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MYERS' THEOREM FOR COMPLETE RIEMANNIAN HYPERSURFACES

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Abstract. In this paper we study the Myers' theorem for orientable Riemannian hypersurfaces under some restrictions on the mean curvature, the second-order mean curvature, or the divergence of the shape operator and give some estimates for the diameter of such hypersurfaces.

1. Introduction

A natural problem in the geometry of hypersurfaces is to explore the interplay between extrinsic and intrinsic geometric data of the hypersurfaces and their geometric and topological properties. A famous result in this direction is the Myers' theorem, which states that if the Ricci tensor of a manifold has a lower bound $(n-1)a > 0$, then the manifold is compact and its diameter is less than $\frac{\pi}{\sqrt{a}}$ [8]. We are interested in the influences of the extrinsic geometric data on the compactness and diameter of an orientable Riemannian hypersurface; including the mean curvature, the second-order mean curvature, the divergence of the shape operator, and the sectional curvature of the ambient manifold. This can be applied also to the space-like hypersurfaces in the Lorentzian manifolds.

Myers' theorem has been generalized in many ways. One of the most instances was done using the Bakry-Emery Ricci tensor. For a Riemannian manifold (M, g) and $f \in C^\infty(M)$, the Bakry-Emery Ricci tensor is defined by $Ric_f = Ric + Hessf$. This is a natural and good substitute for the Ricci tensor. Many interesting results in differential geometry were proved and extended for this tensor in weighted manifolds [10]. Some of the most well-known results in this regard are Myers' theorem [2–4, 6, 10], volume comparison theorems [7, 10, 11], eigenvalue estimates [4, 11], Li-Yau Harnack inequalities [5, 13], Cheeger-Gromoll splitting theorem [3, 10, 11], Betti number estimate, etc.

First, we state a generalization of Myers' theorem which is proved by Wei and Wylie [10].

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Theorem 1.1. *Let (M^n, g) be a complete and connected Riemannian manifold, $f \in C^\infty(M)$, and $H, K > 0$ be constants. If $|f| \leq K$ satisfies the following inequality*

$$Ric_f = Ric + Hessf \geq (n-1)Hg > 0,$$

then M is compact and its diameter satisfies

$$diam(M) \leq \frac{\pi}{\sqrt{H}} + \frac{4K}{(n-1)H}.$$

Limoncu obtained a better estimate for the diameter as follows [6].

Theorem 1.2. *Let (M^n, g) be a complete and connected Riemannian manifold, $H, K > 0$ some constants, and V a smooth vector field on M with $|V| < K$. If $Ric + L_V g \geq (n-1)Hg > 0$, then M is compact and*

$$diam(M) \leq \frac{4\pi}{(n-1)H}\left(\sqrt{2}K + \sqrt{2K^2 + (n-1)^2 H}\right).$$

If we set $V = \frac{1}{2}\nabla f$, then the above condition yields

$$Ric + Hessf \geq (n-1)Hg > 0,$$

with $|\nabla f| \leq 4K^2$ and

$$diam(M) \leq \frac{\pi}{(n-1)H}\left(\sqrt{\frac{K}{2}} + \sqrt{\frac{K}{2} + (n-1)^2 H}\right).$$

He also obtained the following result in [6].

Theorem 1.3. *Let (M^n, g) be a complete and connected Riemannian manifold, and p be a fixed point, and let $r(x) = dist(x, p)$. If $f \in C^\infty(M)$ satisfies*

$$Ric_f = Ric + Hessf \geq (n-1)Hg > 0$$

for some constants $H, K > 0$ with $|\nabla f|^2 \leq \frac{K}{r(x)^2}$, then M is compact and its diameter satisfies

$$diam_p(M) \leq \frac{\pi\sqrt{4\sqrt{K} + n - 1}}{(n-1)H}.$$

Theorem 1.2 has been improved by Wu in [12]. Tadano proved the following theorem in [9].

Theorem 1.4. *Let (M, g) be a complete and connected Riemannian manifold of dimension n and $f \in C^\infty(M)$. If $|f| \leq K$ satisfies*

$$Ric_f = Ric + Hessf \geq (n-1)Hg > 0$$

for some constants $H, K > 0$, then M is compact and

$$diam(M) \leq \frac{\pi}{\sqrt{H}}\sqrt{1 + \frac{2\sqrt{2}K}{n-1}}.$$

The main goal of this paper is to show how the geometric data of a complete connected orientable hypersurface can determine its compactness. This data consists of the mean curvature H, the second-order mean curvature H_2, the divergence of the shape operator, and the constantness (non-constantness) of the sectional curvature of the ambient manifold. We are also interested in parameters affecting the compactness when the ambient manifold doesn't have constant sectional curvature. We extend Theorems 1.2, 1.3 and 1.4 to hypersurfaces; and obtain new results about the diameter of these hypersurfaces. We use the ∞-Bakry-Emery Ricci tensor $Ric_f = Ric + Hessf$ by replacing a certain function on the hypersurface with the energy function f. The function that we want to replace should carry the extrinsic geometric data of the hypersurfaces, hence we use the functions $\frac{1}{2}|A|^2$. Imposing some restrictions on both of the mean curvature and the second-order mean curvature, or the divergence of the shape operator, we get certain estimates of the diameter of the hypersurface.

Our main results are as follows. The first is a compactness result by restriction on the shape operator.

