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STABILIZATION OF 2D g-NAVIER-STOKES EQUATIONS

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Abstract. We study the stabilization of 2D g-Navier-Stokes equations in bounded domains with no-slip boundary conditions. First, we stabilize an unstable stationary solution by using finite-dimensional feedback controls, where the designed feedback control scheme is based on the finite number of determining parameters such as determining Fourier modes or volume elements. Second, we stabilize the long-time behavior of solutions to 2D g-Navier-Stokes equations under action of fast oscillating-in-time external forces by showing that in this case there exists a unique time-periodic solution and every solution tends to this periodic solution as time goes to infinity.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider the following 2D g-Navier-Stokes equations

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & x\in\Omega, t>0,\\ \nabla\cdot(gu) = 0, & x\in\Omega, t>0,\\ u(x,t) = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases}$$

where $u = u(x,t)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, and u_0 is the initial velocity.

The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D Navier-Stokes problem in a 3D thin domain $\Omega_g = \Omega\times(0,g)$ (see e.g. [21]). As mentioned in that paper, good properties of the 2D g-Navier-Stokes equations can lead to an initial study of the 3D Navier-Stokes equations in the thin domain Ω_g. In the last years, the existence and long-time behavior of solutions in terms of existence of attractors for 2D g-Navier-Stokes equations have been studied extensively in both autonomous and non-autonomous

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cases, see e.g. [1, 2, 8–11, 15–17, 21, 22] and references therein. The stability and stabilization of stationary solutions to 2D g-Navier-Stokes equations by using an internal feedback control with support large enough were studied recently in [19].

In this paper we continue studying the stabilization of long-time behavior of solutions to 2D g-Navier-Stoks equations. To do this, we assume that the function g satisfies the following assumption:

(**G**) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \leq g(x) \leq M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A is defined in Section 2 below).

The aim of the present paper is twofold. First, we show that any arbitrary stationary solution to problem (1.1) can be stabilized by using a finite-dimensional feedback controller. This approach was first introduced in [4] for the reaction-diffusion equations, and then developed for some other dissipative partial differential equations in [12–14]. Here the designed feedback control scheme takes advantage of the fact that such systems possess finite number of determining parameters such as determining Fourier modes or volume elements. Second, we show that if the external force is oscillating fast enough in time but not necessarily having small magnitude of spatial norms, then there exists a unique time periodic solution such that any solution to the 2D g-Navier-Stokes equations converges to that time periodic solution with exponential speed in time. Such an approach was introduced in [7] and developed for the 3D Navier-Stokes-Voigt equations in [3].

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some results on function spaces and operators related to 2D g-Navier-Stokes equations which will be used in the paper. In Section 3, we show that any unstable stationary solution to 2D g-Navier-Stokes equations can be exponentially stabilized by using finite-dimensional feedback controls. Stabilizing the long-time behavior of solutions to 2D g-Navier-Stokes equations by fast oscillating-in-time forces is given in the last section.

2. Preliminaries

Let $\mathbb{L}^2(\Omega, g) = (L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \ u, v \in \mathbb{L}^2(\Omega, g),$$

and

$$((u, v))_g = \int_\Omega \sum_{j=1}^2 \nabla u_j \cdot \nabla v_j g dx, \ u = (u_1, u_2), \ v = (v_1, v_2) \in \mathbb{H}_0^1(\Omega, g),$$

and norms $|u|^2 = (u,u)_g$, $\|u\|^2 = ((u,u))_g$. By the assumption (**G**), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla\cdot(gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega, g)$, and by V_g the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega, g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle\cdot,\cdot\rangle_g$ for duality pairing between V_g and V_g'.

As in [21], we recall the g-Stokes operator $A : V_g \to V_g'$ defined by

$$\langle Au, v\rangle_g = ((u,v))_g \quad \text{for all } u, v \in V_g.$$

Then $A = -P_g\Delta$ and $D(A) = H^2(\Omega, g) \cap V_g$, where P_g is the ortho-projector from $\mathbb{L}^2(\Omega, g)$ onto H_g. We also define the operator $B : V_g \times V_g \to V_g'$ by

$$\langle B(u,v), w\rangle_g = b(u,v,w) \quad \text{for all } u, v, w \in V_g,$$

where

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx.$$

It is easy to check that if $u, v, w \in V_g$, then

$$b(u,v,w) = -b(u,w,v),\ b(u,v,v) = 0.$$

We have the following Poincaré inequality

$$\|u\|^2 \geq \lambda_1 |u|^2, \forall u \in V_g, \tag{2.1}$$

$$|Au|^2 \geq \lambda_1 \|u\|^2, \forall u \in D(A), \tag{2.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator A.

As in [21], we consider the operator $C : V_g \to H_g$ defined by

$$(Cu, v)_g = ((\frac{\nabla g}{g}\cdot\nabla)u, v)_g = b(\frac{\nabla g}{g}, u, v),\ \forall v \in V_g.$$

Since $-\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - (\frac{\nabla g}{g}\cdot\nabla)u$, we have

$$(-\Delta u, v)_g = ((u,v))_g + ((\frac{\nabla g}{g}\cdot\nabla)u, v)_g = \langle Au, v\rangle_g + ((\frac{\nabla g}{g}\cdot\nabla)u, v)_g, \forall u, v \in V_g.$$

We now recall some known results which will be used in the paper.

Lemma 2.1 ([1])**.** *We have*

$$|b(u,v,w)| \leq \begin{cases} c_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, & \forall u,v,w \in V_g,\\ c_2|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|, & \forall u \in V_g, v \in D(A), w \in H_g,\\ c_3|u|^{1/2}|Au|^{1/2}\|v\||w|, & \forall u \in D(A), v \in V_g, w \in H_g,\\ c_4|u|\|v\||w|^{1/2}|Aw|^{1/2}, & \forall u \in H_g, v \in V_g, w \in D(A), \end{cases}$$

where $c_i, i = 1,\dots,4,$ are appropriate positive constants.

Lemma 2.2 ([6]). *Let $u \in L^2(0,T;V_g)$. Then the function Cu defined by*

$$(Cu(t),v)_g = ((\frac{\nabla g}{g}\cdot\nabla)u,v)_g = b(\frac{\nabla g}{g},u,v), \forall v \in V_g,$$

belongs to $L^2(0,T;H_g)$, and hence also belongs to $L^2(0,T;V_g')$. Moreover,

$$|Cu(t)| \leq \frac{|\nabla g|_\infty}{m_0}\|u(t)\| \text{ for a.e. } t \in (0,T),$$

and

$$\|Cu(t)\|_* \leq \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u(t)\| \text{ for a.e. } t \in (0,T).$$

Definition 2.1. Let $f \in H_g$ be given. A strong stationary solution to problem (1.1) is an element $u^* \in D(A)$ such that

$$\nu Au^* + \nu Cu^* + B(u^*,u^*) = f \text{ in } H_g.$$

The following result was proved in [18, 19].

