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EXTENSION OF HUYGENS TYPE INEQUALITIES FOR BESSEL AND MODIFIED BESSEL FUNCTIONS

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Abstract. In this note our aim is to extend the Huygens type inequalities to the Bessel and modified Bessel functions of the first kind. Our main motivation to write this note is a recent publication of Zhu, which we wish to complement.

1. Introduction

Bessel and modified Bessel functions of the first kind J_ν and I_ν appear frequently in various problems of applied mathematics. Because of this their properties are worth studying also from the point of view of analytic inequalities. For a long list of applications concerning properties involving Bessel and modified Bessel functions of the first kind, and some other special functions, we refer to the papers [3, 4, 8–10, 12, 15–17] and to the references therein. The inequality

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>3 \tag{1}$$

which holds for all $x\in(0,\pi/2)$ is known in literature as Huygens's inequality [6]. The hyperbolic counterpart of (1) was established in [11] as follows:

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3,\ x>0. \tag{2}$$

The inequalities (1) and (2) were respectively refined in [6] as

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>2\frac{x}{\sin x}+\frac{x}{\tan x}>3 \tag{3}$$

for $0<x<\frac{\pi}{2}$ and

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>2\frac{x}{\sinh x}+\frac{x}{\tanh x}>3,\ x\neq 0. \tag{4}$$

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Recently, in [19], Zhu give some new inequalities of the Huygens type for circular functions, hyperbolic functions, and the reciprocals of circular and hyperbolic functions, as follows:

Theorem A. *The following inequalities*

$$(1-p)\frac{x}{\sin x}+p\frac{x}{\tan x}>1>(1-q)\frac{x}{\sin x}+q\frac{x}{\tan x} \tag{5}$$

hold for all $x\in(0,\pi/2)$ *if and only if* $p\le 1/3$ *and* $q\ge 1-2/\pi$.

Theorem B. *The following inequalities*

$$(1-p)\frac{\sin x}{x}+p\frac{\tan x}{x}>1>(1-q)\frac{\sin x}{x}+q\frac{\tan x}{x} \tag{6}$$

hold for all $x\in(0,\pi/2)$ *if and only if* $p\ge 1/3$ *and* $q\le 0$.

Theorem C. *The following inequalities*

$$(1-p)\frac{\sinh x}{x}+p\frac{\tanh x}{x}>1>(1-q)\frac{\sinh x}{x}+q\frac{\tanh x}{x} \tag{7}$$

hold for all $x\in(0,\infty)$ *if and only if* $p\le 1/3$ *and* $q\ge 1$.

Theorem D. *The following inequalities*

$$(1-p)\frac{x}{\sinh x}+p\frac{x}{\tanh x}>1>(1-q)\frac{x}{\sinh x}+q\frac{x}{\tanh x} \tag{8}$$

hold for all $x\in(0,\infty)$ *if and only if* $p\ge 1/3$ *and* $q\le 0$.

In this note, we present a generalizations of inequalities (5) and (6) to Bessel functions of the first kind. Moreover, we extend and sharpen inequalities (7) and (8) for the modified Bessel functions of the first kind.

2. Lemmas

In the proof of the main results we will need the following two lemmas. The first lemma is about the monotonicity of two power series, see [14] for more details.

Lemma 2.1. *Let* a_n *and* b_n $(n=0,1,2,\ldots)$ *be real numbers, and let the power series* $A(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $B(x)=\sum_{n=0}^{\infty}b_nx^n$ *be convergent for* $|x|<R$. *If* $b_n>0$ *for* $n=0,1,\ldots,$ *and if* (a_n/b_n) *is strictly increasing* (*or decreasing*) *for* $n=0,1,2,\ldots,$ *then the function* $A(x)/B(x)$ *is strictly increasing* (*or decreasing*) *on* $(0,R)$.

The second lemma is the so-called monotone form of l'Hospital's rule, see [2, 7, 13] for a proof.

Lemma 2.2. *Let* $f,g:[a,b]\to\mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g^{'}\neq 0$ *on* (a,b). *If* $\frac{f'}{g'}$ *is increasing* (*or decreasing*) *on* (a,b), *then the functions* $(f(x)-f(a))/(g(x)-g(a))$ *and* $(f(x)-f(b))/(g(x)-g(b))$ *are also increasing* (*or decreasing*) *on* (a,b).

