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THE SET OF ZOLL METRICS IS NOT PRESERVED BY SOME GEOMETRIC FLOWS

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Abstract. The geodesics on the round 2-sphere S^2 are all simple closed curves of equal length. In 1903 Otto Zoll introduced other Riemannian surfaces with the same property. After that, his name is attached to the Riemannian manifolds whose geodesics are all simple closed curves of the same length. The question that "whether or not the set of Zoll metrics on 2-sphere S^2 is connected?" is still an outstanding open problem in the theory of Zoll manifolds. In the present paper, continuing the work of D. Jane for the case of the Ricci flow, we show that a naive application of some famous geometric flows does not work to answer this problem. In fact, we identify an attribute of Zoll manifolds and prove that along the geometric flows this quantity no longer reflects a Zoll metric. At the end, we will establish an alternative proof of this fact.

1. Introduction

By definition, a Zoll manifold M is a Riemannian manifold (M, g) whose geodesics are all simple closed curves of equal length. The round 2-sphere is a primary example of Zoll manifolds. A question naturally raised is to find other surfaces and manifolds with the same property. In 1903 Otto Zoll discovered that S^2 admits many such metrics besides the obvious metrics of constant curvature [11].

It is shown that there is a one-to-one correspondence between the set of odd functions $[-1, 1] \to (-1, 1)$ vanishing at ± 1 and the set of Zoll metrics on S^2 [8]. Indeed, in terms of cylindrical coordinates $(z, \theta) \in [-1, 1] \times [0, 2\pi]$,

$$g = \frac{[1 + f(z)]^2}{1 - z^2} dz^2 + (1 - z^2) d\theta^2, \tag{1.1}$$

defines a Zoll metric on S^2 for any smooth odd function

$$f : [-1, 1] \to (-1, 1), \qquad f(-z) = -f(z),$$

which vanishes at the end-points of the interval.

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A C_L-manifold is a Zoll manifold with common period $2\pi L$. In this paper, we use the following result of Alan Weinstein. Let (S^n, can) be the n-dimensional sphere equipped with its canonical metric structure.

Theorem 1.1 (Weinstein [10]). *Suppose that (M, g) is an n-dimensional C_L-manifold. Then the ratio*

$$i(M,g) = \frac{\text{vol}(M,g)}{L^n \text{vol}(\mathrm{S^n}, \text{can})}$$

is an integer, now called Weinstein's integer.

For basic references on Zoll manifolds, as well as a more detailed account of the history, one may consult [2] and also [1]. It is unknown if the space of Zoll metrics on S^2 is connected. It is natural to try to address this problem by using a Ricci flow to construct a path g_t in the space of Zoll metrics from any given metric g to the round metric g_0. However it was shown by D. Jane that this method does not work because the Ricci flow does not preserve the condition of being a Zoll metric. In the present paper, we generalize the result of D. Jane to other examples of geometric flows, namely, besides the Ricci flow, we consider the Yamabe and the Ricci-Bourgoignon flows, as well as their hyperbolic versions. More precisely, we prove that those geometric flows produce non-Zoll metrics when applied to the Gambir's gong. Also, we present two proofs for our main theorem. Our second proof not only generalizes D. Jane's result, but also it is completely technical and different from D. Jane's one. In the next paragraph, we summarize some definitions and details on geometric flows with emphasis on those aspects that we will need in the paper.

The notion of the Ricci flow was introduced by R. Hamiltion in 1982 as the pillar of a programme designed to prove Poincare's and Thurston's conjecture. Because of numerous applications of this concept in both geometry and theoretical physics, specially in subjects related to gravity, this flow and other related flows are the subjects of great interest for many authors. A geometric flow is an evolutionary equation for the metrics of a Riemannian manifold M as follows

$$\frac{\partial g_{ij}}{\partial t} = S_{ij}(g), \qquad g_{ij}(0) = g_0, \tag{1.2}$$

and, hyperbolic geometric flows are formed as in the following

$$\frac{\partial^2 g_{ij}}{\partial t^2} = S_{ij}(g), \qquad g_{ij}(0) = g_0, \quad \frac{\partial g_{ij}}{\partial t}(0) = Q_{ij}, \tag{1.3}$$

where, $S_{ij}(g)$ is a symmetric tensor field of type $(0, 2)$ depending on the metric $g = g(t)$ and, Q_{ij} is also a symmetric tensor field of type $(0, 2)$. Geometric flows (1.2) and (1.3) are called flows with heat and wave kernel, respectively. It should be noticed that, for special cases of the tensor field $S_{ij}(g)$, the obtained evolution equations are of special interest, in particular,