Theorem 2.1. *If $f \in H_g$, then problem* (1.1) *admits at least one strong stationary solution u^* satisfying*

$$\|u^*\| \leq \frac{1}{\lambda_1^{1/2}\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}|f|. \tag{2.3}$$

Moreover, if the following condition holds

$$\nu^2\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2 > \frac{c_1|f|}{\lambda_1}, \tag{2.4}$$

where c_1 is the constant in Lemma 2.1, then the strong stationary solution u^* to problem (1.1) *is unique and globally exponentially stable.*

3. Stabilization of 2D g-Navier-Stokes equations by using finite-dimensional feedback controls

Let u^* be a strong stationary solution to problem (1.1). By Theorem 2.1, it is known that if condition (2.4) does not hold, that is, when

$$\nu^2\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2 \leq \frac{c_1|f|}{\lambda_1},$$

then problem (1.1) may have more than one stationary solution and thus the solution u^* may be unstable. In this section, we will stabilize the stationary solution u^* by using an interpolant operator I_h as a feedback controller.

We consider the following controlled 2D g-Navier-Stokes equations with interpolant operator I_h:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = -\mu I_h(u-u^*) + f, & x\in\Omega, t>0,\\
\nabla\cdot(gu) = 0, & x\in\Omega, t>0,\\
u(x,t) = 0, & x\in\partial\Omega, t>0,\\
u(x,0) = u_0, & x\in\Omega,
\end{cases}
\tag{3.1}
$$

where $f = f(x) \in H_g$ is given.

We assume that the feedback controller $I_h : V_g \to H_g$ is an interpolant operator that approximates the identity with error of order h, i.e., it satisfies the following estimate

$$
|\varphi - I_h(\varphi)|^2 \le \frac{M_0}{m_0} c_0^2 h^2 \|\varphi\|^2, \quad \forall \varphi \in V_g, \tag{3.2}
$$

where the positive constant $c_0 > 0$ is chosen such that $c_0^2 \ge \gamma_0$ with γ_0 is as in [5] satisfying

$$
\|\varphi - I_h(\varphi)\|^2_{L^2(\Omega)^2} \le \gamma_0 h^2 \|\varphi\|^2_{H^1(\Omega)^2}
$$

for all $\varphi \in H^1(\Omega)^2$. A typical example of such an operator I_h is the projection onto Fourier modes (see, for instance, [5] for other examples).

It is obvious that the stationary solution u^* obtained in Theorem 2.1 is also a solution of system (3.1) with initial datum u^*. We will stabilize the stationary solution u^*.

The following theorem is the main result of this section.

Theorem 3.1. *Let $f \in H_g$ and let u^* be any strong stationary solution to* (1.1) *obtained in Theorem 2.1. Suppose that I_h satisfies* (3.2), *μ and h are positive parameters such that*

$$
\frac{M_0}{m_0}\mu c_0^2 h^2 < \nu \quad \text{and} \quad \mu > 2\nu \frac{|\nabla g|^2_\infty}{m_0^2} + 2\frac{c_1^2 |f|^2}{\lambda_1 \nu^3 \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right)^2}. \tag{3.3}
$$

Then for each $u_0 \in H_g$ given, there exists a unique weak solution u to system (3.1) *such that for any $T > 0$,*

$$
u \in C([0,T]; H_g) \cap L^2(0,T;V_g), \quad \frac{du}{dt} \in L^2(0,T;V_g'),
$$

and

$$
|u(t) - u^*|^2 \le e^{-\eta t}|u_0 - u^*|^2, \quad \forall t \ge 0, \tag{3.4}
$$

where $\eta = \mu - 2\nu \frac{|\nabla g|^2_\infty}{m_0^2} - 2\frac{c_1^2|f|^2}{\lambda_1\nu^3\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2} > 0$ due to condition (3.3).

Proof. We set $z = u - u^*$ and rewrite system (3.1) as

$$\begin{cases} z' + \nu Az + \nu Cz + Bz + B_0 z + \mu P_g I_h(z) = 0, \\ z(0) = u(0) - u^* =: z_0, \end{cases} \tag{3.5}$$

where P_g is the orthogonal projector from $\mathbb{L}^2(\Omega, g)$ onto H_g, and $B_0 z$ is defined by

$$\langle B_0 z, w\rangle_g = b(u^*, z, w) + b(z, u^*, w) \quad \text{for all } w \in V_g.$$

(i) Existence. We prove the existence of a weak solution z to problem (3.5) by using the Galerkin method.

Let $\{w_j\}_{j=1}^{\infty}$ be a basis of $D(A)$ consisting of eigenfunctions of the g-Stokes operator A. Using the n-dimensional Galerkin approximation $z_n = \sum_{j=1}^{n} z_{nj}(t) w_j$, then the equation for z_n is

$$\begin{cases} z_n' + \nu Az_n + \nu P_n Cz_n + P_n Bz_n + P_n B_0 z_n + \mu P_n P_g I_h(z_n) = 0, \\ z_n(0) = P_n z_0, \end{cases} \tag{3.6}$$

where

$$P_n z = \sum_{j=1}^{n} (z, w_j) w_j.$$

We try to find a bound on $|z_n|$ uniform in n. Taking the inner product of (3.6) with z_n and noticing that $P_n z_n = z_n$ and $\langle Bz_n, z_n\rangle_g = 0$, we have

$$(z_n', z_n)_g + \nu\langle Az_n, z_n\rangle_g + \nu(Cz_n, z_n)_g + \langle B_0 z_n, z_n\rangle_g + \mu(I_h(z_n), z_n)_g = 0$$

or

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|z_n|^2 + \nu\|z_n\|^2 \\ &\quad + \nu(Cz_n, z_n)_g + b\left(u^*, z_n, z_n\right) + b\left(z_n, u^*, z_n\right) + \mu(I_h(z_n), z_n)_g = 0. \end{aligned}$$

Since $b\left(u^*, z_n, z_n\right) = 0$, we have

$$\frac{1}{2}\frac{d}{dt}|z_n|^2 + \nu\|z_n\|^2 = -\nu(Cz_n, z_n)_g - b\left(z_n, u^*, z_n\right) - \mu(I_h(z_n), z_n)_g. \tag{3.7}$$

Using (3.2) and the Cauchy inequality, we have

$$\begin{aligned} -\mu(I_h(z_n), z_n)_g &= \mu(z_n - I_h(z_n), z_n)_g - \mu|z_n|^2 \\ &\le \mu|z_n - I_h(z_n)|\,|z_n| - \mu|z_n|^2 \\ &\le \frac{\mu}{2}|z_n - I_h(z_n)|^2 - \frac{\mu}{2}|z_n|^2 \\ &\le \frac{M_0 \mu c_0^2 h^2}{2m_0}\|z_n\|^2 - \frac{\mu}{2}|z_n|^2. \end{aligned} \tag{3.8}$$