Theorem 1.5. *Let $(\Sigma^n, g_\Sigma) \subseteq M^{n+1}(c)$ (c is the sectional curvature of the ambient manifold) be a complete orientable Riemannian hypersurface with the shape operator A; and let $p \in \Sigma$ be a fixed point, and $r(x) := dist_\Sigma(p, x)$. If for some constants $K, d > 0$ and integrable function $g : \mathbb{R} \to \mathbb{R}$ with $\left(\int_0^{+\infty} g^2(r)dr\right)^{\frac{1}{2}} \leq K$ we have*

$$\left|div(A^2) - AdivA\right|^2(r(x)) \leq g(r) \text{ and } Ric_{\frac{1}{2}|A|^2} \geq ((n-1)d)\, g_\Sigma,$$

then

$$diam_\Sigma \leq 2\sqrt{\theta_0},$$

where θ_0 is the first positive root of the quartic equation

$$-(n-1)du^4 + \sqrt{\frac{3}{8}}Ku^3 + n\pi^2 = 0.$$

We prove the following compactness result by restricting on the mean curvature H and the second-order mean curvature H_2.

Theorem 1.6. *Let $(\Sigma^n, g_\Sigma) \subset M^{n+1}$ be a complete connected orientable hypersurface with the shape operator A, $p \in \Sigma$ be a fixed point, and $r(x) := dist_\Sigma(p, x)$; and let $H^2(r(x)) \leq g_1(r)$, where $g_1 : \mathbb{R} \to \mathbb{R}$ is an integrable function with $\left(\int_0^{+\infty} g_1^2(r)dr\right)^{\frac{1}{2}} \leq K_1$. If for some constant $d > 0$, $Ric_{\frac{1}{2}|A|^2} \geq ((n-1)d)\, g_\Sigma$, and one of the following conditions holds,*

a) *$HessH_2 \geq \beta g_\Sigma$ and $(n-1)d + \frac{n(n-1)}{2}\beta > 0$ for some constant $\beta \geq 0$; (in this item the condition $(n-1)d > 0$ is superfluous.)*

b) $|\nabla H_2(r(x))| \le g_2(r)$, *where* $g_2 : \mathbb{R} \to \mathbb{R}$ *is an integrable function with* $\left(\int_0^{+\infty} g_2^2(r)dr\right)^{\frac{1}{2}} \le K_2$;

c) $H_2^2(r(x)) \le g_2(r)$, *where* $g_2 : \mathbb{R} \to \mathbb{R}$ *is an integrable function with* $\int_0^{+\infty} g_2(r)dr \le K_2$,

then Σ *is compact and its diameter is approximated by*

$$diam_\Sigma \le 2\sqrt{\theta_0},$$

where θ_0 *is the first positive root of* $p(u)$. *The polynomial* $p(u)$ *corresponding to each condition is as follows,*

a) $p(u) = -\left((n-1)d + \frac{n(n-1)\beta}{2}\right)u^5 + (n-1)\pi^2 u + \sqrt{2}n^2\pi^2 K_1$;

b) $p(u) = -(n-1)du^5 + \frac{n(n-1)}{\sqrt{2}}K_2\pi u^2 + (n-1)\pi^2 u + \sqrt{2}n^2\pi^2 K_1$;

c) $p(u) = -(n-1)du^8 + K_2 u^6 + (n-1)\pi^2 u^4 + \sqrt{2}n^2\pi^2 K_1 u^3 + 2\pi^4$.

Note that, in Theorem 3.1 when a hypersurface $\Sigma^n \subset M^{n+1}$ is compact and the ambient manifold M has a constant sectional curvature, we get an estimate for its diameter by putting a restriction on the mean value of $\left|div(A^2) - AdivA\right|^4$ on every maximal radial geodesic. Also in Theorem 3.4 we get an estimate of the diameter of a compact hypersurface, by putting some restrictions on the mean values of H^4 and H_2^2 on every maximal radial geodesic.

2. Preliminaries

In this section, we compute $Hess(\frac{1}{2}|A|^2)$ by two methods. The first method is simple and doesn't differ if the sectional curvature of the ambient manifold is constant or not. The second method, shows the parameters which affect $Hess(\frac{1}{2}|A|^2)$ when the ambient manifold doesn't have a constant sectional curvature. In order to do this, we present some definitions and lemmas.

Definition ([1])**.** Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface with the shape operator A, and let $\{e_p\}$ be a local orthonormal frame field on Σ. Then we define:

a) The mean curvature of Σ^n, by $\frac{1}{n}Trace(A)$ and denote it by H.

b) The second-order mean curvature of Σ^n, by $\frac{2}{n(n-1)}\sum_{i<j}\lambda_i\lambda_j$ where $A = diag(\lambda_1, \dots, \lambda_n)$, and denote it by H_2.

c) The scalar curvature of Σ^n at a point x, by $\sum_{i<j} K(e_i, e_j)$, where K is the sectional curvature of Σ^n and $\{e_i\}$ is a basis of $T_x\Sigma$; and we denote it by $scal(x)$.

d) The divergence of the shape operator A, by $\sum_p \left(\nabla_{e_p}A\right)e_p$, and denote it by $divA$.

e) The square of the norm of the shape operator, by $|A|^2 := Trace(A.A^t)$.

Definition. Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, A be the corresponding shape operator, and

$\bar{R}$ be the curvature tensor of the ambient manifold M; and let $\{e_p\}$ be a local orthonormal frame field on Σ. We define the tensor $\overline{Ric}_A$ by

$$\overline{Ric}_A(N,X) := \sum\nolimits_p \left\langle \bar{R}(N,Ae_p)e_p, X\right\rangle.$$

To compute the gradient of the function $\frac{1}{2}|A|^2$, we need the following lemma.

Lemma 2.1. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface with the shape operator A, and $X \in \Gamma(T\Sigma)$ be a smooth vector field on Σ^n. Then*

a) $\langle(\nabla_X A), A\rangle = Trace((\nabla_X A)A) = \left\langle div(A^2) - AdivA, X\right\rangle + \overline{Ric}_A(N,X)$;

b) $\langle divA, X\rangle = n\nabla_X H - \overline{Ric}(X,N)$.