3. Extensions of Huygens type inequalities to Bessel functions

In this section, our aim is to extend the inequalities (5) and (6) to Bessel functions of the first kind. For this suppose that $\nu > -1$ and consider the function $\mathcal{J}_\nu : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n\geq} \frac{\left(\frac{-1}{4}\right)^n}{(\nu+1)_n n!} x^{2n},$$

where Γ is the gamma function, $(\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)$ for each $n \geq 0$, is the well-known Pochhammer (or Appell) symbol, and J_ν defined by

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)},$$

stands for the Bessel function of the first kind of order ν. It is worth mentioning that in particular the function J_ν reduces to some elementary functions, like sine and cosine. More precisely, in particular we have:

$$\mathcal{J}_{-1/2}(x) = \sqrt{\pi/2}.x^{1/2} J_{-1/2}(x) = \cos x, \tag{9}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2}.x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \tag{10}$$

respectively, which can verified easily by using the series representation of the function J_ν and of the cosine and sine functions, respectively. Taking into account the relations (9) and (10) as we mentioned above, the inequalities (5) and (6) can be rewritten in terms of $\mathcal{J}_{-1/2}(x)$ and $\mathcal{J}_{1/2}(x)$. For example, using (9) and (10) the inequalities (5) and (6) can be rewritten as

$$(1-p)\frac{1}{\mathcal{J}_{1/2}(x)} + p\frac{\mathcal{J}_{-1/2}(x)}{\mathcal{J}_{1/2}(x)} > 1 > (1-q)\frac{1}{\mathcal{J}_{1/2}(x)} + q\frac{\mathcal{J}_{-1/2}(x)}{\mathcal{J}_{1/2}(x)} \tag{11}$$

and

$$(1-p)\mathcal{J}_{1/2}(x) + p\frac{\mathcal{J}_{1/2}(x)}{\mathcal{J}_{-1/2}(x)} > 1 > (1-q)\mathcal{J}_{1/2}(x) + q\frac{\mathcal{J}_{1/2}(x)}{\mathcal{J}_{-1/2}(x)} \tag{12}$$

and thus it is natural to ask what is the general form of the inequalities (5) and (6) for arbitrary ν.

Our first main result reads as follows.

Theorem 3.1. *Let $\nu > -1$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens types inequalities*

$$(1-p)\frac{1}{\mathcal{J}_{\nu+1}(x)} + p\frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)} > 1 > (1-q)\frac{1}{\mathcal{J}_{\nu+1}(x)} + q\frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}, \tag{13}$$

hold for all $x \in (0, j_{\nu,1})$ if and only if $p \leq \frac{\nu+1}{\nu+2}$ and $q \geq 1 - \mathcal{J}_\nu(j_{\nu,1})$.

Proof. We define the function $F_\nu(x)$ on $(0, j_{\nu,1})$ by

$$F_\nu(x) = \frac{\frac{1}{\mathcal{J}_{\nu+1}(x)} - 1}{\frac{1}{\mathcal{J}_{\nu+1}(x)} - \frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}} = \frac{1 - \mathcal{J}_{\nu+1}(x)}{1 - \mathcal{J}_\nu(x)} = \frac{h_{\nu,1}(x)}{h_{\nu,2}(x)},$$

where $f_{\nu,1}(x) = 1 - \mathcal{J}_{\nu+1}(x)$ and $f_{\nu,2}(x) = 1 - \mathcal{J}_\nu(x)$. Now, by again using the differentiation formula

$$\mathcal{J}'_\nu(x) = -\frac{x}{2(\nu+1)}\mathcal{J}_{\nu+1}(x) \tag{14}$$

and the infinite product representation [18, p. 498],

$$\mathcal{J}_\nu(x) = \prod_{n\geq 1}\left(1 - \frac{x^2}{j^2_{\nu,n}}\right) \tag{15}$$

we obtain that

$$\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)} = \frac{(\nu+1)\mathcal{J}_{\nu+2}(x)}{(\nu+2)\mathcal{J}_{\nu+1}(x)} = 4(\nu+1)\sum_{n\geq 1}\frac{1}{j^2_{\nu+1,n} - x^2}.$$