Furthermore, by the Cauchy inequality, the first estimate in Lemma 2.1 and Lemma 2.2, we have

$$-\nu(Cz_n, z_n)_g - b\left(z_n, u^*, z_n\right) \le \nu|Cz_n|\,|z_n| + c_1\|u^*\||z_n|\|z_n\|$$

$$\le \nu \frac{|\nabla g|_\infty}{m_0} \|z_n\|\, |z_n| + c_1 \|u^*\| \|z_n\| \|z_n\|$$

$$\le \frac{\nu}{2}\|z_n\|^2 + \nu \frac{|\nabla g|_\infty^2}{m_0^2}|z_n|^2 + \frac{c_1^2\|u^*\|^2}{\nu}|z_n|^2. \tag{3.9}$$

Combining (3.8) and (3.9) we deduce from (3.7) that

$$\frac{1}{2}\frac{d}{dt}|z_n|^2 + \nu\|z_n\|^2 \le \frac{\nu}{2}\|z_n\|^2 + \nu \frac{|\nabla g|_\infty^2}{m_0^2}|z_n|^2 + \frac{c_1^2\|u^*\|^2}{\nu}|z_n|^2 + \frac{M_0 \mu c_0^2 h^2}{2m_0}\|z_n\|^2 - \frac{\mu}{2}|z_n|^2.$$

This implies that

$$\frac{d}{dt}|z_n|^2 + \left(\nu - \frac{M_0}{m_0}\mu c_0^2 h^2\right)\|z_n\|^2 \le \left(-\mu + 2\nu\frac{|\nabla g|_\infty^2}{m_0^2} + 2\frac{c_1^2\|u^*\|^2}{\nu}\right)|z_n|^2. \tag{3.10}$$

Integrating both sides from 0 to t, we obtain

$$|z_n(t)|^2 + \left(\nu - \frac{M_0}{m_0}\mu c_0^2 h^2\right)\int_0^t \|z_n(s)\|^2 ds + \left(\mu - 2\nu \frac{|\nabla g|_\infty^2}{m_0^2} - 2\frac{c_1^2\|u^*\|^2}{\nu}\right)\int_0^t |z_n(s)|^2 ds \le |z(0)|^2,$$

so that

$$|z_n(t)|^2 + \left(\nu - \frac{M_0}{m_0}\mu c_0^2 h^2\right)\int_0^t \|z_n(s)\|^2 ds + \left(\mu - 2\nu\frac{|\nabla g|_\infty^2}{m_0^2} - 2\frac{c_1^2|f|^2}{\lambda_1\nu^3\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2}\right)\int_0^t |z_n(s)|^2 ds \le |z(0)|^2, \tag{3.11}$$

where we have used the estimate (2.3) for the stationary solution u^*.

By hypothesis (3.3), the second and the third terms in the left-hand side of (3.11) are positive and this implies that $\{z_n\}$ is bounded in $L^\infty(0,T;H_g) \cap L^2(0,T;V_g)$. These uniform bounds allow us to use the Alaoglu theorem to find a subsequence (which we shall relabel as z_n) such that

$$z_n \rightharpoonup^* z \quad \text{in} \quad L^\infty(0,T;H_g),$$
$$z_n \rightharpoonup z \quad \text{in} \quad L^2(0,T;V_g).$$

Next, we need to obtain a uniform bound for the derivative $\frac{dz_n}{dt}$.

Note that from (3.2) we also obtain

$$|I_h(z_n)| \le |z_n - I_h(z_n)| + |z_n| \le \sqrt{\frac{M_0}{m_0}}c_0 h\|z_n\| + |z_n|.$$

Since

$$\frac{dz_n}{dt} = -\nu A z_n - \nu P_n C z_n - P_n B z_n - P_n B_0 z_n - \mu P_n P_g I_h(z_n),$$

then by Lemmas 2.1 and 2.2, one can check that $\{\frac{dz_n}{dt}\}$ is bounded in $L^2(0,T;V_g')$. Since $\{z_n\}$ is bounded in $L^2(0,T;V_g)$ and $\{\frac{dz_n}{dt}\}$ is bounded in $L^2(0,T;V_g')$, by applying the Aubin-Lions compactness lemma, there is a subsequence $\{z_n\}$ (after relabeling) that converges to z strongly in $L^2(0,T;H_g)$. Hence, arguing as in the case of the 2D Navier-Stokes equations (see e.g. [20, p. 248]), we have

$$\begin{aligned} P_n B z_n &\rightharpoonup Bz \quad \text{in} \quad L^2(0,T;V_g'), \\ P_n B_0 z_n &\rightharpoonup B_0 z \quad \text{in} \quad L^2(0,T;V_g'). \end{aligned}$$

Thus, we have

$$\frac{dz}{dt} + \nu A z + \nu C z + Bz + B_0 z + \mu P_g I_h(z) = 0 \text{ in } L^2(0,T;V_g'). \tag{3.12}$$

Finally, to show that $z(0) = z_0$, we choose an arbitrary test function $\varphi \in C^1([0,T];V_g)$ with $\varphi(T) = 0$. Taking the inner product of (3.12) with φ and integrating by parts in t the first term, we have

$$\begin{aligned} &-\int_0^T (z(t),\varphi'(t))_g\,dt + \nu\int_0^T ((z(t),\varphi(t)))_g\,dt + \nu\int_0^T (Cz(t),\varphi(t))_g\,dt \\ &+\int_0^T b(z(t),z(t),\varphi(t))\,dt + \int_0^T b(z(t),u^*,\varphi(t))\,dt + \mu\int_0^T (P_g I_h(z(t)),\varphi(t))_g\,dt \\ &= (z(0),\varphi(0))_g\,. \end{aligned}$$

Doing the same for the Galerkin approximations,

$$\begin{aligned} &-\int_0^T (z_n(t),\varphi'(t))_g\,dt + \nu\int_0^T ((z_n(t),\varphi(t)))_g\,dt + \nu\int_0^T (Cz_n(t),\varphi(t))_g\,dt \\ &+\int_0^T b(z_n(t),z_n(t),\varphi(t))\,dt + \int_0^T b(z_n(t),u^*,\varphi(t))\,dt + \mu\int_0^T (P_g I_h(z_n(t)),\varphi(t))_g\,dt \\ &= (z_n(0),\varphi(0))_g\,, \end{aligned}$$

and then letting $n \to \infty$, we obtain

$$\begin{aligned} &-\int_0^T (z(t),\varphi'(t))_g\,dt + \nu\int_0^T ((z(t),\varphi(t)))_g\,dt + \nu\int_0^T (Cz(t),\varphi(t))_g\,dt \\ &+\int_0^T b(z(t),z(t),\varphi(t))\,dt + \int_0^T b(z(t),u^*,\varphi(t))\,dt + \mu\int_0^T (P_g I_h(z(t)),\varphi(t))_g\,dt \\ &= (z_0,\varphi(0))_g \end{aligned}$$

since $z_n(0) = P_n z_0 \to z_0$. Therefore, $z(0) = z_0$, and this implies that z is a weak solution to (3.5).