Proof. For part (a), suppose that $\{e_p\}$ is a local orthonormal frame field on Σ, by Codazzi equation

$$(\nabla_X A)\, Y = (\nabla_Y A)\, X + \left(\bar{R}(Y,X)N\right)^T.$$

So

$$\begin{aligned}\langle(\nabla_X A), A\rangle &= Trace((\nabla_X A)A) = \sum\nolimits_p \langle(\nabla_X A)\, Ae_p, e_p\rangle \\ &= \sum\nolimits_p \langle(\nabla_X A)\, e_p, Ae_p\rangle = \sum\nolimits_p \left\langle\left(\nabla_{e_p} A\right) X - \bar{R}(X,e_p)N, Ae_p\right\rangle \\ &= \sum\nolimits_p \left\langle\left(\nabla_{e_p} A\right) Ae_p, X\right\rangle + \sum\nolimits_p \left\langle \bar{R}(N,Ae_p)e_p, X\right\rangle \\ &= \sum\nolimits_p \left\langle\left(\nabla_{e_p} A^2\right) e_p, X\right\rangle - \left\langle A\left(\sum\nolimits_p \left(\nabla_{e_p} A\right) e_p\right), X\right\rangle \\ &\quad + \overline{Ric}_A(N,X) \\ &= \left\langle div(A^2) - AdivA, X\right\rangle + \overline{Ric}_A(N,X).\end{aligned}$$

For (b) see [1]. □

Corollary 2.2. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. Then for any vector field $X \in \Gamma(T\Sigma)$ we have $\overline{Ric}(X,N) = 0$. Therefore*

a) $divA = n\nabla H$;

b) *if the mean curvature is constant, then $divA = 0$.*

Now, we compute $Hess(|A|^2)$ by the following proposition.

Proposition 2.3. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. For the smooth vector fields $X, Y \in \Gamma(T\Sigma)$ we have*

a) $$\frac{1}{2}Hess(|A|^2)(X,Y) = n^2\langle\nabla H, X\rangle\langle\nabla H, Y\rangle + n^2 HHessH(X,Y) - \frac{n(n-1)}{2}HessH_2(X,Y);$$

b) *If Y is a parallel vector field, then*

$$\frac{1}{2}Hess(|A|^2)(X,Y) = \left\langle \nabla_X \left(div(A^2) - AdivA\right), Y\right\rangle + X.\left(\overline{Ric}_A(N,Y)\right).$$

Proof. For (a), by $|A|^2 = (nH)^2 - n(n-1)H_2$, we have

$$\nabla(\frac{1}{2}|A|^2) = n^2 H\nabla H - \frac{n(n-1)}{2}\nabla H_2.$$

So

$$\frac{1}{2}Hess(|A|^2)(X,Y) = n^2\langle\nabla H, X\rangle\langle\nabla H, Y\rangle + n^2 HHessH(X,Y) \\ - \frac{n(n-1)}{2}HessH_2(X,Y).$$

For (b), suppose that x is an arbitrary point. For simplicity, suppose that $\{e_p\}$ is a local orthonormal frame field on Σ such that $\nabla_{e_p}e_q(x) = 0$, and $X, Y \in \Gamma(T\Sigma)$ are parallel vector fields. Then

$$\nabla(\frac{1}{2}|A|^2) = \sum\nolimits_p \left(e_p.|A|^2\right)e_p = \frac{1}{2}\sum\nolimits_p \left\langle\nabla_{e_p}A, A\right\rangle e_p \\ = \sum\nolimits_p \left(\left\langle div(A^2) - AdivA, e_p\right\rangle + \overline{Ric}_A(N, e_p)\right)e_p.$$

So

$$\begin{aligned}\frac{1}{2}Hess(|A|^2)(X,Y) &= \sum\nolimits_p \left\langle\nabla_X\left[\left\langle div(A^2) - AdivA, e_p\right\rangle e_p\right], Y\right\rangle \\ &\quad + \sum\nolimits_p \left\langle\nabla_X\left[\overline{Ric}_A(N,e_p)e_p\right], Y\right\rangle \\ &= \sum\nolimits_p \left\langle\left(X.\left\langle div(A^2) - AdivA, e_p\right\rangle\right)e_p, Y\right\rangle \\ &\quad + X.\sum\nolimits_p \left\langle\left(\overline{Ric}_A(N,e_p)\right)e_p, Y\right\rangle \\ &= \sum\nolimits_p \left\langle\left\langle\nabla_X\left(div(A^2) - AdivA\right), e_p\right\rangle e_p, Y\right\rangle \\ &\quad + X.\sum\nolimits_p \left\langle\left(\overline{Ric}_A(N,e_p)\right)e_p, Y\right\rangle \\ &= \left\langle\nabla_X\left(div(A^2) - AdivA\right), Y\right\rangle \\ &\quad + X.\left(\overline{Ric}_A(N, \sum\nolimits_p\langle e_p, Y\rangle e_p)\right) \\ &= \left\langle\nabla_X\left(div(A^2) - AdivA\right), Y\right\rangle + X.\left(\overline{Ric}_A(N,Y)\right). \quad \square\end{aligned}$$

Corollary 2.4. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. If M has constant sectional curvature, then*

$$Hess(\frac{1}{2}|A|^2)(X,Y) = n^2\langle\nabla H, X\rangle\langle\nabla H, Y\rangle + n^2 HHessH(X,Y) \\ - Hess\,(scal)\,(X,Y).$$