Therefore,

$$\left(\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)}\right)' = 8(\nu+1)\sum_{n\geq 1}\frac{x}{(j^2_{\nu+1,n} - x^2)^2}.$$

From this, we deduce that the function $\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)}$ is increasing on $(0, j_{\nu,1})$. Thus, the function $F_\nu(x)$ is also increasing on $(0, j_{\nu,1})$ by means of Lemma 2.1. Moreover,

$$\lim_{x\to 0^+} F_\nu(x) = \frac{\nu+1}{\nu+2} \quad \text{and} \quad \lim_{x\to j_{\nu,1}} F_\nu(x) = 1 - \mathcal{J}_{\nu+1}(j_{\nu,1}).$$

With this the proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $-1 < \nu \leq 0$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens type inequalities*

$$(1-p)\mathcal{J}_{\nu+1}(x) + p\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} > 1 > (1-q)\mathcal{J}_{\nu+1}(x) + q\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \tag{16}$$

hold for all $x \in (0, j_{\nu,1})$ if and only if $p \geq \frac{\nu+1}{\nu+2}$ and $q \leq 0$.

Proof. Let $\nu > -1$, we consider the function

$$G_\nu(x) = \frac{1 - \mathcal{J}_{\nu+1}(x)}{\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} - \mathcal{J}_{\nu+1}(x)}, \quad 0 < x < j_{\nu,1}.$$

For $0 < x < j_{\nu,1}$, let

$$g_{\nu,1}(x) = 1 - \mathcal{J}_{\nu+1}(x) \quad \text{and} \quad g_{\nu,2}(x) = \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} - \mathcal{J}_{\nu+1}(x).$$

From the differentiation formula (14), we get

$$\frac{g'_{\nu,1}(x)}{g'_{\nu,2}(x)} = \frac{1}{1+\frac{1}{\mathcal{J}_\nu(x)}\left(\frac{\nu+2}{\nu+1}.\frac{\mathcal{J}^2_{\nu+1}(x)}{\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}-1\right)} = \frac{1}{1+\frac{L_\nu(x)}{\mathcal{J}_\nu(x)}},$$

where

$$L_\nu(x) = \frac{\nu+2}{\nu+1}.\frac{\mathcal{J}^2_{\nu+1}(x)}{\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)} - 1.$$

On other hand, by using the Turán type inequality [3, Eq. 2.9]

$$\mathcal{J}^2_{\nu+1}(x) - \mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x) > 0, \tag{17}$$

where $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$, we obtain that the function $L_\nu(x)$ is positive on $(0, j_{\nu,1})$.

Elementary calculations reveal that

$$L'_\nu(x) = \frac{(\nu+2)x\mathcal{J}_{\nu+1}(x)}{(\nu+1)\mathcal{J}^2_\nu(x)\mathcal{J}_{\nu+2}(x)}\left[\frac{\mathcal{J}^2_{\nu+1}(x)}{2(\nu+1)} - \frac{\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{\nu+2}\right] + \frac{(\nu+2)x\mathcal{J}^2_{\nu+1}(x)\mathcal{J}_{\nu+2}(x)}{2(\nu+3)(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}^2_{\nu+2}(x)}. \tag{18}$$

Using the fact that $\mathcal{J}_{\nu+1}(x) \geq \mathcal{J}_\nu(x) > 0$ for all $x \in (0, j_{\nu,1})$ and the Turán type inequality (17), we get

$$L'_\nu(x) \geq \frac{-\nu x\mathcal{J}^3_{\nu+1}(x)}{2(\nu+1)^2\mathcal{J}^2_\nu(x)\mathcal{J}^2_{\nu+2}(x)} + \frac{(\nu+1)x\mathcal{J}^2_{\nu+1}(x)\mathcal{J}_{\nu+3}(x)}{2(\nu+3)(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}^2_{\nu+2}(x)}. \tag{19}$$