(ii) Stabilization. We now prove the exponential stability of u^*. Using (3.5) and the fact that $b(z, z, z) = b(u^*, z, z) = 0$, we arrive at

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\|z\|^2 = -b(z, u^*, z) - \nu(Cz, z)_g - \mu(I_h(z), z)_g. \tag{3.13}$$

Therefore, as in (3.8)-(3.10) and (3.13) we obtain

$$\frac{d}{dt}|z|^2 + \left(\nu - \frac{M_0}{m_0}\mu c_0^2 h^2\right)\|z\|^2 + \left(\mu - 2\nu\frac{|\nabla g|_\infty^2}{m_0^2} - 2\frac{c_1^2\|u^*\|^2}{\nu}\right)|z|^2 \le 0.$$

Hence, by using hypothesis (3.3) and estimate (2.3), we obtain

$$\frac{d}{dt}|z|^2 + \left(\mu - 2\nu\frac{|\nabla g|_\infty^2}{m_0^2} - 2\frac{c_1^2|f|^2}{\lambda_1\nu^3\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2}\right)|z|^2 \le 0.$$

This implies that

$$|z(t)|^2 \le e^{-\eta t}|z(0)|^2,$$

where

$$\eta = \mu - 2\nu\frac{|\nabla g|_\infty^2}{m_0^2} - 2\frac{c_1^2|f|^2}{\lambda_1\nu^3\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)^2} > 0.$$

Hence, the inequality (3.4) holds true. □

4. Stabilization of 2D g-Navier-Stokes equations by using fast oscillating-in-time external forces

In this section, we consider the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F(x, \omega t), & x \in \Omega, t > 0, \\ \nabla\cdot(gu) = 0 & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{4.1}$$

We give the following assumption on the external force:

(F) *For any positive constant $\omega > 0$, we assume that the force term $F(x, \omega t)$ is a time periodic function with period T_{per} having the following structure: There exists a time periodic function $h(x, \omega t)$ with period T_{per} such that*

$$\begin{cases} \dfrac{1}{\omega}h_t(x, \omega t) = -F(x, \omega t) & \text{in } \Omega\times\mathbb{R}^+, \\ \nabla\cdot(gh) = 0 & \text{in } \Omega\times\mathbb{R}^+, \\ h = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We also assume that
$F \in L^\infty(0, T_{per}; D(A))$ and $\|F\|_{L^\infty(0,T_{per};D(A))}$ with upper bound is independent of ω.

Moreover, we assume that $h \in L^{\infty}(0, T_{per}; D(A))$ *and there exists a positive constant* L_h *independent of* ω *such that*

$$\|h\|^2_{L^{\infty}(0,T_{per};D(A))} \le L_h \|F\|^2_{L^{\infty}(0,T_{per};D(A))}. \tag{4.3}$$

Let us give an example about the external force F. Let

$$F(x, \omega t) = f(x) \sin \omega t,$$

where $f \in D(A)$ and ω is a positive constant. Then $h(x, \omega t)$ has the form

$$h(x, \omega t) = f(x) \cos \omega t.$$

It is clear that $h \in L^{\infty}(0, T_{per}; D(A))$ and h satisfies (4.3), where $T_{per} = 2\pi/\omega$.

More general, we can take the external force F of the form $F(x, \omega t) = f(x)\varphi(\omega t)$, where $f \in D(A)$ and $\varphi(t)$ is a real-valued periodic continuous function with period T. Then F satisfies the Assumption **(F)** with $T_{per} = T/\omega$.

First, we prove the existence of a time periodic solution to (4.1).

Theorem 4.1. *Let hypothesis* (**F**) *hold. Then there exists* $\omega_0 > 0$ *depending on* $\nu, c_1, c_3, \lambda_1, L_h$ *and* $\|F\|_{L^{\infty}(0,T_{per};D(A))}$ *such that for any* $\omega \ge \omega_0$*, the system* (4.1) *has a* T_{per}*-periodic solution* u_{per} *satisfying*

$$\|u_{per}(t)\| \le \frac{\nu \lambda_1^{1/2}}{2c_1} \left(1 - \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} \right) \quad \text{for all } t \in [0, T_{per}], \tag{4.4}$$

where c_1, c_3 *are the constants in Lemma 2.1.*

Proof. From assumption (**F**) we particularly have $F \in L^2(0, T_{per}; H_g)$. Therefore, for each initial datum $u_0 \in V_g$ given, there exists a unique strong solution $u \in C([0, T_{per}]; V_g)$ to system (4.1) such that $u(0) = u_0$ (see e.g. [2]).

We now prove that system (4.1) has at least one T_{per}-periodic solution $u_{per} \in L^{\infty}(0, T_{per}; V_g)$ satisfying (4.4). We set

$$u = y - \frac{1}{\omega} h(x, \omega t), \tag{4.5}$$

and rewrite system (4.1) as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p = \dfrac{1}{\omega} h_t(x, \omega t) + F(x, \omega t) & \\ -\dfrac{\nu}{\omega} \Delta h - \dfrac{1}{\omega^2} (h \cdot \nabla) h + \dfrac{1}{\omega} [(y \cdot \nabla) h + (h \cdot \nabla) y] & \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot (g y) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+. \end{cases}$$

Using (4.2), we obtain

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p & \\ = -\dfrac{\nu}{\omega} \Delta h - \dfrac{1}{\omega^2} (h \cdot \nabla) h + \dfrac{1}{\omega} [(y \cdot \nabla) h + (h \cdot \nabla) y] & \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot (g y) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases}$$

or

$$y' + \nu Ay + \nu Cy + By = \frac{\nu}{\omega}Ah + \frac{\nu}{\omega}Ch - \frac{1}{\omega^2}Bh + \frac{1}{\omega}B_0 h, \tag{4.6}$$

where $B_0 h$ is defined by

$$\langle B_0 h, v\rangle_g = b(y,h,v) + b(h,y,v) \text{ for all } v \in V_g.$$

Taking the inner product of (4.6) with y and noticing that $\langle By, y\rangle_g = 0$, we have

$$\begin{aligned}(y',y)_g + \nu\langle Ay,y\rangle_g + \nu(Cy,y)_g = &\frac{\nu}{\omega}\langle Ah,y\rangle_g + \frac{\nu}{\omega}(Ch,y)_g \\ &- \frac{1}{\omega^2}\langle Bh,y\rangle_g + \frac{1}{\omega}\langle B_0 h,y\rangle_g,\end{aligned}$$

or

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|y|^2 + \nu\|y\|^2 + \nu(Cy,y)_g = &\frac{\nu}{\omega}\langle Ah,y\rangle_g + \frac{\nu}{\omega}(Ch,y)_g \\ &- \frac{1}{\omega^2}b(h,h,y) + \frac{1}{\omega}b(y,h,y) + \frac{1}{\omega}b(h,y,y).\end{aligned}$$

Since $b(h,y,y)=0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|y|^2 + \nu\|y\|^2 = &-\nu(Cy,y)_g + \frac{\nu}{\omega}\langle Ah,y\rangle_g + \frac{\nu}{\omega}(Ch,y)_g \\ &- \frac{1}{\omega^2}b(h,h,y) + \frac{1}{\omega}b(y,h,y).\end{aligned} \tag{4.7}$$