Proof. Suppose that λ_i's are eigenvalues of A. By the assumptions, M has a constant sectional curvature, so by Gauss equation, we have $\lambda_i\lambda_j = K(e_i,e_j) - c$. Thus

$$\frac{n(n-1)}{2}H_2 = \sum\nolimits_{i<j}\lambda_i\lambda_j = \sum\nolimits_{i<j}K(e_i,e_j) - c = scal - \frac{n(n-1)}{2}c.$$

So $\frac{n(n-1)}{2}HessH_2 = Hess(scal)$. □

Proposition 2.5. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. Then for a parallel vector field $Y \in \Gamma(T\Sigma)$ we have*

$$\begin{aligned} X.\left(\overline{Ric}_A(N,Y)\right) = &\sum\nolimits_p \left\langle \left(\bar\nabla_X\bar R\right)(N,Ae_p)e_p, Y\right\rangle - \overline{Ric}_A(AX,Y)\\ &+ \overline{Ric}_{(\nabla_X A)}(N,Y) + \left\langle \bar R(N,A^2X)N,Y\right\rangle\\ &+ \langle AX,Y\rangle\,\overline{Ric}_A(N,N). \end{aligned}$$

Proof. Suppose that x is an arbitrary point; for simplicity, assume that $\{e_p\}$ is a local orthonormal frame field with $\nabla_{e_i}e_j(x) = 0$. So

$$\begin{aligned} &X.\left(\overline{Ric}_A(N,Y)\right)\\ =&\sum\nolimits_p \left(\left\langle\left(\bar\nabla_X\bar R\right)(N,Ae_p)e_p,Y\right\rangle + \left\langle \bar R(\bar\nabla_X N, Ae_p)e_p, Y\right\rangle\right)\\ &+\sum\nolimits_p\left(\left\langle \bar R(N,\bar\nabla_X(Ae_p))e_p,Y\right\rangle + \left\langle \bar R(N,Ae_p)\bar\nabla_X e_p, Y\right\rangle\right)\\ &+\sum\nolimits_p\left\langle \bar R(N,(Ae_p))e_p,\bar\nabla_X Y\right\rangle\\ =&\sum\nolimits_p \left(\left\langle\left(\bar\nabla_X\bar R\right)(N,Ae_p)e_p,Y\right\rangle + \left\langle \bar R(-AX, Ae_p)e_p, Y\right\rangle\right)\\ &+\sum\nolimits_p\left\langle \bar R(N,(\nabla_X A)\,e_p + S\,(X,Ae_p))e_p,Y\right\rangle\\ &+\sum\nolimits_p\left(\left\langle \bar R(N,Ae_p)S\,(X,e_p)\,,Y\right\rangle + \left\langle \bar R(N,(Ae_p))e_p, S\,(X,Y)\right\rangle\right)\\ =&\sum\nolimits_p \left\langle\left(\bar\nabla_X\bar R\right)(N,Ae_p)e_p,Y\right\rangle - \overline{Ric}_A(AX,Y)\\ &+\overline{Ric}_{(\nabla_X A)}(N,Y) + 0 + \sum\nolimits_p\left\langle \bar R(N,Ae_p)\,\langle AX,e_p\rangle\,N,Y\right\rangle\\ &+\langle AX,Y\rangle\sum\nolimits_p\left\langle \bar R(N,(Ae_p))e_p,N\right\rangle\\ =&\sum\nolimits_p \left\langle\left(\bar\nabla_X\bar R\right)(N,Ae_p)e_p,Y\right\rangle - \overline{Ric}_A(AX,Y)\\ &+\overline{Ric}_{(\nabla_X A)}(N,Y) + \left\langle \bar R(N,A^2X)N,Y\right\rangle\\ &+\langle AX,Y\rangle\,\overline{Ric}_A(N,N), \end{aligned}$$

where $S\,(X,Y)$ is the second fundamental form of Σ. □

By the following definition, we recall an extension of Ricci tensor which is called the ∞-Bakry-Emery Ricci tensor. This tensor field with the weighted

Laplacian $\Delta_f = \Delta - \langle \nabla f, - \rangle$ plays an important role in the study of the geometry of weighted manifolds. We use this tensor field for hypersurfaces and replace $\frac{1}{2}|A|^2$ with f, where A is the shape operator of the hypersurface.

Definition ([10])**.** The ∞-Bakry-Emery Ricci tensor is defined by

$$Ric_f = Ric + Hessf.$$

So we define the tensor $Ric_{\frac{1}{2}|A|^2}$ as

$$Ric_{\frac{1}{2}|A|^2} := Ric + Hess(\frac{1}{2}|A|^2).$$

Corollary 2.6. *Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. Then for any parallel smooth vector fields $X, Y \in \Gamma(T\Sigma)$ we have*

$$\begin{aligned}
Ric_{\frac{1}{2}|A|^2}(X,Y) &= Ric(X,Y) + Hess(\frac{1}{2}|A|^2)(X,Y) \\
&= Ric(X,Y) + \left\langle \nabla_X \left(div(A^2) - A div A \right), Y \right\rangle \\
&\quad + X. \left(\overline{Ric}_A(N,Y) \right) \\
&= Ric(X,Y) + n^2 \langle \nabla H, X \rangle \langle \nabla H, Y \rangle + n^2 H Hess H(X,Y) \\
&\quad - \frac{n(n-1)}{2} Hess H_2(X,Y).
\end{aligned}$$

Remark 2.7. Let $\Sigma^n \subset M^{n+1}$ be an orientable Riemannian hypersurface, N be a unit normal vector field on Σ^n, and A be the corresponding shape operator. Then for smooth vector fields $X, Y \in \Gamma(T\Sigma)$, the followings are well-known.

a) $Ric(X,Y) = \overline{Ric}(X,Y) - \left\langle \bar{R}(X,N)N, Y \right\rangle + nH \langle AX, Y \rangle - \langle AX, AY \rangle$. So when the ambient manifold M has constant sectional curvature c, we have

$$Ric(X,Y) = (n-1)\, c \langle X, Y \rangle + nH \langle AX, Y \rangle - \langle AX, AY \rangle .$$

b) $2scal = 2\overline{scal} - 2\overline{Ric}(N,N) + n(n-1)H_2$.