This implies that the function $L_\nu(x)$ is increasing on $(0, j_{\nu,1})$ for all $-1 < \nu \leq 0$. In addition, since the function $x \longmapsto \mathcal{J}_\nu(x)$ is decreasing ([3, Theorem 3]) on $(0, j_{\nu,1})$, we gave that the function $\frac{L_\nu(x)}{\mathcal{J}_\nu(x)}$ is increasing too on $(0, j_{\nu,1})$, as a product of two positives increasing functions. Then, the function $\frac{g'_{\nu,1}(x)}{g'_{\nu,2}(x)}$ is decreasing on $(0, j_{\nu,1})$. Then, the function

$$G_\nu(x) = \frac{g_{\nu,1}(x)}{g_{\nu,2}(x)} = \frac{g_{\nu,1}(x) - g_{\nu,1}(0)}{g_{\nu,2}(x) - g_{\nu,2}(0)}.$$

is decreasing on $(0, j_{\nu,1})$, from Lemma 2.1.

At the same time, we can write the function $G_\nu(x)$ in the following form

$$G_\nu(x) = \frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}.F_\nu(x).$$

Then,

$$\lim_{x\to 0^+} G_\nu(x) = F_\nu(0) = \frac{\nu+1}{\nu+2} \text{ and } \lim_{x\to j_{\nu,1}} G_\nu(x) = 0,$$

and with this the proof of inequalities (16) is done. □

Remark 3.3. Since $j_{-1/2,1} = \frac{\pi}{2}$ we find that the inequalities (13) and (16) is a generalization of inequalities (5) and (6).

4. Extensions of the Huygens type inequalities to modified Bessel functions

In this section, we present a generalization of inequalities (7) and (8). For $\nu > -1$ let us consider the function $\mathcal{I}_\nu : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x) = \sum_{n\geq} \frac{\left(\frac{1}{4}\right)^n}{(\nu+1)_n n!} x^{2n},$$

where I_ν is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)} \quad \text{for all } x \in \mathbb{R}.$$

It is worth mentioning that in particular we have

$$\mathcal{I}_{-1/2}(x) = \sqrt{\pi/2}.x^{1/2} I_{-1/2}(x) = \cosh x, \tag{20}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2}.x^{1/2} I_{-1/2}(x) = \frac{\sinh x}{x}. \tag{21}$$

Thus, the function I_ν is of special interest in this paper because inequalities (7) and (8) is actually equivalent to

$$(1-p)\,\mathcal{I}_{1/2}(x) + p\frac{\mathcal{I}_{1/2}(x)}{\mathcal{I}_{-1/2}(x)} > 1 > (1-q)\,\mathcal{I}_{1/2}(x) + q\frac{\mathcal{I}_{1/2}(x)}{\mathcal{I}_{-1/2}(x)} \tag{22}$$

for all $x \in (0, \infty)$ if and only if $p \leq \frac{-1/2+1}{-1/2+2} = 1/3$ and $q \geq 1$, and

$$(1-p)\,\frac{1}{\mathcal{I}_{-1/2+1}(x)} + p\frac{\mathcal{I}_{-1/2}(x)}{\mathcal{I}_{-1/2+1}(x)} > 1 > (1-q)\,\frac{1}{\mathcal{I}_{-1/2+1}(x)} + q\frac{\mathcal{I}_{-1/2}(x)}{\mathcal{I}_{-1/2+1}(x)} \tag{23}$$

for all $x \in (0, \infty)$ if and only if $p \geq \frac{-1/2+1}{-1/2+2} = 1/3$ and $q \leq 0$.

In view of inequalities (22) and (23) it is natural to ask: what is the analogue of this inequalities for modified Bessel functions of the first kind? In order to answer this question we prove the following results.

Theorem 4.1. *Let $\nu > -1$. Then the following inequalities*

$$(1-p)\,\mathcal{I}_{\nu+1}(x) + p\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} > 1 > (1-q)\,\mathcal{I}_{\nu+1}(x) + q\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} \tag{24}$$

hold for all $x \in (0, \infty)$ if and only if $p \leq \frac{\nu+1}{\nu+2}$ and $q \geq 1$.