Using Lemma 2.2 and Poincaré inequalities (2.1), (2.2) we obtain

$$\nu(Cy,y)_g \le \nu|Cy|\,|y| \le \nu\frac{|\nabla g|_\infty}{m_0}\|y\|\,|y| \le \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|y\|^2, \tag{4.8}$$

and

$$\begin{aligned}\frac{\nu}{\omega}(Ch,y)_g &\le \frac{\nu|\nabla g|_\infty}{\omega m_0}\|h\||y| \\ &\le \frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1^{1/2}}|Ah||y| \le \frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1}\|h\|_{L^\infty(0,T_{per};D(A))}\|y\|.\end{aligned} \tag{4.9}$$

By Lemma 2.1 and Poincaré inequalities (2.1), (2.2), we get

$$\begin{aligned}\frac{1}{\omega}b(y,h,y) &\le \frac{c_1}{\omega}|y|^{1/2}\|y\|^{1/2}\|h\||y|^{1/2}\|y\|^{1/2} \\ &\le \frac{c_1}{\omega\lambda_1^{1/2}}|Ah||y|\|y\| \le \frac{c_1}{\omega\lambda_1}\|h\|_{L^\infty(0,T_{per};D(A))}\|y\|^2,\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}-\frac{1}{\omega^2}b(h,h,y) &\le \frac{c_1}{\omega^2}|h|^{1/2}\|h\|^{1/2}\|h\||y|^{1/2}\|y\|^{1/2} \\ &\le \frac{c_1}{\omega^2\lambda_1^{5/4}}|Ah||y|^{1/2}\|y\|^{1/2}\end{aligned}$$

$$\leq \frac{c_1}{\omega^2 \lambda_1^{3/2}} \|h\|^2_{L^\infty(0,T_{per};D(A))} \|y\|. \tag{4.11}$$

We also have

$$\begin{aligned}
\frac{\nu}{\omega} \langle Ah, y \rangle_g &= \frac{\nu}{\omega} \int_\Omega \nabla h \cdot \nabla y g dx \\
&\leq \frac{\nu}{\omega} \|h\| \|y\| \\
&\leq \frac{\nu}{\omega \lambda_1^{1/2}} |Ah| \|y\| \leq \frac{\nu}{\omega \lambda_1^{1/2}} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\|. \qquad (4.12)
\end{aligned}$$

From (4.7)-(4.12) we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |y|^2 + \nu \|y\|^2 \\
&\leq \nu \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \|y\|^2 + \frac{\nu}{\omega \lambda_1^{1/2}} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| \\
&\quad + \frac{\nu |\nabla g|_\infty}{\omega m_0 \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| + \frac{c_1}{\omega^2 \lambda_1^{3/2}} \|h\|^2_{L^\infty(0,T_{per};D(A))} \|y\| \\
&\quad + \frac{c_1}{\omega \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |y|^2 + \nu \gamma_0 \|y\|^2 \\
&\leq \frac{\nu}{\omega \lambda_1^{1/2}} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| + \frac{\nu |\nabla g|_\infty}{\omega m_0 \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| \\
&\quad + \frac{c_1}{\omega^2 \lambda_1^{3/2}} \|h\|^2_{L^\infty(0,T_{per};D(A))} \|y\| + \frac{c_1}{\omega \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\|^2, \qquad (4.13)
\end{aligned}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}$. Using (4.3), we have

$$\frac{c_1}{\omega \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\|^2 \leq \frac{c_1 \sqrt{L_h}}{\omega \lambda_1} \|F\|_{L^\infty(0,T_{per};V_g)} \|y\|^2. \tag{4.14}$$

By the Cauchy inequality, we obtain

(4.15)

$$\frac{\nu}{\omega \lambda_1^{1/2}} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| \leq \frac{\nu \gamma_0}{8} \|y\|^2 + \frac{2\nu}{\omega^2 \gamma_0 \lambda_1} \|h\|^2_{L^\infty(0,T_{per};D(A))},$$

(4.16)

$$\frac{\nu |\nabla g|_\infty}{\omega m_0 \lambda_1} \|h\|_{L^\infty(0,T_{per};D(A))} \|y\| \leq \frac{\nu \gamma_0}{8} \|y\|^2 + \frac{2\nu |\nabla g|^2_\infty}{\omega^2 m_0^2 \gamma_0 \lambda_1^2} \|h\|^2_{L^\infty(0,T_{per};D(A))},$$

(4.17)

$$\frac{c_1}{\omega^2 \lambda_1^{3/2}} \|h\|^2_{L^\infty(0,T_{per};D(A))} \|y\| \leq \frac{\nu \gamma_0}{8} \|y\|^2 + \frac{2c_1^2}{\omega^4 \nu \gamma_0 \lambda_1^3} \|h\|^4_{L^\infty(0,T_{per};D(A))}.$$

Taking

$$\omega \geq \frac{8c_1\sqrt{L_h}}{\nu\gamma_0\lambda_1}\|F\|_{L^\infty(0,T_{per};D(A))} \tag{4.18}$$

then (4.14) becomes

$$\frac{c_1}{\omega\lambda_1}\|h\|_{L^\infty(0,T_{per};D(A))}\|y\|^2 \leq \frac{\nu\gamma_0}{8}\|y\|^2. \tag{4.19}$$

Using (4.15)-(4.17) and (4.19) then we implies from (4.13) that

$$\begin{aligned}\frac{d}{dt}|y|^2 + \nu\gamma_0\|y\|^2 \leq{}& \frac{4\nu}{\omega^2\gamma_0\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))} + \frac{4\nu|\nabla g|^2_\infty}{\omega^2 m_0^2\gamma_0\lambda_1^2}\|h\|^2_{L^\infty(0,T_{per};D(A))}\\ &+ \frac{4c_1^2}{\omega^4\nu\gamma_0\lambda_1^3}\|h\|^4_{L^\infty(0,T_{per};D(A))}.\end{aligned} \tag{4.20}$$

Therefore, using the bound (4.3), we deduce from (4.20) that

$$\frac{d}{dt}|y|^2 + \nu\gamma_0\|y\|^2 \leq M, \tag{4.21}$$

where

$$M = \frac{4L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\gamma_0\lambda_1}\left(\nu + \frac{\nu|\nabla g|^2_\infty}{m_0^2\lambda_1} + \frac{c_1^2 L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\nu\lambda_1^2}\right). \tag{4.22}$$

Therefore, by using the Poincaré (2.1), we obtain

$$\frac{d}{dt}|y|^2 + \nu\gamma_0\lambda_1|y|^2 \leq M.$$

Hence, by virtue of Gronwall's inequality, we have

$$|y(t)|^2 \leq |y(0)|^2 e^{-\nu\gamma_0\lambda_1 t} + \left(1 - e^{-\nu\gamma_0\lambda_1 t}\right)\frac{M}{\nu\gamma_0\lambda_1}. \tag{4.23}$$