- If $S = -2\text{Ric}$, then (1.2) is the Ricci flow and, (1.3) is the hyperbolic Ricci flow,
- If $S = Rg$, then (1.2) is the Yamabe flow and, (1.3) is the hyperbolic Yamabe flow,
- If $S = -2(\text{Ric} - \rho Rg)$, then (1.2) is the Ricci-Bourgoignon flow,

where, Ric is the Ricci tensor of the manifold, R its scalar curvature and, ρ is a real constant. Short time existence and uniqueness of solutions for these geometric flows were studied in [3], [4], [6]. In spite of numerous applications and effectiveness of theses flows in analysis of Riemannian manifolds, in [7] it is shown that a naive application of the Ricci flow is not sufficient to answer the above question. In this paper we extend the results of [7] by showing that the set of Zoll metrics on S^2 is not preserved by the other geometric flows. Here, we have considered a more general case and hyperbolic geometric flows are also taken into account. The main theorem of this paper is as follows.

Theorem 1.2. *The set of Zoll metrics is not preserved by the above geometric flows.*

2. Proof of Theorem 1.2

This section is devoted to prove the main theorem. The argument given here consists of considering some attribute of Zoll manifolds and showing that along the geometric flows this quantity no longer reflects a Zoll metric.

Let $\{g_t\}_t$ be a volume normalised continuous family of Zoll metrics on a surface of revolution Σ. Also, let's assume $l_{g(t)}$ denotes the common period of the metric g_t. It is easy to check that the map

$$g_t \mapsto l_{g(t)}$$

is a continuous map. By Theorem 1.1, the image of this map is a discrete set. Since, every continuous function to a discrete set is constant, the common period of g_0 is the same as the common period of g_t for all t.

Let $l(t)$ be the length of a closed geodesic γ_t with respect to the metric g_t. According to the previous paragraph, the value of $l(t)$ is independent of t, but for a particular Zoll manifold, we will show that the derivative $l'(0)$ (or $l''(0)$, in the case of hyperbolic geometric flows) along the geometric flows is non-zero, which leads a contradiction.

Now, we give a Zoll metric so that the first or the second derivative of $l(t)$ is not zero at $t = 0$. The Zoll surface of revolution with the meridian curve

$$r(z) = \left(\sqrt{1 + \sqrt{1 - 16(cz)^2}} - 1\right)/c,$$

where $c = \sqrt{2} - 1$, known as Gambir's gong Σ [5], has volume 4π, and the period of every geodesic is 2π. Also, this surface has the average curvature $\overline{K} = 1$. In fact, a straightforward calculation shows that for a surface of revolution with parametrisation

$$\sigma(z, \theta) = \big(r(z)\cos(\theta), r(z)\sin(\theta), z\big)$$

the Gaussian curvature K is given by

$$K(z,\theta) = K(z) = \frac{-r''(z)}{r(z)\big(r'(z)^2+1\big)^2}. \tag{2.1}$$

Now, a straightforward computation shows that $\overline{K} = 1$.

Let us assume $\Sigma = (S^2, g_0)$, where

$$g_0 = (1 + r'(z)^2)dz^2 + r(z)^2 d\theta^2.$$

Besides the rotational symmetry, Σ has a reflectional symmetry in the $z = 0$ plane. So the curve of intersection of Σ and the $z = 0$ plane is a geodesic which we denote it by γ_0. This reflectional symmetry will be preserved along the geometric flows. Hence $\gamma_t = \gamma_0$ is a geodesic of g_t for all t. In fact, the geometric flows preserve any symmetries that are present in the initial metric. To see this, note that each symmetry is an isometric diffeomorphism of the initial metric. So the pull-back of a solution of the geometric flows by this diffeomorphism gives a solution to the geometric flow. Since the symmetry is an isometry of the initial metric, these are solutions of the geometric flows with the same initial data, and so are identical.