Proof. Let $\nu > -1$, we define the function H_ν on $(0, \infty)$ by

$$H_\nu(x) = \frac{\mathcal{I}_{\nu+1}(x) - 1}{\mathcal{I}_{\nu+1}(x) - \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)}} = \frac{\mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_{\nu+1}(x)} = \frac{h_{\nu,1}(x)}{h_{\nu,2}(x)},$$

where $h_{\nu,1}(x) = \mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_\nu(x)$ and $h_{\nu,2}(x) = \mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_{\nu+1}(x)$. By using the differentiation formula [18, p. 79]

$$\mathcal{I}'_\nu(x) = \frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x) \tag{25}$$

can easily show that

$$h'_{\nu,1}(x) = \frac{x}{2(\nu+1)}\mathcal{I}^2_{\nu+1}(x) + \frac{x}{2(\nu+2)}\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x) - \frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x), \tag{26}$$

and

$$h'_{\nu,2}(x) = \frac{x}{2(\nu+1)}\mathcal{I}^2_{\nu+1}(x) + \frac{x}{2(\nu+2)}\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x) - \frac{x}{2(\nu+2)}\mathcal{I}_{\nu+2}(x). \tag{27}$$

Using the Cauchy product

$$I_\mu(x)I_\nu(x) = \sum_{n\geq 0} \frac{\Gamma(\nu+\mu+2n+1)x^{\nu+\mu+2n}}{2^{\mu+\nu+2n}\Gamma(n+1)\Gamma(\nu+\mu+n+1)\Gamma(\mu+n+1)\Gamma(\nu+n+1)} \tag{28}$$

we obtain

$$h'_{\nu,1}(x) = \sum_{n\geq 0} A_n(\nu)x^{2n} \tag{29}$$

and

$$h'_{\nu,2}(x) = \sum_{n\geq 0} B_n(\nu)x^{2n}, \tag{30}$$

where

$$A_n(\nu) = \frac{\Gamma(\nu+1)\left(\Gamma(\nu+2)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+3)\right)}{2^{2n+1}\Gamma(n+1)\Gamma(\nu+n+2)\Gamma(\nu+n+3)\Gamma(2\nu+n+3)} \tag{31}$$

and

$$B_n(\nu) = \frac{\Gamma(\nu+2)\left(\Gamma(\nu+1)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+2)\right)}{2^{2n+1}\Gamma(n+1)\Gamma(\nu+n+2)\Gamma(\nu+n+3)\Gamma(2\nu+n+3)}. \tag{32}$$

Now, we define the sequence $C_n = \frac{A_n}{B_n}$ for $n = 0, 1, \ldots$, thus

$$C_n(\nu) = \frac{\Gamma(\nu+1)\left(\Gamma(\nu+2)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+3)\right)}{\Gamma(\nu+2)\left(\Gamma(\nu+1)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+2)\right)}.$$

So, for $\nu > -1$ and $n = 0, 1, \ldots$, we get

$$\begin{aligned} \frac{C_{n+1}(\nu)}{C_n(\nu)} &= \frac{[\Gamma(\nu+2)\Gamma(2\nu+2n+6) - \Gamma(2\nu+n+4)\Gamma(\nu+n+4)]}{[\Gamma(\nu+1)\Gamma(2\nu+2n+6) - \Gamma(2\nu+n+4)\Gamma(\nu+n+3)]} \\ &\quad \times \frac{[\Gamma(\nu+1)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+2)]}{[\Gamma(\nu+2)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+3)]} \\ &= \frac{K_n^1(\nu)}{K_n^2(\nu)}, \end{aligned} \tag{33}$$

where

$$\begin{aligned}
K_n^1(\nu) = &\ [\Gamma(\nu+2)\Gamma(2\nu+2n+6)-\Gamma(2\nu+n+4)\Gamma(\nu+n+4)]\\
&\times[\Gamma(\nu+1)\Gamma(2\nu+2n+4)-\Gamma(2\nu+n+3)\Gamma(\nu+n+2)]\\
=&\ \underbrace{\Gamma(\nu+1)\Gamma(\nu+2)\Gamma(2\nu+2n+6)\Gamma(2\nu+2n+4)}_{A_1}\\
&-\underbrace{\Gamma(\nu+2)\Gamma(2\nu+2n+6)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)}_{B_1}\\
&-\underbrace{\Gamma(\nu+1)\Gamma(2\nu+n+4)\Gamma(\nu+n+4)\Gamma(2\nu+2n+4)}_{C_1}\\
&+\underbrace{\Gamma(2\nu+n+4)\Gamma(\nu+n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)}_{D_1}
\end{aligned}$$