Setting $R = \frac{1}{\sqrt{12}}\left[\frac{\nu\lambda_1^{1/2}}{c_1}\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\right]$. We will prove that for any $y(0)$ such that $|y(0)| \leq R$, we also have $|y(T_{per})| \leq R$. Indeed, for ω satisfying (4.18), we have from (4.23) that

$$|y(T_{per})|^2 \leq e^{-\nu\gamma_0\lambda_1 T_{per}}R^2 + \left(1 - e^{-\nu\gamma_0\lambda_1 T_{per}}\right)\frac{M}{\nu\gamma_0\lambda_1}. \tag{4.24}$$

Since the definition of M in (4.22), we can find $\omega_1 > 0$ large enough such that

$$\frac{M}{\nu\gamma_0\lambda_1} \leq R^2, \quad \forall \omega \geq \omega_1. \tag{4.25}$$

So, combining this with (4.24), we get

$$|y(T_{per})| \leq R$$

for

$$\omega \geq \max\left\{\omega_1, \frac{8c_1\sqrt{L_h}}{\nu\gamma_0\lambda_1}\|F\|_{L^\infty(0,T_{per};D(A))}\right\}. \tag{4.26}$$

Since the ball $\{|\cdot| \leq R\} \subset H_g$ is a convex set in an Hilbert space, and it is compact in the weak topology. Hence, by using the Tychonoff theorem we conclude that the mapping $y(0) \mapsto y(T_{per})$ has a fixed point y^*. Then the solution y_{per} of system (4.6) with initial datum y^* is a T_{per}-periodic solution satisfying

$$|y_{per}(0)| \leq R.$$

Then, by (4.23) and (4.25), we obtain

$$|y_{per}(t)|^2 \leq e^{-\nu\gamma_0\lambda_1 t}R^2 + \left(1 - e^{-\nu\gamma_0\lambda_1 t}\right)\frac{M}{\nu\gamma_0\lambda_1} \leq R^2, \quad \forall t \in [0, T_{per}], \tag{4.27}$$

for ω is large enough satisfying (4.26).

Besides, integrating (4.21) over $[0, T_{per}]$ with respect to time variable and considering the time periodicity of y_{per}, we obtain

$$\int_0^{T_{per}} \|y_{per}(t)\|^2 dt \leq \frac{MT_{per}}{\nu\gamma_0}.$$

Hence, there exists $t^* \in [0, T_{per})$ such that

$$\|y_{per}(t^*)\|^2 \leq \frac{M}{\nu\gamma_0}. \tag{4.28}$$

Now integrating (4.21) over $[t^*, T_{per}]$ we get

$$|y_{per}(T_{per})|^2 - |y_{per}(t^*)|^2 + \nu\gamma_0\int_{t^*}^{T_{per}} \|y_{per}(t)\|^2 dt \leq MT_{per} \quad \text{for all } t \in [t^*, T_{per}].$$

Thus, by using (4.27), we have

$$\int_{t^*}^{T_{per}} \|y_{per}(t)\|^2 dt \leq \frac{MT_{per} + R^2}{\nu\gamma_0}, \quad t \in [t^*, T_{per}]. \tag{4.29}$$

Integrating from 0 to t in (4.21) yields

$$\begin{aligned} |y_{per}(t)|^2 + \nu\gamma_0\int_0^t \|y_{per}(s)\|^2 ds &\leq |y_{per}(0)|^2 + MT_{per} \\ &\leq R^2 + MT_{per}, \ \forall t \in [0, T_{per}], \end{aligned}$$

and therefore

$$\int_0^t \|y_{per}(s)\|^2 ds \leq \frac{MT_{per} + R^2}{\nu\gamma_0}, \ \forall t \in [0, T_{per}]. \tag{4.30}$$

Taking the inner product of (4.6) with Ay_{per}, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|y_{per}\|^2 + \nu|Ay_{per}|^2 = &-\nu(Cy_{per}, Ay_{per})_g + \frac{\nu}{\omega}(Ch, Ay_{per})_g \\ &- b(y_{per}, y_{per}, Ay_{per}) - \frac{1}{\omega^2}b(h, h, Ay_{per}) \end{aligned}$$

$$+\frac{1}{\omega}b(y_{per},h,Ay_{per})+\frac{1}{\omega}b(h,y_{per},Ay_{per})$$
$$+\frac{\nu}{\omega}(Ah,Ay_{per})_g. \tag{4.31}$$

Using Lemma 2.2 and the Poincaré inequality (2.2), we obtain

$$-\nu(Cy_{per},Ay_{per})_g \le \nu\frac{|\nabla g|_\infty}{m_0}\|y_{per}\||Ay_{per}| \le \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}|Ay_{per}|^2, \tag{4.32}$$

and

$$\begin{aligned}\frac{\nu}{\omega}(Ch,Ay_{per})_g &\le \frac{\nu|\nabla g|_\infty}{\omega m_0}\|h\||Ay_{per}|\\ &\le \frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1^{1/2}}|Ah||Ay_{per}|\\ &\le \frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1^{1/2}}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}|.\end{aligned} \tag{4.33}$$

By Lemma 2.1 and Poincaré inequalities (2.1), (2.2), we obtain

$$-b(y_{per},y_{per},Ay_{per}) \le c_3|y_{per}|^{1/2}\|y_{per}\||Ay_{per}|^{3/2}, \tag{4.34}$$

$$\begin{aligned}-\frac{1}{\omega^2}b(h,h,Ay_{per}) &\le \frac{c_3}{\omega^2}|h|^{1/2}|Ah|^{1/2}\|h\||Ay_{per}|\\ &\le \frac{c_3}{\omega^2\lambda_1}|Ah|^2|Ay_{per}|\\ &\le \frac{c_3}{\omega^2\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}|Ay_{per}|,\end{aligned} \tag{4.35}$$

$$\begin{aligned}\frac{1}{\omega}b(y_{per},h,Ay_{per}) &\le \frac{c_3}{\omega}|y_{per}|^{1/2}|Ay_{per}|^{1/2}\|h\||Ay_{per}|\\ &\le \frac{c_3}{\omega\lambda_1^{1/2}}|y_{per}|^{1/2}|Ah||Ay_{per}|^{3/2}\\ &\le \frac{c_3}{\omega\lambda_1^{1/2}}\|h\|_{L^\infty(0,T_{per};D(A))}|y_{per}|^{1/2}|Ay_{per}|^{3/2},\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}\frac{1}{\omega}b(h,y_{per},Ay_{per}) &\le \frac{c_2}{\omega}|h|^{1/2}\|h\|^{1/2}\|y_{per}\|^{1/2}|Ay_{per}|^{3/2}\\ &\le \frac{c_2}{\omega\lambda_1^{3/4}}|Ah|\ \|y_{per}\|^{1/2}|Ay_{per}|^{3/2}\\ &\le \frac{c_3}{\omega\lambda_1^{3/4}}\|h\|_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^{1/2}|Ay_{per}|^{3/2}.\end{aligned} \tag{4.37}$$