In the above items, Ric and $scal$ are the Ricci and scalar curvature of Σ; and $\overline{Ric}$ and $\overline{scal}$ are the Ricci and scalar curvature of M.

3. Myers' theorem for Riemannian hypersurfaces

In this section, we prove Theorems 1.5 and 1.6; and show some of their consequences. First, we extend Theorem 1.3. In [6] the author used the modified Laplacian to get the result; here we get another estimate with a different approach.

Proof of Theorem 1.5. At first, note that the ambient manifold has a constant sectional curvature, so for any vector field X, we have $\overline{Ric}_A(N,X) \equiv 0$. Thus

$$Ric_{\frac{1}{2}|A|^2}(X,Y) = Ric(X,Y) + \frac{1}{2} Hess(|A|^2)(X,Y)$$

$$= Ric(X,Y) + \left\langle \nabla_X \left(div(A^2) - AdivA\right), Y\right\rangle.$$

Now suppose that $\gamma(t)$ is the unit minimal geodesic from the fixed point p to an arbitrary point $x \in \Sigma\backslash cut(p)$ with length l. By considering a parallel orthonormal frame $\{E_1 = \gamma', E_2, \dots, E_n\}$ along γ, and the smooth function $f(t) = \sin(\frac{\pi t}{l})$, and the index form

$$I(fE_i, fE_i) = \int_0^l \Big(|f'E_i|^2 - \langle R(fE_i,\gamma')\gamma', fE_i\rangle\Big)dt,$$

we have

$$\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) = \int_0^l \Big((n-1)f'^2 - f^2 Ric(\gamma',\gamma')\Big)dt.$$

For simplicity, we set $V = div(A^2) - AdivA$, so

$$Ric_{\frac{1}{2}|A|^2}(\gamma',\gamma') = Ric(\gamma',\gamma') + \frac{d}{dt}\langle V, \gamma'\rangle.$$

By the hypothesis $Ric_{\frac{1}{2}|A|^2}(\gamma',\gamma') \geq (n-1)d$, so

$$-Ric(\gamma',\gamma') \leq -(n-1)d + \frac{d}{dt}\langle V, \gamma'\rangle.$$

Thus

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) &= \int_0^l \Big((n-1)f'^2 - f^2 Ric(\gamma',\gamma')\Big)dt \\ &\leq \int_0^l \Big((n-1)f'^2 - (n-1)f^2 d + f^2\frac{d}{dt}\langle V,\gamma'\rangle\Big)dt \\ &= (n-1)\int_0^l \Big(f'^2 - f^2 d\Big)dt + \int_0^l f^2\frac{d}{dt}\langle V,\gamma'\rangle dt \\ &= \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \int_0^l f^2\frac{d}{dt}\langle V,\gamma'\rangle dt.\end{aligned} \tag{1}$$

Now, we get an estimate for $\int_0^l f^2\frac{d}{dt}\langle V,\gamma'\rangle dt$,

$$\begin{aligned}\int_0^l f^2\frac{d}{dt}\langle V,\gamma'\rangle dt &= -2\int_0^l ff'\langle V,\gamma'\rangle dt \\ &\leq \int_0^l (f')^2 dt + \int_0^l f^2\langle V,\gamma'\rangle^2 dt \\ &\leq \frac{\pi^2}{2l} + \left(\int_0^l f^4(t)dt\right)^{\frac{1}{2}}\left(\int_0^l |V|^4 dt\right)^{\frac{1}{2}} \\ &\leq \frac{\pi^2}{2l} + \left(\int_0^l f^4(t)dt\right)^{\frac{1}{2}}\left(\int_0^l g(t)^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{\pi^2}{2l} + \sqrt{\frac{3l}{8}}K.$$

By (1) we have

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\pi^2}{2l} + \sqrt{\frac{3l}{8}}K \\ &= \frac{n\pi^2}{2l} - \frac{(n-1)dl}{2} + \sqrt{\frac{3l}{8}}K \\ &= \frac{1}{2l}(n\pi^2 - (n-1)dl^2 + l\sqrt{\frac{3l}{8}}K).\end{aligned}$$

Thus

$$diam_\Sigma \leq 2\sqrt{\theta_0},$$

where θ_0 is the first positive root of the quartic equation $-(n-1)du^4+\sqrt{\frac{3}{8}}Ku^3+n\pi^2 = 0.$ □

If the hypersurface $\Sigma^n \subseteq M^{n+1}$ is compact, we can estimate the diameter of the hypersurface by the mean value of $\left|div(A^2) - AdivA\right|^4(x)$ on the maximal geodesics which are the geodesics from the point p to a point of $cut(p)$.

