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UPPER BOUND OF SECOND HANKEL DETERMINANT FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we introduce and investigate a subclass $\Im_{\Sigma}(\alpha,\beta,\gamma)$ of analytic and bi-univalent functions of complex order in the open unit disk U in complex plane. Here, we obtain an upper bound for the second Hankel determinant of the functions belonging to this class. Moreover, several interesting conclusions of the results obtained here are also discussed.

1. Introduction and definitions

In this section, first of all let us give the necessary information.

We will denote by $\mathcal{A}$ the class of the functions $f:\mathbb{C}\to\mathbb{C}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$. Also, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ that are univalent in U.

Some of the important subclasses of S include the class $\Re(\alpha,\beta)$ that is defined as

$$\Re(\alpha,\beta)=\left\{f\in S:\operatorname{Re}\left[f'(z)+\beta zf''(z)\right]>\alpha,z\in U\right\},\ \alpha\in[0,1),\beta\geq 0.$$

Gao and Zhou [14] investigated the class $\Re(\alpha,\beta)$ and showed some mapping properties of this subclass. Yang and Liu [34], proved that $\Re(\alpha,\beta)\subset S$ if $2(1-\alpha)\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{\beta n+1}\leq 1$.

In the special case, we have subclass $\Re(\beta)$;

$$\Re(\beta)=\left\{f\in S:\operatorname{Re}\left[f'(z)+\beta zf''(z)\right]>0,z\in U\right\},\ \beta\geq 0$$

for $\alpha=0$.

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Early, Altintaş *et al.* [1] investigated a subclass $\Re(\alpha, \beta, \gamma)$ consisting of functions $f \in T$, which satisfy the conditions

$$\left| \frac{1}{\gamma} \left[f'(z) + \beta z f''(z) - 1 \right] \right| \leq \alpha, \ z \in U, \ \beta \geq 0, \ \alpha \in (0, 1], \ \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}.$$

Here T is the class of the functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk U. They were found a necessary and sufficient condition for the functions belonging to this class.

It is well-known that (see, for example [10]) every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \ z \in U \text{ and}$$
$$f\left(f^{-1}(w)\right) = w, \ w \in D = \{w : |w| < r_0(f)\}, \ r_0(f) \geq \frac{1}{4},$$

where $f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots$.

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent. Let Σ denote the class of bi-univalent functions in U given (1.1).

In 1967, Lewin [22] showed that for every function $f \in \Sigma$ of the form (1.1) the second coefficient satisfies the estimate $|a_2| < 1.51$. In 1967, Brannan and Clunie [2] conjectured that $|a_2| < \sqrt{2}$ for each $f \in \Sigma$. In 1984, Tan [31] obtained the bound for $|a_2|$, namely, that $|a_2| < 1.485$, which is the best known estimate for the function class Σ. In 1985, Kedzierawski [18] proved the Brannan-Clunie conjecture for bi-starlike functions. Brannan and Taha [3] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in the classes of bi-starlike functions of order α and bi-convex functions of order α.

The study of bi-univalent functions was revived, in recently years, by Srivastava *et al.* [30] and a considerably large number of sequels to the work of Srivastava *et al.* [30] have appeared in the literature. In particular, several results on coefficient estimates for the initial coefficients $|a_2|$, $|a_3|$ and $|a_4|$ were proved for various subclasses of Σ (see, for example, [7,13,16,25,28,29,32,33]).

Recently, Deniz [8] and Kumar *et al.* [20] both extended and improved the results of Brannan and Taha [3] by generalizing their classes by means of the principle of subordination between analytic functions.

Despite the numerous studies mentioned above, the problem of estimating the coefficients $|a_n|$, $n = 2, 3, \ldots$ for the general class functions Σ is still open (see, also [29] in this connection).

One of the important tools in the theory of univalent functions is Hankel determinant which are utilized, for example, in showing that a function of bounded characteristic in U; that is, a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [4]. The Hankel determinants $H_q(n)$, $n, q \in \mathbb{N}$ of the function f are defined by (see [23])

$$H_q(n) = \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \quad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we know that the functional $H_2(1) = a_3 - a_2^2$ is known as the Fekete-Szeg's functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$ where, μ is some real number (see [11]). Estimating for the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szeg's problem. In 1969, Keogh and Merkes [19] solved the Fekete-Szeg's problem for the classes starlike and convex functions. Someone can see the Fekete-Szeg's problem for the classes of starlike functions of order β and convex functions of order β at special cases in the paper of Orhan *et al.* [24]. On the other hand, recently, Çağlar and Aslan (see [5]) have obtained Fekete-Szeg's inequality for a subclass of bi-univalent functions. Also, Zaprawa (see [35, 36]) have studied on Fekete-Szeg's problem for some subclasses of bi-univalent functions. In special cases, he gave the Fekete-Szeg's problem for the subclasses bi-starlike functions of order β and bi-convex functions of order β.