We have

$$\frac{\nu}{\omega}(Ah,Ay_{per})_g \le \frac{\nu}{\omega}|Ah||Ay_{per}| \le \frac{\nu}{\omega}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}|. \tag{4.38}$$

From (4.31)-(4.38) we get

$$\frac{1}{2}\frac{d}{dt}\|y_{per}\|^2 + \nu\gamma_0|Ay_{per}|^2 \leq \frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1^{1/2}}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}| + c_3|y_{per}|^{1/2}\|y_{per}\||Ay_{per}|^{3/2} + \frac{c_3}{\omega^2\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}|Ay_{per}| + \frac{c_3}{\omega\lambda_1^{1/2}}|y_{per}|^{1/2}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}|^{3/2} + \frac{c_2}{\omega\lambda_1^{3/4}}\|h\|_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^{1/2}|Ay_{per}|^{3/2} + \frac{\nu}{\omega}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}|. \tag{4.39}$$

Using the Cauchy inequality, we obtain

$$\frac{\nu|\nabla g|_\infty}{\omega m_0\lambda_1^{1/2}}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}| \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{2\nu|\nabla g|^2_\infty}{\omega^2 m_0^2\gamma_0\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}, \tag{4.40}$$

$$\frac{c_3}{\omega^2\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}|Ay_{per}| \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{2c_3^2}{\omega^4\nu\gamma_0\lambda_1^2}\|h\|^4_{L^\infty(0,T_{per};D(A))}, \tag{4.41}$$

$$\frac{\nu}{\omega}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}| \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{2\nu}{\omega^2\gamma_0}\|h\|^2_{L^\infty(0,T_{per};D(A))}. \tag{4.42}$$

By using the Young inequality, we get

$$c_3|y_{per}|^{1/2}\|y_{per}\||Ay_{per}|^{3/2} \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{54c_3^4}{\nu^3\gamma_0^3}|y_{per}|^2\|y_{per}\|^4, \tag{4.43}$$

$$\frac{c_2}{\omega\lambda_1^{3/4}}\|h\|_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^{1/2}|Ay_{per}|^{3/2} \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{54c_2^4}{\omega^4\nu^3\gamma_0^3\lambda_1^3}\|h\|^4_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^2, \tag{4.44}$$

$$\frac{c_3}{\omega\lambda_1^{1/2}}|y_{per}|^{1/2}\|h\|_{L^\infty(0,T_{per};D(A))}|Ay_{per}|^{3/2} \leq \frac{\nu\gamma_0}{8}|Ay_{per}|^2 + \frac{54c_3^4}{\omega^4\nu^3\gamma_0^3\lambda_1^2}|y_{per}|^2\|h\|^4_{L^\infty(0,T_{per};D(A))}. \tag{4.45}$$

Substituting estimates from (4.40) to (4.45) into (4.39), we have

$$\frac{1}{2}\frac{d}{dt}\|y_{per}\|^2 + \nu\gamma_0|Ay_{per}|^2 \leq \frac{3\nu\gamma_0}{4}|Ay_{per}|^2 + \frac{2\nu|\nabla g|^2_\infty}{\omega^2 m_0^2\gamma_0\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}$$

$$+ \frac{2c_3^2}{\omega^4\nu\gamma_0\lambda_1^2}\|h\|^4_{L^\infty(0,T_{per};D(A))}$$
$$+ \frac{54c_2^4}{\omega^4\nu^3\gamma_0^3\lambda_1^3}\|h\|^4_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^2$$
$$+ \frac{2\nu}{\omega^2\gamma_0}\|h\|^2_{L^\infty(0,T_{per};D(A))} + \frac{54c_3^4}{\nu^3\gamma_0^3}|y_{per}|^2\|y_{per}\|^4$$
$$+ \frac{54c_3^4}{\omega^4\nu^3\gamma_0^3\lambda_1^2}|y_{per}|^2\|h\|^4_{L^\infty(0,T_{per};D(A))}.$$

Thus

$$\begin{aligned}
&\frac{d}{dt}\|y_{per}\|^2 + \frac{1}{2}\nu\gamma_0|Ay_{per}|^2 \\
&\le 2\Big[\frac{2\nu|\nabla g|^2_\infty}{\omega^2 m_0^2\gamma_0\lambda_1} + \frac{2c_3^2}{\omega^4\nu\gamma_0\lambda_1^2}\|h\|^2_{L^\infty(0,T_{per};D(A))} + \frac{2\nu}{\omega^2\gamma_0} \\
&\quad + \frac{54c_3^4R^2}{\omega^4\nu^3\gamma_0^3\lambda_1^2}\|h\|^2_{L^\infty(0,T_{per};D(A))}\Big]\|h\|^2_{L^\infty(0,T_{per};D(A))} \\
&\quad + \frac{108c_2^4}{\omega^4\nu^3\gamma_0^3\lambda_1^3}\|h\|^4_{L^\infty(0,T_{per};D(A))}\|y_{per}\|^2 + \frac{108c_3^4R^2}{\nu^3\gamma_0^3}\|y_{per}\|^4.
\end{aligned}$$

Here we have used (4.27). And therefore

$$\begin{aligned}
&\frac{d}{dt}\|y_{per}\|^2 + \frac{1}{2}\nu\gamma_0|Ay_{per}|^2 \\
&\le \frac{4L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\gamma_0}\Big[\frac{\nu|\nabla g|^2_\infty}{m_0^2\lambda_1} + \frac{c_3^2L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\nu\lambda_1^2} \\
&\quad + \nu + \frac{27c_3^4L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\nu^3\gamma_0^2\lambda_1^2}R^2\Big] \\
&\quad + \frac{108c_2^4L_h^2\|F\|^4_{L^\infty(0,T_{per};D(A))}}{\omega^4\nu^3\gamma_0^3\lambda_1^3}\|y_{per}\|^2 + \frac{108c_3^4}{\nu^3\gamma_0^3}R^2\|y_{per}\|^4
\end{aligned}$$

thanks to (4.3). Neglecting the term $\frac{1}{2}\nu\gamma_0|Ay_{per}|^2$, we find that

$$\frac{d}{dt}\|y_{per}\|^2 \le M_1 + \Big(M_2 + \frac{108c_3^4}{\nu^3\gamma_0^3}R^2\|y_{per}\|^2\Big)\|y_{per}\|^2, \tag{4.46}$$

where

$$\begin{aligned}
M_1 = \frac{4L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\gamma_0}\Big[&\frac{\nu|\nabla g|^2_\infty}{m_0^2\lambda_1} + \frac{c_3^2L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\nu\lambda_1^2} \\
&+ \nu + \frac{27c_3^4L_h\|F\|^2_{L^\infty(0,T_{per};D(A))}}{\omega^2\nu^3\gamma_0^2\lambda_1^2}R^2\Big],
\end{aligned}$$

and

$$M_2 = \frac{108c_2^4 L_h^2 \|F\|^4_{L^\infty(0,T_{per};D(A))}}{\omega^4 \nu^3 \gamma_0^3 \lambda_1^3}.$$