We assume that the geodesic $\gamma_0 : [0, 2\pi] \to S^2$ has unit speed. Now we differentiate the length of γ_t along the geometric flows:

$$l'(0) = \frac{\partial}{\partial t}\Big|_{t=0} l(t) = \frac{\partial}{\partial t}\Big|_{t=0} \int_0^{2\pi} \sqrt{g_t(\gamma_t'(s), \gamma_t'(s))}ds.$$

As mentioned before, $\gamma_t = \gamma_0$ is not a function of t, hence

$$l'(0) = \int_0^{2\pi} \frac{\frac{\partial}{\partial t}|_{t=0} g_t(\gamma_0'(s), \gamma_0'(s))}{2\sqrt{g_0(\gamma_0'(s), \gamma_0'(s))}}ds.$$

In the denominator we have $g_0(\gamma_0'(s), \gamma_0'(s)) = 1$, since we choose γ_0 to be a unit speed geodesic. Using the normalized geometric flow equation $\frac{\partial g_{ij}}{\partial t} = S_{ij} - \frac{1}{2}\mathrm{tr}_{g_0}(S) g_{0ij}$ (which preserves the volume of Σ), in the numerator, we obtain

$$l'(0) = \frac{1}{2}\int_0^{2\pi} \big[S(\gamma_0'(s), \gamma_0'(s)) - \frac{1}{2}\mathrm{tr}_{g_0}(S)\big]ds.$$

Case 1: (Ricci Flow) Let $S_{ij} = -2\mathrm{Ric}_{ij}$. We know that $\mathrm{Ric}_{ij} = K g_{ij}$ and $R = 2K$ in $\dim M = 2$. So

$$l'(0) = -\int_0^{2\pi} \big(K(\gamma_0(s)) - \overline{K}\big)ds.$$

Because of the rotational symmetry, the Gaussian curvature of g_0 is constant along γ_0, and we denote it by κ. Using (2.1) we get

$$\kappa = K(0) = 4(2 - \sqrt{2}).$$

Hence $l'(0) = 2\pi(1 - 4(2 - \sqrt{2})) \neq 0$.
Case 2: (Yamabe Flow) In this case we have $S_{ij} = -Rg_{ij}$. Hence

$$l'(0) = -\int_0^{2\pi} \big(K(\gamma_0(s)) - \overline{K}\big)ds.$$

Hence $l'(0) = 2\pi(1 - 4(2 - \sqrt{2}))$.
Case 3: (Ricci-Bourgoignon Flow) Let $S_{ij} = -2(\mathrm{Ric}_{ij} - \rho R g_{ij})$ with $\rho < 1/2$. So

$$l'(0) = (2\rho - 1)\int_0^{2\pi} \big(K(\gamma_0(s)) - \overline{K}\big)ds.$$

For the (normalized) hyperbolic geometric flows equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = S_{ij}(g), \qquad g_{ij}(0) = g_0, \quad \frac{\partial g_{ij}}{\partial t}(0) = Q_{ij},$$

a similar argument shows that

$$\begin{aligned} l''(0) = \frac{\partial}{\partial t}\Big|_{t=0} l'(t) = \frac{\partial}{\partial t}\Big|_{t=0} \int_0^{2\pi} \frac{\frac{\partial}{\partial t} g_t(\gamma'(s), \gamma'(s))}{2\sqrt{g_t(\gamma'(s), \gamma'(s))}} ds. \\ = \frac{1}{2}\int_0^{2\pi} S(\gamma_0'(s), \gamma_0'(s))ds - \frac{1}{4}\int_0^{2\pi} [Q(\gamma_0'(s), \gamma_0'(s))]^2 ds \\ = I + II, \end{aligned}$$

where I is the first integral and II is the second one. It is clear that value of II is a non-positive real number, hence the previous computations show that the set of Zoll metrics is not preserved by hyperbolic Ricci flow and hyperbolic Yamabe flow. We can also deduce that the hyperbolic geometric flows are much worse choice than the heat kernel geometric flows to answer the underlying question.