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2 a_4 - a_3^2$. The bounds for the second Hankel determinant $H_2(2)$ obtained for the classes starlike and convex functions in [17]. Lee *et al.* [21] obtained the sharp bound for $|H_2(2)|$ by generalizing their classes by means of the principle of subordination between analytic functions. In their paper [21], one can find the sharp bound of $|H_2(2)|$ for the functions in the classes of starlike functions of order β and convex functions of order β. Recently, Çağlar *et al.* [6], Deniz *et al.* [9] and Orhan *et al.* [26] found the upper bound of the functional $|H_2(2)|$ for a subclasses of bi-univalent functions.

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

Definition 1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\Im_\Sigma(\alpha, \ \beta, \gamma)$, $\alpha \in [0, 1)$, $\beta \geq 0, \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$ if the following conditions are satisfied

$$\Re\left\{1 + \frac{1}{\gamma}\left[f'(z) + \beta z f''(z) - 1\right]\right\} > \alpha, \ z \in \mathcal{U}$$

and

$$\Re\left\{1 + \frac{1}{\gamma}\left[g'(w) + \beta w g''(w) - 1\right]\right\} > \alpha, \ w \in D,$$

where the function g is given by $g(w) = f^{-1}(w)$.

For $\beta = 0$ and $\gamma = 1$, we have $\Im_\Sigma(\alpha, 0, 1) = \aleph_\Sigma(\alpha)$. The class $\aleph_\Sigma(\alpha)$ investigated by Srivastava *et al.* [28]. Also, Çağlar *et al.* [6] found the upper bound of $|H_2(2)|$ for the functions belonging to the class $\aleph_\Sigma(\alpha)$.

Recently, Frasin [12] investigated subclass $\Im_\Sigma(\alpha, \beta, 1) = H_\Sigma(\alpha, \beta)$, $\alpha \in [0, 1)$, $\beta > 0$ with condition $2(1-\alpha)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\beta n+1} \leq 1$. He found estimates on two first coefficients for the functions in this class.

The object of the present paper is to find the upper bound of the functional $|H_2(2)|$ for the functions f belonging to the class $\Im_\Sigma(\alpha, \beta, \gamma)$.

To prove our main results, we need require the following lemmas.

Lemma 1 (see, for example, [27]). *If $p \in P$, then $|p_n| \leqq 2$, $n = 1, 2, 3, \ldots$, where P is the family of all functions p, analytic in U, for which $\Re(p(z)) > 0$, $z \in U$ and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \ z \in U. \tag{1.2}$$

Lemma 2 (see, for example, [15]). *If the function $p \in P$ is given by the series* (1.2)*, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x, \tag{1.3}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)z \tag{1.4}$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

2. Main results

In this section, we prove the following theorem on upper bound of the second Hankel determinant of the function class $\Im_\Sigma(\alpha, \beta, \gamma)$.

Theorem 1. *Let the function $f(z)$ given by* (1.1) *be in the class $\Im_\Sigma(\alpha, \beta, \gamma)$, $\alpha \in [0, 1)$, $\gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$, $\beta \in [0, 1]$. Then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \max\{G(2^-), G(t_0)\}, & \text{if } \tau \in (0, \tau_0), \\ G(2^-), & \text{if } \tau \in [\tau_0, +\infty), \end{cases} \tag{2.1}$$

where

$$\tau_0 = \tau_0(\alpha, \beta) = \frac{(1+\beta)\left[(1+\beta)(1+3\beta) + \sqrt{d}\right]}{6(1-\alpha)(1+2\beta)(1+3\beta)},$$

$$d = (1+\beta)(1+3\beta)\left[36(1+2\beta)^2 - 15(1+\beta)(1+3\beta)\right],$$

$$G(2^-) = \frac{(1-\alpha)^2\tau^2}{2(1+\beta)^4(1+3\beta)}\left[2(1-\alpha)^2(1+3\beta)\tau^2 + (1+\beta)^3\right],$$

$$G(t_0) = \frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2} - \frac{(1-\alpha)^2\tau^2 b(\alpha,\beta,\tau)}{144(1+2\beta)^2(1+3\beta)a(\alpha,\beta,\tau)},$$