Theorem 3.1. *Let $\Sigma^n \subseteq M^{n+1}(c)$ be a compact orientable Riemannian hypersurface with the shape operator A, $p \in \Sigma$ be a fixed point; and let $\bar{V}(x) \leq K^2$ for a constant K, where $\bar{V}(x)$ is the mean value of $\left|div(A^2) - AdivA\right|^4(x)$ on the maximal geodesic passing through the points p and x. If*

$$Ric_{\frac{1}{2}|A|^2} \geq ((n-1)d)\, g_\Sigma \ \text{ and } \ (n-1)d + \sqrt{3/8}K > 0,$$

then

$$diam_\Sigma \leq 2\sqrt{\frac{n\pi^2}{(n-1)d + \sqrt{3/8}K}}.$$

Proof. Suppose that $\gamma(t)$ is a unit minimal geodesic from the fixed point p to an arbitrary point $x \in cut(p)$ with the length l. As in the proof of the previous theorem we set $V = div(A^2) - AdivA$. Now by (1)

$$\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \int_0^l f^2 \frac{d}{dt}\langle V, \gamma'\rangle dt;$$

also

$$\begin{aligned}\int_0^l f^2 \frac{d}{dt}\langle V, \gamma'\rangle dt &\leq \frac{\pi^2}{2l} + \left(\int_0^l f^4(t)dt\right)^{\frac{1}{2}} \left(\int_0^l |V|^4 dt\right)^{\frac{1}{2}} \\ &= \frac{\pi^2}{2l} + \sqrt{(3/8)\,\bar{V}l},\end{aligned}$$

where $\overline{V} = \frac{1}{l}\int_0^l |V|^4 dt$ is the mean value of $|V|^4$ on the maximal geodesic. So

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\pi^2}{2l} + \sqrt{3/8}lK \\ &= \frac{1}{2l}(n\pi^2 - \left((n-1)d + \sqrt{3/8}K\right)l^2).\end{aligned}$$

Thus $l \leq \sqrt{\frac{n\pi^2}{(n-1)d+\sqrt{3/8}K}}$ and we have

$$diam_\Sigma \leq 2\sqrt{\frac{n\pi^2}{(n-1)d + \sqrt{3/8}K}}.$$

□

Remark 3.2. When the ambient manifold doesn't have a constant sectional curvature, Proposition 2.5 shows that how the extrinsic data such as $\overline{Ric}_A(N,N)$, $(\bar{\nabla}_X \bar{R})$, $T_1(W,Y) = \overline{Ric}_A(AW,Y)$, $T_2(Y) = \overline{Ric}_{(\nabla_X A)}(N,Y)$ and $T_3(W,Y) = \left\langle \bar{R}(N, A^2W)N, Y\right\rangle$ affects the compactness (it also influences other topological and geometrical features).

Now we prove Theorem 1.6. Note that the required conditions are independent of the sectional curvature of the ambient manifold.

Proof of Theorem 1.6. Suppose that $\gamma(t)$ is the unit minimal geodesic by the length l, from the fixed point p to an arbitrary point $x \in \Sigma \backslash cut(p)$. By considering a parallel orthonormal frame $\{E_1 = \gamma', E_2, \ldots, E_n\}$ along γ, the smooth function $f(t) = \sin(\frac{\pi t}{l})$, and the index form of $\gamma(t)$, we have

$$I(fE_i, fE_i) = \int_0^l \left(|f'E_i|^2 - \langle R(fE_i,\gamma')\gamma', fE_i\rangle\right)dt.$$

So

$$\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) = \int_0^l \left((n-1){f'}^2 - f^2 Ric(\gamma',\gamma')\right)dt.$$

By the assumptions, $Ric_{\frac{1}{2}|A|^2}(\gamma',\gamma') \geq (n-1)d$; so

$$\begin{aligned}-Ric(\gamma',\gamma') \leq\ & -(n-1)d + n^2\langle \nabla H, \gamma'\rangle^2 + n^2 H Hess H(\gamma',\gamma') \\ & - \frac{n(n-1)}{2} Hess H_2(\gamma',\gamma').\end{aligned}$$

Thus we have

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) = & \int_0^l \left((n-1){f'}^2 - f^2 Ric(\gamma',\gamma')\right)dt \\ \leq & (n-1)\int_0^l \left({f'}^2 - f^2 d\right)dt + n^2 \int_0^l f^2\left\langle \nabla H, \gamma'\right\rangle^2 dt \\ & + n^2 \int_0^l f^2 H Hess H(\gamma',\gamma')dt \\ & - \frac{n(n-1)}{2}\int_0^l f^2 Hess H_2(\gamma',\gamma')dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + n^2 \int_0^l f^2 \left(H'\right)^2 dt \\
&\quad + n^2 \int_0^l f^2 H H'' dt - \frac{n(n-1)}{2} \int_0^l f^2 {H_2}'' dt,
\end{aligned}
$$

where $H(t) = H(\gamma(t))$ and $H_2(t) = H_2(\gamma(t))$. Now by

$$
\begin{aligned}
n^2 \int_0^l f^2 H H'' dt &= n^2 f^2 H H'\big|_0^l - n^2 \int_0^l \left(f^2 H\right)' H' dt \\
&= -n^2 \int_0^l \left(f^2\right)' H H' dt - n^2 \int_0^l f^2 (H')^2 dt,
\end{aligned}
$$

we have the following estimate;

$$
\begin{aligned}
\sum\nolimits_{i=2}^n I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) - n^2 \int_0^l \left(f^2\right)' H H' dt \\
- \frac{n(n-1)}{2} \int_0^l f^2 {H_2}'' dt. \qquad (2)
\end{aligned}
$$

By estimating $-n^2 \int_0^l \left(f^2\right)' H H' dt$, we get

$$
\begin{aligned}
-n^2 \int_0^l \left(f^2\right)' H H' dt &= -n^2 \left(f^2\right)' \frac{H^2}{2}\bigg|_0^l + \frac{n^2}{2} \int_0^l \left(f^2\right)'' H^2 dt \\
&= \frac{n^2}{2} \int_0^l \left(f^2\right)'' H^2 dt \\
&\leq \frac{n^2}{2} \left(\int_0^l \left(\left(f^2\right)''\right)^2 dt \right)^{\frac{1}{2}} \left(\int_0^l H^4 dt \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2} n^2 \pi^2}{2l\sqrt{l}} \left(\int_0^l H^4 dt \right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum\nolimits_{i=2}^n I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2} n^2 \pi^2}{2l\sqrt{l}} \left(\int_0^l H^4 dt \right)^{\frac{1}{2}} \\
- \frac{n(n-1)}{2} \int_0^l f^2 {H_2}'' dt. \qquad (3)
\end{aligned}
$$

