_3)e_1=0,\ \bar R(e_2,e_3)e_2=0,\ \bar R(e_2,e_3)e_3=0, \tag{85}$$

$$\bar{R}(e_1, e_3)e_1 = 0,\ \bar{R}(e_1, e_3)e_2 = 0,\ \bar{R}(e_1, e_3)e_3 = 0.$$

From these curvature tensors, we calculate the Ricci tensors and the scalar curvatures as follows:

$$S(e_1, e_1) = -2,\ S(e_2, e_2) = -2,\ S(e_3, e_3) = 2,\ r = -2,$$

$$\bar{S}(e_1, e_1) = -4,\ \bar{S}(e_2, e_2) = -4,\ \bar{S}(e_3, e_3) = 0,\ \bar{r} = -8.$$

From the expressions of the curvature tensors and the Ricci tensors, we have

$$\bar{P}(e_1, e_2)e_3 = \bar{P}(e_1, e_3)e_3 = \bar{P}(e_2, e_3)e_3 = 0,$$

that is, the manifold M is ξ-projectively flat with respect to the g-Tanaka-Webster connection. Hence Remark 6.4 is verified.

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