$$t_0 = (1+\beta)\sqrt{\frac{b(\alpha,\beta,\tau)}{-a(\alpha,\beta,\tau)}},$$

$$\begin{aligned} a(\alpha,\beta,\tau) &= 9(1-\alpha)^2(1+2\beta)^2(1+3\beta)\tau^2 \\ &\quad - 3(1-\alpha)(1+\beta)^2(1+2\beta)(1+3\beta)\tau \\ &\quad + (1+\beta)^3\left[4(1+\beta)(1+3\beta) - 9(1+2\beta)^2\right], \\ b(\alpha,\beta,\tau) &= 6(1-\alpha)(1+2\beta)(1+3\beta)\tau \\ &\quad + (1+\beta)\left[27(1+2\beta)^2 - 16(1+\beta)(1+3\beta)\right], \\ \tau &= |\gamma|. \end{aligned}$$

Proof. Let $f \in \Im_\Sigma(\alpha,\beta,\gamma)$, $\alpha \in [0,1)$, $\gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$, $\beta \in [0,1]$ and $g = f^{-1}$. Then,

$$1 + \frac{1}{\gamma}\left[f'(z) + \beta z f''(z) - 1\right] = \alpha + (1-\alpha)p(z) \tag{2.2}$$

and

$$1 + \frac{1}{\gamma}\left[g'(w) + \beta w g''(w) - 1\right] = \alpha + (1-\alpha)q(w), \tag{2.3}$$

where functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ are in the class P.

Comparing the coefficients in (2.2) and (2.3), we have

$$a_2 = \frac{\gamma(1-\alpha)}{2(1+\beta)}p_1, \tag{2.4}$$

$$a_3 = \frac{\gamma(1-\alpha)}{3(1+2\beta)}p_2, \tag{2.5}$$

$$a_4 = \frac{\gamma(1-\alpha)}{4(1+3\beta)}p_3, \tag{2.6}$$

and

$$-a_2 = \frac{\gamma(1-\alpha)}{2(1+\beta)}q_1, \tag{2.7}$$

$$2a_2^2 - a_3 = \frac{\gamma(1-\alpha)}{3(1+2\beta)}q_2, \tag{2.8}$$

$$-5a_2^3 + 5a_2a_3 - a_4 = \frac{\gamma(1-\alpha)}{4(1+3\beta)}q_3. \tag{2.9}$$

From (2.4) and (2.7), we find that

$$p_1 = -q_1 \tag{2.10}$$

and

$$\frac{\gamma(1-\alpha)}{2(1+\beta)}p_1 = a_2 = -\frac{\gamma(1-\alpha)}{2(1+\beta)}q_1. \tag{2.11}$$

Now, from (2.5), (2.8) and (2.11), we get

$$a_3 = \frac{\gamma^2(1-\alpha)^2}{4(1+\beta)^2}p_1^2 + \frac{\gamma(1-\alpha)}{6(1+2\beta)}(p_2-q_2). \tag{2.12}$$

Also, from (2.6), (2.9), (2.11) and (2.12), we find that

$$a_4 = \frac{5\gamma^2(1-\alpha)^2}{24(1+\beta)(1+2\beta)}p_1(p_2-q_2) + \frac{\gamma(1-\alpha)}{8(1+3\gamma)}(p_3-q_3). \tag{2.13}$$

Thus, from (2.11), (2.12) and (2.13), we can easily establish that

$$\begin{aligned} a_2a_4 - a_3^2 = & -\frac{\gamma^4(1-\alpha)^4}{16(1+\beta)^4}p_1^4 + \frac{\gamma^3(1-\alpha)^3}{48(1+2\beta)(1+\beta)^2}p_1^2(p_2-q_2) \\ & + \frac{\gamma^2(1-\alpha)^2}{16(1+3\beta)(1+\beta)}p_1(p_3-q_3) - \frac{\gamma^2(1-\alpha)^2}{36(1+2\beta)^2}(p_2-q_2)^2. \end{aligned} \tag{2.14}$$

In view of Lemma 2, since (see (2.10)) $p_1 = -q_1$, we write

$$\left.\begin{aligned} 2p_2 &= p_1^2 + (4-p_1^2)x, \\ 2q_2 &= q_1^2 + (4-q_1^2)y \end{aligned}\right\} \Longrightarrow p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{2.15}$$

and

$$\left.\begin{aligned} 4p_3 &= p_1^3 + 2(4-p_1^2)p_1x - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)z, \\ 4q_3 &= q_1^3 + 2(4-q_1^2)q_1y - (4-q_1^2)q_1y^2 + 2(4-q_1^2)(1-|y|^2)w \end{aligned}\right\} \Longrightarrow$$

$$\begin{aligned} p_3 - q_3 = & \frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(x+y) - \frac{(4-p_1^2)p_1}{4}(x^2+y^2) \\ & + \frac{4-p_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right] \end{aligned} \tag{2.16}$$

for some x, y, z, w with $|x| \le 1, |y| \le 1, |z| \le 1, |w| \le 1$.