Thus, from (4.46), by using the Gronwall inequality, we have for all $t \in [t^*, T_{per}]$,

$$\|y_{per}(T_{per})\|^2 \le \|y_{per}(t^*)\|^2 \exp\Big(\int_{t^*}^{T_{per}} \big(M_2 + \frac{108c_3^4}{\nu^3\gamma_0^3} R^2 \|y_{per}(\tau)\|^2\big) d\tau \Big) + \int_{t^*}^{T_{per}} M_1 \exp\Big(\int_{t^*}^{T_{per}} \big(M_2 + \frac{108c_3^4}{\nu^3\gamma_0^3} R^2 \|y_{per}(\tau)\|^2\big) d\tau \Big) ds.$$

Thus, from (4.28), (4.29) when ω satisfying (4.26), we get

$$\|y_{per}(T_{per})\|^2 \le K_0,$$

where

$$K_0 = \frac{M}{\nu\gamma_0} \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big) + M_1 T_{per} \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big).$$

Hence, since $y_{per}(T_{per}) = y_{per}(0)$, we have

$$\|y_{per}(0)\|^2 \le K_0.$$

Applying the Gronwall inequality to (4.46), we get for all $t \in [0, T_{per}]$,

$$\begin{aligned} \|y_{per}(t)\|^2 &\le \|y_{per}(0)\|^2 \exp\Big(\int_0^t \big(M_2 + \frac{108c_3^4}{\nu^3\gamma_0^3} R^2 \|y_{per}(\tau)\|^2\big) d\tau \Big) \\ &\quad + \int_0^t M_1 \exp\Big(\int_0^t \big(M_2 + \frac{108c_3^4}{\nu^3\gamma_0^3} R^2 \|y_{per}(\tau)\|^2\big) d\tau \Big) ds \\ &\le K_0 \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big) \\ &\quad + M_1 T_{per} \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big) \end{aligned} \tag{4.47}$$

for ω satisfying (4.26). Here we have used estimate (4.30).

By the definitions of M, M_1, M_2, K_0 and T_{per}, we can take $\omega \ge \omega_2 > 0$ large enough such that

$$K_0 \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big) + M_1 T_{per} \exp\Big(M_2 T_{per} + \frac{108c_3^4}{\nu^4\gamma_0^4}(MT_{per} + R^2)R^2\Big) \le R^2.$$

Thus, we deduce from (4.47) that

$$\|y_{per}(t)\|^2 \le R^2, \quad \forall t \in [0, T_{per}], \tag{4.48}$$

for $\omega \geq \max\left\{\frac{8c_1\sqrt{L_h}}{\nu\gamma_0\lambda_1}\|F\|_{L^\infty(0,T_{per};V_g)}, \omega_1, \omega_2\right\}$. From (4.5) then $u_{per} = y_{per} - \frac{1}{\omega}h(x,\omega t)$ is a T_{per}-periodic solution to (4.1). Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and (4.48), we obtain

$$\begin{aligned}\|u_{per}(t)\|^2 &\leq 2\left(\|y_{per}(t)\|^2 + \frac{1}{\omega^2}\|h\|^2\right)\\ &\leq 2R^2 + \frac{2}{\omega^2\lambda_1}\|h\|^2_{L^\infty(0,T_{per};D(A))}\\ &\leq 2R^2 + \frac{2L_h}{\omega^2\lambda_1}\|F\|^2_{L^\infty(0,T_{per};D(A))}. \quad (4.49)\end{aligned}$$

Now, if ω satisfies

$$\omega \geq \frac{\sqrt{2L_h}}{R\lambda_1^{1/2}}\|F\|_{L^\infty(0,T_{per};D(A))},$$

then by (4.49) we obtain

$$\|u_{per}(t)\|^2 \leq 3R^2 = \left[\frac{\nu\lambda_1^{1/2}}{2c_1}\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\right]^2, \ \forall t \in [0, T_{per}],$$

for

$$\omega \geq \omega_0 = \max\left\{\frac{8c_1\sqrt{L_h}}{\nu\gamma_0\lambda_1}\|F\|_{L^\infty(0,T_{per};V_g)}, \omega_1, \omega_2, \frac{\sqrt{2L_h}}{R\lambda_1^{1/2}}\|F\|_{L^\infty(0,T_{per};D(A))}\right\}.$$

This proves (4.4). The proof is complete. □

Remark 4.1. When proving the existence of a time periodic solution y_{per} satisfying (4.4), we need to estimate the term $b(h,h,Ay_{per})$ in (4.35) and the term $(Ah, Ay_{per})_g$ in (4.38). Then the term $|Ah|$ arises, and so we need the assumption $F \in L^\infty(0,T_{per};D(A))$ due to the relation (4.3) between h and F.

We now prove the global exponential stability of the time periodic solution u_{per}.

Theorem 4.2. *Let hypothesis* (**F**) *hold and let* $u_0 \in V_g$ *be given. Then any solution* $u(\cdot)$ *to system* (4.1) *with initial datum* u_0 *satisfies*

$$|u(t) - u_{per}(t)|^2 \leq e^{-\eta t}|u_0 - u_{per}(0)|^2, t \geq 0,$$

where $\eta = \nu\lambda_1\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) > 0$ *and* u_{per} *is the time periodic solution obtained in Theorem 4.1. In particular, the periodic solution* u_{per} *must be unique.*

Proof. Setting $z = u - u_{per}$, we have

$$\begin{cases} z' + \nu Az + \nu Cz + Bz + B_0 z = 0,\\ z(0) = u_0 - u_{per}(0),\end{cases}$$

where $B_0 z$ is defined by

$$\langle B_0 z, v\rangle_g = b(z, u_{per}, v) + b(u_{per}, z, v) \quad \text{for all } v \in V_g.$$

Multiplying the first equation by z and using the fact that

$$b(z, z, z) = b(u_{per}, z, z) = 0,$$

we arrive at

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\|z\|^2 = -\nu(Cz, z)_g - b(z, u_{per}, z).$$

Using Lemmas 2.1, 2.2 and the Poincaré inequality (2.1), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\|z\|^2 &\le \nu\frac{|\nabla g|_\infty}{m_0}\|z\|\,|z| + c_1|z|\|z\|\|u_{per}\| \\ &\le \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|z\|^2 + \frac{c_1\|u_{per}\|}{\lambda_1^{1/2}}\|z\|^2.\end{aligned}$$

Using the estimate (4.4) for the periodic solution u_{per} and the Poincaré inequality (2.1), we have

$$\frac{d}{dt}|z|^2 + \nu\lambda_1\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)|z|^2 \le 0.$$

This implies that $|z(t)|^2 \le e^{-\eta t}|z(0)|^2$, $\forall t \ge 0$, where $\eta = \nu\lambda_1\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) > 0$. This completes the proof. □

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