and

$$\begin{aligned}
K_n^2(\nu) = &\ [\Gamma(\nu+1)\Gamma(2\nu+2n+6)-\Gamma(2\nu+n+4)\Gamma(\nu+n+3)]\\
&\times[\Gamma(\nu+2)\Gamma(2\nu+2n+4)-\Gamma(2\nu+n+3)\Gamma(\nu+n+3)]\\
=&\ \underbrace{\Gamma(\nu+1)\Gamma(\nu+2)\Gamma(2\nu+2n+6)\Gamma(2\nu+2n+4)}_{A_1}\\
&-\underbrace{\Gamma(\nu+1)\Gamma(2\nu+2n+6)\Gamma(2\nu+n+3)\Gamma(\nu+n+3)}_{B_2}\\
&-\underbrace{\Gamma(\nu+2)\Gamma(2\nu+n+4)\Gamma(\nu+n+3)\Gamma(2\nu+2n+4)}_{C_2}\\
&+\underbrace{\Gamma(2\nu+n+4)\Gamma^2(\nu+n+3)\Gamma(2\nu+n+3)}_{D_2}.
\end{aligned}$$

Thus

$$K_n^1(\nu)-K_n^2(\nu)=(B_2-B_1)+(C_2-C_1)+(D_1-D_2).$$

A simple calculation we obtain

$$B_2-B_1=(n+1)\Gamma(\nu+1)\Gamma(\nu+n+2)\Gamma(2\nu+n+3)\Gamma(2\nu+2n+6)$$

and

$$C_2-C_1=-(n+2)\Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+4)\Gamma(\nu+n+3)$$

and

$$D_1-D_2=\Gamma(2\nu+n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+3)\Gamma(\nu+n+2)\geq 0.$$

Then simple computations lead to

$$\begin{aligned}
&B_2-B_1+(C_2-C_1)\\
&=\Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)\\
&\quad\times[(n+1)(2\nu+2n+5)(2\nu+2n+4)-(n+2)(2\nu+2n+3)(\nu+n+2)]
\end{aligned}$$

$\geq P_n(\nu)(n+1)\Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)$,

where

$$P_n(\nu) = 2\nu^2 + (4n+11)\nu + 2n^2 + 11n + 14$$
$$= (\nu+n+2)(2\nu+2n+7) > 0$$

for all $\nu > -1$ and $n \in \mathbb{N}$. Therefore, the sequence $(C_n)_n$ is increasing, we obtain that the function $\frac{h'_{\nu,1}(x)}{h'_{\nu,2}(x)}$ is increasing on $(0, \infty)$ too (by Lemma 2.2). Thus $H_\nu(x) = \frac{h_{\nu,1}(x)-h_{\nu,1}(0^+)}{h_{\nu,2}(x)-h_{\nu,2}(0^+)}$ is also increasing on $(0,\infty)$ by Lemma 2.1. So,

$$\lim_{x\to 0^+} H_\nu(x) = C_0(\nu) = \frac{\nu+1}{\nu+2},$$

and using the asymptotic formula [1, p. 377]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{4\nu^2-1}{1!(8x)} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} - \cdots\right]$$

which holds for large values of x and for fixed $\nu > -1$, we obtain

$$\lim_{x\to\infty} H_\nu(x) = 1.$$

So, the proof of Theorem 4.1 is complete. □

Theorem 4.2. *Let $\nu > -1$. Then the following inequalities*

$$(1-p)\frac{1}{\mathcal{I}_{\nu+1}(x)} + p\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} > 1 > (1-q)\frac{1}{\mathcal{I}_{\nu+1}(x)} + q\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} \tag{34}$$

hold for all $x \in (0,\infty)$ if and only if $p \geq \frac{\nu+1}{\nu+2}$ and $q \leq 0$.