According to Lemma 1, we may assume without any restriction that $t \in [0,2]$ where $t = |p_1|$.

Thus, substituting the expressions (2.15) and (2.16) in (2.14), using the triangle inequality, and letting $|\gamma| = \tau, |x| = \xi, |y| = \eta$, we obtain

$$\left|a_2a_4 - a_3^2\right| \le C_1(t)(\xi+\eta)^2 + C_2(t)(\xi^2+\eta^2) + C_3(t)(\xi+\eta) + C_4(t), \ t \in [0,2],$$

where

$$\begin{aligned} C_1(t) &= \frac{(1-\alpha)^2\tau^2(4-t^2)^2}{144(1+2\beta)^2} \ge 0, \\ C_2(t) &= \frac{(1-\alpha)^2\tau^2(4-t^2)(t-2)t}{64(1+\beta)(1+3\beta)} \le 0, \\ C_3(t) &= \frac{(1-\alpha)^2\tau^2t^2(4-t^2)}{96(1+\beta)^2(1+2\beta)(1+3\beta)} \\ & \quad \times [(1-\alpha)(1+3\beta)\tau + 3(1+\beta)(1+2\beta)] \ge 0, \\ C_4(t) &= \frac{(1-\alpha)^2\tau^2t}{32(1+\beta)^4(1+3\beta)}\left\{\left[2(1-\alpha)^2(1+3\beta)\tau^2 + (1+\beta)^3\right]t^3\right. \end{aligned}$$

$$+2(4-t^2)(1+\beta)^3\big\} \geq 0.$$

Let us define the function $F : \Omega \to \mathbb{R}$ as follows

$$F(\xi,\eta) = C_1(t)(\xi+\eta)^2+C_2(t)(\xi^2+\eta^2)+C_3(t)(\xi+\eta)+C_4(t), (\xi,\eta) \in \Omega, \tag{2.17}$$

where $\Omega = \big\{(\xi,\eta) \in \mathbb{R}^2 : \xi,\eta \in [0,1]\big\}$ for fixed $t \in [0,2]$.

Now, we need to maximize the function $F(\xi,\eta)$ in the closed square Ω. Since the coefficients of the function $F(\xi,\eta)$ is dependent to the variable t, we must investigate the maximum of $F(\xi,\eta)$ respect to t taking into account these cases $t = 0$, $t \in (0,2)$ and $t = 2$.

1. For $t = 0$, we write

$$F(\xi,\eta) = \frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}(\xi+\eta)^2,\ (\xi,\eta) \in \Omega.$$

From the equations

$$F'_\xi(\xi,\eta) = \frac{2(1-\alpha)^2\tau^2}{9(1+2\beta)^2}(\xi+\eta) = 0,\ F'_\eta(\xi,\eta) = \frac{2(1-\alpha)^2\tau^2}{9(1+2\beta)^2}(\xi+\eta) = 0,\ (\xi,\eta) \in \Omega,$$

we can easily see that the function $F(\xi,\eta)$ cannot have a critical point in the interior of the square Ω; that is, $F(\xi,\eta)$ cannot have a local maximum in the interior of the square Ω.

Now, we need to maximize the function $F(\xi,\eta)$ on the boundary of the square Ω.

For $\xi = 0, \eta \in [0,1]$, we write

$$F(0,\eta) = \frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}\eta^2 := \varphi_1(\eta), \eta \in [0,1].$$

Since $\varphi_1'(\eta) = \frac{2\tau^2(1-\alpha)^2}{9(1+2\beta)^2}\eta \geq 0$ for each $\eta \in [0,1]$, the function $\varphi_1(\eta)$ is an increasing function on $[0,1]$. Hence,

$$\max\{F(0,\eta) : \eta \in [0,1]\} = \frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}. \tag{2.18}$$

Now, let us $\eta = 0,\ \xi \in [0,1]$. In this case, similarly to the previous case, we can easily write

$$\max\{F(\xi,0) : \xi \in [0,1]\} = \frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}.$$

For $\xi = 1,\ \eta \in [0,1]$, we have

$$F(1,\eta) = \frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}(1+\eta)^2,\ \eta \in [0,1].$$