Now, we get the estimation of the diameter of the hypersurface under each condition. For (a) we have

$$
\begin{aligned}
\sum\nolimits_{i=2}^n I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2} n^2 \pi^2}{2l\sqrt{l}} K_1 - \frac{n(n-1)l\beta}{4} \\
&= \frac{1}{2l\sqrt{l}} \left(- \left((n-1)d + \beta \frac{n(n-1)}{2} \right) l^2 \sqrt{l} + (n-1)\pi^2 \sqrt{l} \right)
\end{aligned}
$$

$$+\frac{1}{2l\sqrt{l}}\sqrt{2}n^2\pi^2K_1.$$

So $l \leq \sqrt{\theta_0}$, where θ_0 is the first positive root of

$$p(u) = -\left((n-1)d + \frac{n(n-1)\beta}{2}\right)u^5 + (n-1)\pi^2u + \sqrt{2}n^2\pi^2K_1.$$

For (b), at first

$$\int_0^l f^2(t)H_2''(t)dt = f^2(t)H_2'(t)\big|_0^l - \int_0^l \left(f^2(t)\right)'H_2'(t)dt$$

$$= -\int_0^l \left(f^2(t)\right)'H_2'(t)dt, \tag{4}$$

also

$$\int_0^l \left(f^2\right)'H_2'dt \leq \left(\int_0^l \left(\left(f^2\right)'\right)^2 dt\right)^{\frac{1}{2}}\left(\int_0^l \left(H_2'\right)^2 dt\right)^{\frac{1}{2}}$$

$$= \frac{\pi}{\sqrt{2l}}\left(\int_0^l \left(H_2'\right)^2 dt\right)^{\frac{1}{2}}. \tag{5}$$

So by (3), (4) and (5) we get the following estimation;

$$\begin{aligned}\sum\nolimits_{i=2}^n I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}\left(\int_0^l H^4dt\right)^{\frac{1}{2}} \\ &\quad + \frac{\pi}{\sqrt{2l}}\frac{n(n-1)}{2}\left(\int_0^l \left(H_2'\right)^2 dt\right)^{\frac{1}{2}} \\ &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}K_1 + \frac{n(n-1)\pi}{2\sqrt{2l}}K_2 \\ &= \frac{1}{2l\sqrt{l}}(-(n-1)dl^2\sqrt{l} + \frac{n(n-1)\pi}{\sqrt{2}}K_2l).\end{aligned}$$

So $diam_\Sigma \leq 2\sqrt{\theta_0}$, where θ_0 is the first positive root of

$$p(u) = -(n-1)du^5 + \frac{n(n-1)}{\sqrt{2}}K_2\pi u^2 + (n-1)\pi^2u + \sqrt{2}n^2\pi^2K_1.$$

For (c), at first

$$\begin{aligned}-\frac{n(n-1)}{2}\int_0^l f^2H_2''dt &= -\frac{n(n-1)}{2}f^2\ H_2'\big|_0^l + \frac{n(n-1)}{2}\int_0^l \left(f^2\right)'H_2'dt \\ &= \frac{n(n-1)}{2}\left(f^2\right)'\ H_2\big|_0^l - \frac{n(n-1)}{2}\int_0^l \left(f^2\right)''H_2dt \\ &= -\frac{n(n-1)}{2}\int_0^l \left(f^2\right)''H_2dt. \qquad (6)\end{aligned}$$

By (3) we get

$$\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}\left(\int_0^l H^4 dt\right)^{\frac{1}{2}}$$
$$-\frac{n(n-1)}{2}\int_0^l \left(f^2\right)'' H_2 dt. \tag{7}$$

By estimating $-\frac{n(n-1)}{2}\int_0^l \left(f^2\right)'' H_2 dt$ we have

$$\pm\frac{n(n-1)}{2}\int_0^l \left(f^2\right)'' H_2 dt \leq \frac{n(n-1)}{4}\left(\int_0^l \left(\left(f^2\right)''\right)^2 dt + \int_0^l H_2^2 dt\right)$$
$$= \frac{n(n-1)\pi^4}{2l^3} + \frac{n(n-1)}{4}\int_0^l H_2^2 dt. \tag{8}$$

By (7) and (8) and the assumption, we get

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}\left(\int_0^l H^4 dt\right)^{\frac{1}{2}} \\ &\quad + \frac{n(n-1)\pi^4}{2l^3} + \frac{n(n-1)}{4}\int_0^l H_2^2 dt \\ &= \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}K_1 \\ &\quad + \frac{n(n-1)\pi^4}{2l^3} + \frac{n(n-1)K_2}{4} \\ &= \frac{1}{2l^3}(-(n-1)dl^4 + \frac{n(n-1)K_2}{2}l^3 + (n-1)\pi^2 l^2) \\ &\quad + \frac{1}{2l^3}\left(\sqrt{2}n^2\pi^2 K_1 l\sqrt{l} + n(n-1)\pi^4\right).\end{aligned}$$

So $diam_\Sigma \leq 2\sqrt{\theta_0}$, where θ_0 is the first positive root of

$$p(u) = -(n-1)du^8 + \frac{n(n-1)K_2}{2}u^6 + (n-1)\pi^2 u^4 + \sqrt{2}n^2\pi^2 K_1 u^3 + n(n-1)\pi^4.$$

□

Remark 3.3. For all polynomials $p(u)$ we have $p(0) > 0$ and $\lim_{u\to+\infty} p(u) = -\infty$, and so they have one just positive root.