Proof. Let $\nu > -1$ and $x \in (0,\infty)$, we define the function $\Phi_\nu(x)$ by

$$\Phi_\nu(x) = \frac{\frac{1}{\mathcal{I}_{\nu+1}(x)} - 1}{\frac{1}{\mathcal{I}_{\nu+1}(x)} - \frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}} = \frac{1-\mathcal{I}_{\nu+1}(x)}{1-\mathcal{I}_\nu(x)} = \frac{\varphi_{\nu,1}(x)}{\varphi_{\nu,2}(x)}, \tag{35}$$

where $\varphi_{\nu,1}(x) = 1 - \mathcal{I}_{\nu+1}(x)$ and $\varphi_{\nu,2}(x) = 1 - \mathcal{I}_\nu(x)$. By again using the differentiation formula (25) we get

$$\frac{\varphi'_{\nu,1}(x)}{\varphi'_{\nu,2}(x)} = \frac{\nu+1}{\nu+2}.\frac{\mathcal{I}_{\nu+2}(x)}{\mathcal{I}_{\nu+1}(x)} = \frac{\sum_{n=0}^\infty a_n(\nu)x^{2n}}{\sum_{n=0}^\infty b_n(\nu)x^{2n}}, \tag{36}$$

where $a_n(\nu) = \frac{\Gamma(\nu+2)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+3)}$ and $b_n(\nu) = \frac{\Gamma(\nu+1)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+2)}$. Let

$$c_n(\nu) = \frac{a_n(\nu)}{b_n(\nu)} = \frac{\nu+1}{\nu+n+2} \quad \text{for } n = 0, 1, \ldots.$$

We conclude that $c_n(\nu)$ is decreasing for $n = 0, 1, \ldots$ and $\frac{g'_1(x)}{g'_2(x)}$ is decreasing on $(0,\infty)$ by Lemma 2.1. Thus

$$\Phi_\nu(x) = \frac{\varphi_{\nu,1}(x)}{\varphi_{\nu,2}(x)} = \frac{\varphi_{\nu,1}(x) - \varphi_{\nu,1}(0)}{\varphi_{\nu,2}(x) - \varphi_{\nu,2}(0)}$$

is decreasing on $(0,\infty)$ by Lemma 2.2. Furthermore,

$$\lim_{x\to 0^+}\Phi_\nu(x)=c_0(\nu)=\frac{\nu+1}{\nu+2},$$

and

$$\lim_{x\to\infty}\Phi_\nu(x)=0.$$

Alternatively, inequality (34) can be proved by using the Mittag-Leffler expansion for the modified Bessel functions of first kind, which becomes [5, Eq. 7.9.3]

$$\frac{I_{\nu+1}(x)}{I_\nu(x)}=\sum_{n=1}^{\infty}\frac{2x}{j_{\nu,n}^2+x^2}, \tag{37}$$

where $0<j_{\nu,1}<j_{\nu,2}<\cdots<j_{\nu,n}<\cdots$, are the positive zeros of the Bessel function J_ν, we obtain that

$$\frac{g_1'(x)}{g_2'(x)}=2(\nu+1)\frac{I_{\nu+2}(x)}{xI_{\nu+1}(x)}=4(\nu+1)\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^2+x^2}.$$

Clearly,

$$\left(\frac{g_1'(x)}{g_2'(x)}\right)'=-8(\nu+1)\sum_{n=1}^{\infty}\frac{x}{(x^2+j_{\nu,n}^2)^2}$$

for all $x>0$ and $\nu>-1$, which implies that $G_\nu(x)$ is decreasing for all $\nu>-1$. On the other hand, using the Rayleigh formula [18, p. 502]

$$\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^2}=\frac{1}{4(\nu+1)}, \tag{38}$$

we get

$$\lim_{x\to 0^+}G_\nu(x)=\frac{\nu+1}{\nu+2}.$$

So, the proof of Theorem 4.2 is complete. □

Remark 4.3. Since $\mathcal{I}_{-\frac{1}{2}}(x)=\cosh x$ and $\mathcal{I}_{\frac{1}{2}}(x)=\frac{\sinh x}{x}$, we find that the inequalities (24) and (34) is the generalization of inequalities (7) and (8).

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