Similarly to the previous cases, we have

$$\max\{F(1,\eta) : \eta \in [0,1]\} = \frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2}. \tag{2.19}$$

Now, let us $\eta = 1,\ \xi \in [0,1]$. In this case, similarly to the previous case, we write

$$\max\{F(\xi,1) : \xi \in [0,1]\} = \frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2}.$$

Thus, in the case $t = 0$

$$\max\{F(\xi,\eta) : (\xi,\eta) \in \Omega\} = \max\left\{\frac{(1-\alpha)^2\tau^2}{9(1+2\beta)^2}, \frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2}\right\} = \frac{4\tau^2(1-\alpha)^2}{9(1+2\beta)^2}. \tag{2.20}$$

2. For $t \in (0,2)$, we will examine the maximum of the function $F(\xi,\eta)$ taking into account the sign of $\Delta(\xi,\eta) = F''_{\xi\xi}(\xi,\eta)F''_{\eta\eta}(\xi,\eta) - \left[F''_{\xi\eta}(\xi,\eta)\right]^2$.

By simple computation, we can easily see that

$$\Delta(\xi,\eta) = 4C_2(t)\left[2C_1(t) + C_2(t)\right].$$

Since $C_2(t) \le 0$, and

$$2C_1(t) + C_2(t) = \frac{(1-\alpha)^2(4-t^2)(2-t)\tau^2}{576(1+\beta)(1+3\beta)(1+2\beta)^2}\Phi(t),$$

where $\Phi(t) = A(\beta)t + B(\beta) > 0$ for all $t \in (0,2)$ and $\beta \in [0,1]$, and $A(\beta) = -12\beta^2 - 4\beta - 1$, $B(\beta) = 48\beta^2 + 64\beta + 16$, we conclude that $\Delta(\xi,\eta) < 0$ for all $(\xi,\eta) \in \Omega$.

Thus, the function $F(\xi,\eta)$ cannot have a critical point in Ω. Consequently, the function $F(\xi,\eta)$ cannot have a local maximum in Ω.

Therefore, we must investigate the maximum of the function $F(\xi,\eta)$ on the boundary of the closed square Ω.

For $\xi = 0, \eta \in [0,1]$ (the case $\eta = 0, \xi \in [0,1]$ is examined in a similar manner), we have

$$F(0,\eta) = \left[C_1(t) + C_2(t)\right]\eta^2 + C_3(t)\eta + C_4(t) := \varphi_2(\eta),\ \eta \in [0,1].$$

Now, we need to maximize of the function $\varphi_2(\eta)$ on the closed interval $[0,1]$.

By simple computation, we have

$$\varphi_2'(\eta) = 2\left[C_1(t) + C_2(t)\right]\eta + C_3(t).$$

Now, we will examine the sign of the function $\varphi_2'(\eta)$ depending on the different cases of the sign $C_1(t) + C_2(t)$ as follows.

(i) Let $C_1(t) + C_2(t) \ge 0$, then since $C_3(t) > 0$, $\varphi_2'(\eta) > 0$; that is, $\varphi_2(\eta)$ is an increasing function on the closed interval $[0,1]$.

(ii) Let $C_1(t) + C_2(t) < 0$. In this case, it is clear that

$$2\left[C_1(t) + C_2(t)\right]\eta + C_3(t) \ge 2\left[C_1(t) + C_2(t)\right] + C_3(t)$$

for all $t \in (0,2)$ and $\eta \in [0,1]$. Also, we can easily show that $2\left[C_1(t) + C_2(t)\right] + C_3(t) > 0$ for all $t \in (0,2)$. Thus, we conclude that $\varphi_2'(\eta) > 0$; that is, $\varphi_2(\eta)$ is an increasing function on the closed interval $[0,1]$.

Consequently,

$$\max\{\varphi_2(\eta) : \eta \in [0,1]\} = \varphi_2(1) = \sum_{n=1}^{4} C_n(t);$$

that is,

$$\max\{F(0,\eta) : \eta \in [0,1]\} = \sum_{n=1}^{4} C_n(t). \tag{2.21}$$

For $\xi = 1,\ \eta \in [0,1]$ (the case $\eta = 1, \xi \in [0,1]$ is examined in a similar manner), we have

$$F(1,\eta) = [C_1(t) + C_2(t)]\,\eta^2 + [2C_1(t) + C_3(t)]\,\eta + \sum_{n=1}^{4} C_n(t) := \varphi_3(\eta),\ \eta \in [0,1]\,.$$

Similarly to the previous case, we can easily show that $\varphi_3(\eta)$ is an increasing function on the closed interval $[0,1]$.