By adapting the proof of Theorem 3.1, we can get estimations for the diameter of hypersurfaces by imposing some conditions on the mean values of the mean curvatures.

Theorem 3.4. *Let $\Sigma^n \subseteq M^{n+1}$ be a compact orientable Riemannian hypersurface with the shape operator A, $p \in \Sigma$ be a fixed point, and $r(x) := dist_\Sigma(p, x)$; and let $\overline{H}(x)$ and $\overline{H_2}(x)$ respectively be the mean values of the functions $H^4(x)$*

and $H_2^2(x)$ on the maximal geodesic passing through p and x. If all the following conditions are satisfied,

a) $Ric_{\frac{1}{2}|A|^2} \geq (n-1)dg_\Sigma$ *and* $\frac{nK_2}{2} < d$,
b) $H \leq K_1$ *and* $H_2 \leq K_2$, *where* $K_1,\, K_2 \geq 0$,

then

$$diam(\Sigma) \leq 2\pi\sqrt{\frac{-B+\sqrt{B^2-4Cn}}{2C}},$$

where $B := \left(\left(\sqrt{2}n^2/(n-1)\right)K_1 + 1\right)$ *and* $C := -d + (nK_2/2)$.

Proof. Suppose that $\gamma(t)$ is a unit minimal geodesic by the length l, from the fixed point p to an arbitrary point $x \in cut(p)$. Now as in the proof of the previous theorem, by (6) and (7) we have

$$\begin{aligned}\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}\left(\int_0^l H^4 dt\right)^{\frac{1}{2}} \\ + \frac{n(n-1)\pi^4}{2l^3} + \frac{n(n-1)}{4}\int_0^l H_2^2 dt.\end{aligned}$$

So by the conditions on the mean curvature and the second-order mean curvature, we have

$$\begin{aligned}&\sum\nolimits_{i=2}^{n} I(fE_i, fE_i) \\ &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l}K_1 \\ &\quad + \frac{n(n-1)\pi^4}{2l^3} + \frac{n(n-1)}{4}lK_2 \\ &= \frac{(n-1)}{2l^3}\left(\left(-d + \frac{nK_2}{2}\right)l^4 + \left(\frac{\sqrt{2}n^2}{(n-1)}K_1 + 1\right)\pi^2 l^2 + n\pi^4\right),\end{aligned}$$

and the result follows. $\square$

In the next two corollaries, we prove compactness results for the two important cases below:

a) the hypersurfaces has constant mean curvature;
b) the hypersurfaces has constant second-order mean curvature (in fact $HessH_2 \equiv 0$).

Corollary 3.5. *Let $\Sigma^n \subset M^{n+1}$ be a complete orientable Riemannian hypersurface with constant mean curvature, $p \in \Sigma$ be a fixed point, and $r(x) := dist_\Sigma(p, x)$; and let $g : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $\int_0^{+\infty} g(r)dr = K$. If*

$$Ric_{\frac{1}{2}|A|^2} \geq ((n-1)d)\, g_\Sigma > 0 \quad \textit{and} \quad \left|H_2^2(r(x))\right| \leq g(r),$$

where $d > 0$ is constant, then Σ is compact and $diam_\Sigma \leq 2\sqrt{\theta_0}$, where θ_0 is the first root of

$$p(u) = -du^4 + \frac{n}{4}Ku^3 + \pi^2 u^2 + n\pi^4.$$

Proof. Since the mean curvature H is constant, by (2) and (6) we have

$$\sum_{i=2}^n I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) - \frac{n(n-1)}{2}\int_0^l \left(f^2\right)'' H_2 dt.$$

Thus, by (8) and the assumption on the second-order mean curvature we have

$$\begin{aligned}\sum_{i=2}^n I(fE_i, fE_i) &\leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{n(n-1)\pi^4}{2l^3} \\ &\quad + \frac{n(n-1)}{4}\int_0^l H_2^2 dt \\ &= \frac{(n-1)}{2l^3}\left(-dl^4 + \frac{nK}{4}l^3 + \pi^2 l^2 + n\pi^4\right).\end{aligned}$$

Hence $diam_\Sigma \leq 2\theta_0$, where θ_0 is the first root of

$$p(u) = -du^4 + \frac{n}{4}Ku^3 + \pi^2 u^2 + n\pi^4.$$

□

Corollary 3.6. *Let $\Sigma^n \subset M^{n+1}$ be a Riemannian hypersurface with $HessH_2 \equiv 0$, $p \in \Sigma$ be a fixed point, and $r(x) = dist_\Sigma(p, x)$; and let $g : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $\left(\int_0^{+\infty} (g(r))^2 dr\right)^{\frac{1}{2}} \leq K$. If for a constant $d > 0$*

$$H^2(r(x)) \leq g(r) \quad and \quad Ric_{\frac{1}{2}|A|^2} \geq ((n-1)d)\, g_\Sigma,$$

then Σ is compact and $diam_\Sigma \leq 2\sqrt{\theta_0}$, where θ_0 is the first root of

$$p(u) = -d(n-1)u^5 + (n-1)\pi^2 u + \sqrt{2}n^2\pi^2 K.$$

Proof. By (3) and the assumption on H_2 we have

$$\sum_{i=2}^n I(fE_i, fE_i) \leq \frac{(n-1)}{2l}\left(\pi^2 - dl^2\right) + \frac{\sqrt{2}n^2\pi^2}{2l\sqrt{l}}\left(\int_0^l H^4 dt\right)^{\frac{1}{2}}.$$

So $diam_\Sigma \leq \sqrt{\theta_0}$, where θ_0 is the first root of

$$p(u) = -(n-1)du^5 + (n-1)\pi^2 u + \sqrt{2}n^2\pi^2 K.$$

□

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