3. Alternative proof of Theorem 1.2

In this section, our assumptions are the same as in the previous section. Here, we establish an alternative proof for Theorem 1.2.

In two dimensions, we know that the Ricci curvature can be written in terms of the scalar curvature as $\mathrm{Ric} = \frac{1}{2}Rg$. So, all of our mentioned heat geometric flows are similar to Yamabe flow. In the other hand, the Yamabe flow preserves the conformal structure, then these flows generate conformal metrics.

Consider $g(t) = e^{a(t,x)} g_0$ as a continuous family of Zoll metrics on the surface of revolution Σ, where $a : (-\epsilon, \epsilon) \times \Sigma \to \mathbb{R}$ is a smooth function and for all $x \in \Sigma$ we have $a(0, x) = 0$. We will prove that $a(t, x)$ must vanish, which proves the theorem.

Let $\overline{\Gamma}_{ij}^k$ and Γ_{ij}^k be the Christoffel symbols of metrics $g(t)$ and g_0, respectively. In terms of cylindrical coordinate system $(z, \theta) = (x^1, x^2)$, a straightforward computations show that

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}\left\{\frac{\partial a}{\partial x^i}\delta_j^k + \frac{\partial a}{\partial x^j}\delta_i^k - \frac{\partial a}{\partial x^l}(g_0)_{ij}(g_0)^{kl}\right\},$$

where, δ is the Kronecker delta function. The curve $\gamma(s) = \sigma(\gamma_1(s), \gamma_2(s))$ is a geodesic of the metric $g(t)$ if and only if

$$\begin{cases} \dfrac{d^2\gamma_1}{ds^2} + \overline{\Gamma}^1_{ij}\dfrac{d\gamma_i}{ds}\dfrac{d\gamma_j}{ds} = 0, \\ \dfrac{d^2\gamma_2}{ds^2} + \overline{\Gamma}^2_{ij}\dfrac{d\gamma_i}{ds}\dfrac{d\gamma_j}{ds} = 0. \end{cases}$$

Hence, we can easily obtain

$$\begin{cases} \dfrac{d^2\gamma_1}{ds^2} + \Gamma^1_{ij}\dfrac{d\gamma_i}{ds}\dfrac{d\gamma_j}{ds} + \dfrac{1}{2}\dfrac{\partial a}{\partial x^1}[(\dfrac{d\gamma_1}{ds})^2 - (g_0)_{22}(g_0)^{11}(\dfrac{d\gamma_2}{ds})^2] = 0, \\ \dfrac{d^2\gamma_2}{ds^2} + \Gamma^2_{ij}\dfrac{d\gamma_i}{ds}\dfrac{d\gamma_j}{ds} + \dfrac{1}{2}\dfrac{\partial a}{\partial x^2}[(\dfrac{d\gamma_2}{ds})^2 - (g_0)_{11}(g_0)^{22}(\dfrac{d\gamma_1}{ds})^2] = 0. \end{cases}$$

Let $\gamma(s)$ be a geodesic with respect to metric g_0. Then

$$\begin{cases} \dfrac{\partial a}{\partial x^1}[(\dfrac{d\gamma_1}{ds})^2 - \dfrac{r^2}{1+r'^2}(\dfrac{d\gamma_2}{ds})^2] = 0, \\ \dfrac{\partial a}{\partial x^2}[(\dfrac{d\gamma_2}{ds})^2 - \dfrac{1+r'^2}{r^2}(\dfrac{d\gamma_1}{ds})^2] = 0. \end{cases} \tag{3.1}$$