Therefore,

$$\max\{F(1,\eta) : \eta \in [0,1]\} = 4C_1(t) + 2\,[C_2(t) + C_3(t)] + C_4(t). \tag{2.22}$$

On the other hand, the following inequality holds

$$\sum_{n=1}^{4} C_n(t) < 4C_1(t) + 2\,[C_2(t) + C_3(t)] + C_4(t)$$

for all $t \in (0,2)$.

Consequently, in the case $t \in (0,2)$

$$\max\{F(\xi,\eta) : (\xi,\eta) \in \Omega\} = 4C_1(t) + 2\,[C_2(t) + C_3(t)] + C_4(t).$$

Let us define the function $G : (0,2) \to \mathbb{R}$ as follows:

$$G(t) = 4C_1(t) + 2\,[C_2(t) + C_3(t)] + C_4(t),\ t \in (0,2)\,. \tag{2.23}$$

Substituting the values of $C_n(t),\ n = 1,2,3,4$ in (2.23), we write

$$\begin{aligned} G(t) = {} & \frac{(1-\alpha)^2\tau^2 t^2}{144(1+\beta)^4(1+2\beta)^2(1+3\beta)}\left[a(\alpha,\beta,\tau)t^2 + 2(1+\beta)^2 b(\alpha,\beta,\tau)\right] \\ & + \frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2}, \end{aligned}$$

where

$$\begin{aligned} a(\alpha,\beta,\tau) = {} & 9(1-\alpha)^2(1+2\beta)^2(1+3\beta)\tau^2 \\ & - 3(1-\alpha)(1+\beta)^2(1+2\beta)(1+3\beta)\tau \\ & + (1+\beta)^3\left[4(1+\beta)(1+3\beta) - 9(1+2\beta)^2\right], \\ b(\alpha,\beta,\tau) = {} & 6(1-\alpha)(1+2\beta)(1+3\beta)\tau \\ & + (1+\beta)\left[27(1+2\beta)^2 - 16(1+\beta)(1+3\beta)\right]. \end{aligned}$$

Now, we must investigate maximum of the function $G(t)$ in the open interval $(0,2)$.

By simple computation, we easily show that

$$G'(t)=\frac{(1-\alpha)^2\tau^2 t}{36(1+\beta)^4(1+2\beta)^2(1+3\beta)}\left[a(\alpha,\beta,\tau)t^2+(1+\beta)^2 b(\alpha,\beta,\tau)\right].$$

We will examine the sign of the function $G'(t)$ depending on the different cases of the signs of $a(\alpha,\beta,\tau)$ and $b(\alpha,\beta,\tau)$ as follows.

It is easily to see that $b(\alpha,\beta,\tau)>0$ for all $\tau>0,\ \beta\in[0,1]$ and $\alpha\in[0,1)$.

(j) We can easily show that $a(\alpha,\beta,\tau)>0$ if $\tau\geq\tau_0$, where

$$\tau_0=\tau_0(\alpha,\beta)=\frac{(1+\beta)\left[(1+\beta)(1+3\beta)+\sqrt{d}\right]}{6(1-\alpha)(1+2\beta)(1+3\beta)},$$
$$d=(1+\beta)(1+3\beta)\left[36(1+2\beta)^2-15(1+\beta)(1+3\beta)\right].$$

Thus, $G'(t)>0$ if $\tau\geq\tau_0$; that is, the function $G(t)$ is an increasing function. Therefore,

$$\begin{aligned}\max\{G(t):t\in(0,2)\}&=G(2^-)=C_4(2)\\ &=\frac{\tau^2(1-\alpha)^2}{2(1+\beta)^4(1+3\beta)}\left[2\tau^2(1-\alpha)^2(1+3\beta)+(1+\beta)^3\right]\end{aligned}\tag{2.24}$$

for $\tau\geq\tau_0$.

(jj) It is clear that $a(\alpha,\beta,\tau)<0$ if $\tau<\tau_0$. In this case, $t_0=(1+\beta)\sqrt{\frac{b(\alpha,\beta,\tau)}{-a(\alpha,\beta,\tau)}}$ is a critical point of the function $G(t)$. We can easily show that $t_0\in(0,2)$. Since $G''(t_0)=\frac{-(1-\alpha)^2\tau^2 b(\alpha,\beta,\tau)}{18(1+\beta)^2(1+2\beta)^2(1+3\beta)}<0$, t_0 is a local maximum point of the function $G(t)$.

Therefore,

$$\begin{aligned}\max\{G(t):t\in(0,2)\}&=G(t_0)\\ &=\frac{4(1-\alpha)^2\tau^2}{9(1+2\beta)^2}-\frac{(1-\alpha)^2\tau^2 b(\alpha,\beta,\tau)}{144(1+2\beta)^2(1+3\beta)a(\alpha,\beta,\tau)}.\end{aligned}\tag{2.25}$$

Since $G(0^+)=\frac{4\tau^2(1-\alpha)^2}{9(1+2\beta)^2}<\frac{\tau^2(1-\alpha)^2}{2(1+\beta)(1+3\beta)}$ and $\frac{4\tau^2(1-\alpha)^2}{9(1+2\beta)^2}<G(t_0)$ for all $\tau>0$, $\beta\in[0,1]$, we can easily see that

$$\max\left\{G(0^+),G(t_0),G(2^-)\right\}=\max\left\{G(2^-),G(t_0)\right\}.\tag{2.26}$$

3. Finally, let us $t=2$. In this case, the function $F(\xi,\eta)$ is a constant as follows

$$F(\xi,\eta)=C_4(2)=G(2^-).\tag{2.27}$$

Thus, from (2.20), (2.26), (2.27) and (2.17), the proof of Theorem 1 is completed. □

From Theorem 1, for the special values of the parameters, we can readily deduce the following results.

Corollary 1. *Let the function $f(z)$ given by (1.1) be in the class $\Im_\Sigma(\alpha,\beta,1)$, $\alpha\in[0,1)$, $\beta\in[0,1]$. Then,*

$$\left|a_2a_4-a_3^2\right|\le\begin{cases} G(2^-), & \text{if } \tau_0\in(0,1]\,,\\ \max\left\{G(2^-),G(t_0)\right\}, & \text{if } \tau_0\in(1,+\infty)\,,\end{cases}$$

where

$$\begin{aligned}
\tau_0=\tau_0(\alpha,\beta)&=\frac{(1+\beta)\left[(1+\beta)(1+3\beta)+\sqrt{d}\right]}{6(1-\alpha)(1+2\beta)(1+3\beta)},\\
d&=(1+\beta)(1+3\beta)\left[36(1+2\beta)^2-15(1+\beta)(1+3\beta)\right],\\
G(2^-)&=\frac{(1-\alpha)^2}{2(1+\beta)^4(1+3\beta)}\left[2(1-\alpha)^2(1+3\beta)+(1+\beta)^3\right],\\
G(t_0)&=\frac{4(1-\alpha)^2}{9(1+2\beta)^2}-\frac{(1-\alpha)^2b(\alpha,\beta)}{144(1+2\beta)^2(1+3\beta)a(\alpha,\beta)},\\
t_0&=(1+\beta)\sqrt{\frac{b(\alpha,\beta)}{-a(\alpha,\beta)}},\\
a(\alpha,\beta)&=9(1-\alpha)^2(1+2\beta)^2(1+3\beta)-3(1-\alpha)(1+\beta)^2(1+2\beta)(1+3\beta)\\
&\quad+(1+\beta)^3\left[4(1+\beta)(1+3\beta)-9(1+2\beta)^2\right],\\
b(\alpha,\beta)&=6(1-\alpha)(1+2\beta)(1+3\beta)\\
&\quad+(1+\beta)\left[27(1+2\beta)^2-16(1+\beta)(1+3\beta)\right].
\end{aligned}$$

Corollary 2. *Let the function $f(z)$ given by (1.1) be in the class $\Im_\Sigma(0,\beta,1)$, $\beta\in[0,1]$. Then,*

$$\left|a_2a_4-a_3^2\right|\le\begin{cases} G(2^-), & \text{if } \tau_0\in(0,1]\,,\\ \max\left\{G(2^-),G(t_0)\right\}, & \text{if } \tau_0\in(1,+\infty)\,,\end{cases}$$

where

$$\begin{aligned}
\tau_0=\tau_0(\beta)&=\frac{(1+\beta)\left[(1+\beta)(1+3\beta)+\sqrt{d}\right]}{6(1+2\beta)(1+3\beta)},\\
d&=(1+\beta)(1+3\beta)\left[36(1+2\beta)^2-15(1+\beta)(1+3\beta)\right],\\
G(2^-)&=\frac{1}{2(1+\beta)^4(1+3\beta)}\left[2(1+3\beta)+(1+\beta)^3\right],\\
G(t_0)&=\frac{4}{9(1+2\beta)^2}-\frac{b(\beta)}{144(1+2\beta)^2(1+3\beta)a(\beta)},\\
t_0&=(1+\beta)\sqrt{\frac{b(\beta)}{-a(\beta)}},\\
a(\beta)&=9(1+2\beta)^2(1+3\beta)-3(1+\beta)^2(1+2\beta)(1+3\beta)\\
&\quad+(1+\beta)^3\left[4(1+\beta)(1+3\beta)-9(1+2\beta)^2\right],
\end{aligned}$$

$$b(\beta) = 6(1+2\beta)(1+3\beta) + (1+\beta)\left[27(1+2\beta)^2 - 16(1+\beta)(1+3\beta)\right].$$