The following argument shows that $(\frac{d\gamma_1}{ds})^2 - \frac{r^2}{1+r'^2}(\frac{d\gamma_2}{ds})^2 \neq 0$. In fact, one can easily check that the geodesic equations for g_0 are

$$\begin{cases} \dfrac{d^2\gamma_1}{ds^2} + \dfrac{r'r''}{1+r'^2}(\dfrac{d\gamma_1}{ds})^2 - \dfrac{rr'}{1+r'^2}(\dfrac{d\gamma_2}{ds})^2 = 0, \\ \dfrac{d^2\gamma_2}{ds^2} + 2\dfrac{r'}{r}(\dfrac{d\gamma_1}{ds})(\dfrac{d\gamma_2}{ds}) = 0. \end{cases}$$

Geodesics on the surface of revolution (Σ, g_0) are characterized by the equation $r^2(\gamma_1(s))\frac{d\gamma_2}{ds} =$ constant by Clairaut's Theorem [9]. So, we have three types of geodesics on the surface Σ and for all of them the expression $(\frac{d\gamma_1}{ds})^2 - \frac{r^2}{1+r'^2}(\frac{d\gamma_2}{ds})^2$ is not zero. Hence, along each geodesic curve $\gamma(s)$, from (3.1) we have

$$\frac{\partial a}{\partial x^1}(t, \gamma(s)) = 0, \qquad \frac{\partial a}{\partial x^2}(t, \gamma(s)) = 0.$$

The above equations indicate $a(t, \gamma(s)) = a(t)$. Hence for all $x \in \Sigma$ we have $a(t, x) = a(t)$.

Also, according to Weinstein Theorem 1.1 we must have $l'(0) = 0$, and hence $l(t) = 2\pi$. But

$$l(t) = \int_0^{2\pi} \sqrt{g(t)(\gamma_0(s), \gamma_0(s))}ds = 2\pi e^{a(t)/2}.$$

Therefore $a(t) = 0$, which leads a contradiction, since the fixed points of the Yamabe flows are metrics with constant scalar curvature.

4. Conclusion

The problem that "whether or not the set of Zoll metrics on 2-sphere S^2 is connected?" is an outstanding open problem in the theory of Zoll manifolds. In this paper, we showed a naive application of some famous geometric flows to solve the problem. Our results give a necessary condition for geometric flows on surfaces of revolution to be a candidate to answer the problem. The paper does not indicate any other appropriate geometric flow in order to answer the question. So, this study must be developed in order to give a more comprehensive study of the underlying problem.

References

[1] M. Berger, *A Panoramic View of Riemannian Geometry*, Springer-Verlag, Berlin, 2003. https://doi.org/10.1007/978-3-642-18245-7

[2] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **93**, Springer-Verlag, Berlin, 1978.

[3] G. Catino, L. Gremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri, *The Ricci-Bourguignon flow*, Pacific J. Math. **287** (2017), no. 2, 337–370. https://doi.org/10.2140/pjm.2017.287.337

[4] W.-R. Dai, D.-X. Kong, and K. Liu, *Hyperbolic geometric flow* (I): *short-time existence and nonlinear stability*, Pure Appl. Math. Q. **6** (2010), no. 2, 331–359. https://doi.org/10.4310/PAMQ.2010.v6.n2.a3

[5] B. Gambir, *Surfaces álignes géod ésiques toutés fermées. etude spéciale de cless qui sont de révolution*, Bulletin des Sciences Math., (2 éme Série Tome XLIX), Mars 1925.

[6] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. http://projecteuclid.org/euclid.jdg/1214436922

[7] D. Jane, *The Ricci flow does not preserve the set of Zoll metrics*, Differential Geom. Appl. **54** (2017), part B, 397–401. https://doi.org/10.1016/j.difgeo.2017.07.005

[8] R. Michel, *Problèmes d'analyse géométrique liés à la conjecture de Blaschke*, Bull. Soc. Math. France **101** (1973), 17–69.

[9] A. Pressley, *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2001. https://doi.org/10.1007/978-1-4471-3696-5

[10] A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geometry **9** (1974), 513–517. http://projecteuclid.org/euclid.jdg/1214432547

[11] O. Zoll, *Ueber Flächen mit Scharen geschlossener geodätischer Linien*, Math. Ann. **57** (1903), no. 1, 108–133. https://doi.org/10.1007/BF01449019

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