Corollary 3. *Let the function $f(z)$ given by (1.1) be in the class $\Im_\Sigma(\alpha, 0, 1)$, $\alpha \in [0, 1)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} G(2^-), & \text{if } \alpha \in \left[0, \frac{5-\sqrt{21}}{6}\right], \\ \max\{G(2^-), G(t_0)\}, & \text{if } \alpha \in \left(\frac{5-\sqrt{21}}{6}, 1\right), \end{cases}$$

where

$$\begin{aligned} G(2^-) &= \frac{(1-\alpha)^2}{2}\left[2(1-\alpha)^2 + 1\right], \\ G(t_0) &= \frac{4(1-\alpha)^2}{9} - \frac{(1-\alpha)^2 b(\alpha)}{144a(\alpha)}, \\ t_0 &= \sqrt{\frac{b(\alpha)}{-a(\alpha)}}, \\ a(\alpha) &= 9(1-\alpha)^2 - 3(1-\alpha) - 5, \\ b(\alpha) &= 6(1-\alpha) + 11. \end{aligned}$$

Corollary 4. *Let the function $f(z)$ given by (1.1) be in the class $\Im_\Sigma(0, 0, 1)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \frac{3}{2}.$$

Corollary 5. *Let the function $f(z)$ given by (1.1) be in the class $\Im_\Sigma(0, \beta, \gamma)$, $\gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$, $\beta \in [0, 1]$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} \max\{G(2^-), G(t_0)\}, & \text{if } \tau \in (0, \tau_0), \\ G(2^-), & \text{if } \tau \in [\tau_0, +\infty), \end{cases}$$

where

$$\begin{aligned} \tau_0 = \tau_0(\beta) &= \frac{(1+\beta)\left[(1+\beta)(1+3\beta) + \sqrt{d}\right]}{6(1+2\beta)(1+3\beta)}, \\ d &= (1+\beta)(1+3\beta)\left[36(1+2\beta)^2 - 15(1+\beta)(1+3\beta)\right], \\ G(2^-) &= \frac{\tau^2}{2(1+\beta)^4(1+3\beta)}\left[2(1+3\beta)\tau^2 + (1+\beta)^3\right], \\ G(t_0) &= \frac{4\tau^2}{9(1+2\beta)^2} - \frac{\tau^2 b(\beta,\tau)}{144(1+2\beta)^2(1+3\beta)a(\beta,\tau)}, \\ t_0 &= (1+\beta)\sqrt{\frac{b(\beta,\tau)}{-a(\beta,\tau)}}, \\ a(\beta,\tau) &= 9(1+2\beta)^2(1+3\beta)\tau^2 - 3(1+\beta)^2(1+2\beta)(1+3\beta)\tau \\ &\quad + (1+\beta)^3\left[4(1+\beta)(1+3\beta) - 9(1+2\beta)^2\right], \\ b(\beta,\tau) &= 6(1+2\beta)(1+3\beta)\tau \end{aligned}$$

$$+ (1+\beta)\left[27(1+2\beta)^2 - 16(1+\beta)(1+3\beta)\right],$$
$$\tau = |\gamma|.$$

Corollary 6. *Let the function $f(z)$ given by* (1.1) *be in the class $\Im_{\Sigma}(0,0,\gamma)$, $\gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} \max\{G(2^-), G(t_0)\}, & \text{if } \tau \in \left(0, \frac{1+\sqrt{21}}{6}\right), \\ G(2^-), & \text{if } \tau \in \left[\frac{1+\sqrt{21}}{6}, +\infty\right), \end{cases}$$

where

$$G(2^-) = \frac{\tau^2}{2}\left(2\tau^2+1\right), \ G(t_0) = \frac{4\tau^2}{9} - \frac{\tau^2 b(\tau)}{144a(\tau)}, \ t_0 = \sqrt{\frac{b(\tau)}{-a(\tau)}},$$
$$a(\tau) = 9\tau^2 - 3\tau - 5, \ b(\tau) = 6\tau + 11, \ \tau = |\gamma|.$$

Remark 1. We can easily see that results obtained in Theorem 1 are improvement of the results obtained by earlier researchers.

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