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GENERALIZATIONS OF ALESANDROV PROBLEM AND MAZUR-ULAM THEOREM FOR TWO-ISOMETRIES AND TWO-EXPANSIVE MAPPINGS

Hamid Khodaei and Abdulqader Mohammadi

Abstract. We show that mappings preserving unit distance are close to two-isometries. We also prove that a mapping f is a linear isometry up to translation when f is a two-expansive surjective mapping preserving unit distance. Then we apply these results to consider two-isometries between normed spaces, strictly convex normed spaces and unital C^*-algebras. Finally, we propose some remarks and problems about generalized two-isometries on Banach spaces.

1. Introduction and preliminaries

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, we call r a conservative (or preserved) distance for the mapping f if f preserves distance r; i.e., for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. In case of $r = 1$, especially, we call it a distance one preserving property (DOPP). A mapping $f : X \to Y$ is said to have the strong distance one preserving property (SDOPP) if it has (DOPP) and in addition, if $d_Y(f(x), f(y)) = 1$ implies $d_X(x, y) = 1$ for all $x, y \in X$.

The theory of isometric mappings had its beginning in the classical paper [13] by S. Mazur and S. Ulam who proved that any surjective isometry between real normed spaces is a linear mapping up to translation. The hypothesis of surjectivity is essential. Without this assumption, Baker [3] proved that every isometry from a normed space into a strictly convex normed space is linear up to translation.

Aleksandrov [2] posed the following problem: *Examine whether the existence of a single conservative distance for some mapping f implies that f is an isometry.* This problem is of great significance for the Mazur-Ulam theorem [13].

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Rassias et al. [16–20] proved a series of results on the Aleksandrov problem on normed spaces. Chu et al. [5–12], [15] in linear n-normed spaces, defined the concept of an n-isometry that are suitable to represent the notion of a volume preserving mapping, and generalized the Aleksandrov problem to n-normed spaces.

An operator f on a Hilbert space H is a two-isometry if $f^{*2}f^2 - 2f^*f + Id = 0$, where Id denotes the identity operator. This last condition can be equivalently expressed as $\|f^2x\|^2 - 2\|fx\|^2 + \|x\|^2 = 0$ for all $x \in H$. This formulation allows for an extension of the two-isometry concept to the Banach space setting. As noted by Richter, in [21], the notion of "two-isometry" generalizes, in a natural way, the well-known definition of isometry. Moreover, these generalized isometries do not belong to well studied classes such as contractions and subnormal operators and can be used as dilations for a class of expanding operators. The class of two-isometries has been generalized by the authors in [1]. A generalization of these operators to Banach spaces has been studied in the paper of Sid Ahmed [22] and Bayart [4]. Recently, the concept of two-expansive operators on a Banach space has introduced and studied in slightly generalized form in [23, 24].

Definition 1.1. Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathcal{X}$ be a mapping (not necessarily linear). Then

(1) f is called a two-isometry if, for all $x, y \in \mathcal{X}$,

$$\|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 = 0. \tag{1.1}$$

(2) f is called a two-expansive mapping if, for all $x, y \in \mathcal{X}$,

$$\|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 \leq 0. \tag{1.2}$$

Remark 1.2. Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathcal{X}$ be a mapping. From Definition 1.1, it is clear that every isometry is a two-isometry, but there exists an invertible two-isometry f which is not an isometry and there exists a continuous two-isometry f which is not linear (see [4, 22] and below in Examples 1.3 and 2.6). Also, it is easy to check that if f is an isometry, then it has SDOPP. If f is surjective and has SDOPP, then f is bijective. Note that f is injective. If not, then we could find x, y in $\mathcal{X}$ with $x \neq y$ such that $f(x) = f(y)$. Choose z in $\mathcal{X}$ so that $\|x - z\| = 1$ and $\|z - y\| \neq 1$. Then we would have $\|f(x) - f(z)\| = \|f(y) - f(z)\| = 1$. But then $\|y - z\| = 1$, a contradiction, so f is bijective.

Example 1.3. Let $\alpha \geq 1$ and $f \in B(\ell_2)$ (the algebra of bounded linear operators f on ℓ_2) be a weighted right-shift operator with weight sequence $(\alpha_n)_{n\in\mathbb{N}}$. That is, for $x = (x_k)_{k\in\mathbb{N}} \in \ell_2$

$$(fx)_n = \begin{cases} 0, & \text{if } n = 0, \\ \alpha_n x_{n-1}, & \text{if } n \geq 1, \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is given by $\alpha_n = \sqrt{\frac{1+(n+1)(\alpha^2-1)}{1+n(\alpha^2-1)}}$. Note that for $\alpha \geq 1$, this is well-defined. Then f is a two-isometry. This follows from the fact that $(\alpha_{n+2}\alpha_{n+1})^2 - 2(\alpha_{n+1})^2 = -1$. Note that for $\alpha > 1$ we have $\|fx\| > \|x\|$, for all non-zero x and, in particular, f is not an isometry for $\alpha > 1$.

In this paper, we investigate the two-isometries and two-expansive mappings between normed spaces, strictly convex normed spaces and unital C^*-algebras. We propose some remarks and problems about generalized two-isometries on Banach spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{X}$ be a normed space of dimension at least 2 and f be a mapping having the SDOPP of $\mathcal{X}$ onto itself. Then*:

(i) *f satisfies the condition*

$$(2.1) \quad -(4n+2) < \|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 < 4n + 2$$

for all $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$. Also, f preserves distance n in both directions for $n \in \mathbb{N}$.

(ii) *If f is a two-expansive mapping, then f is a linear isometry up to translation.*

Proof. (i) From the SDOPP and the surjectivity of f, and Remark 1.2, we see that f is bijective, and both f and f^{-1} preserve unit distance. In proving inequality (2.1) we will use the following notations. With x in $\mathcal{X}$ and $r > 0$, $B(x,r) = \{z : \|z - x\| \leq r\}$, $B^\circ(x,r) = \{z : \|z - x\| < r\}$ and $C_x(n, n+1] = \{z : n < \|z - x\| \leq n+1\}$. Given x in $\mathcal{X}$ and n in $\mathbb{N} \setminus \{1\}$, let z be an element of $B(x,n)$. Since $\dim X > 1$, we can find a sequence $x = x_0, x_1, \ldots, x_n = z$ such that $\|x_{i+1} - x_i\| = 1$ for $i = 0, 1, \ldots, n-1$. Thus

$$\|f(x) - f(z)\| \leq \sum_{i=0}^{n-1} \|f(x_{i+1}) - f(x_i)\| = n.$$

Similarly, we get

$$\|f^2(x) - f^2(z)\| \leq n.$$

Hence

$$f(B(x,n)) \subset B(f(x), n), \quad f(B(f(x),n)) \subset B\left(f^2(x), n\right).$$

The same argument applies to f^{-1} in place of f to obtain

$$f^{-1}(B(f(x),n)) \subset B(x,n).$$

Thus, for all $x \in \mathcal{X}$ and $n = 2, 3, \ldots$, we have

$$f(B(x,n)) = B(f(x), n), \quad f(B(f(x),n)) = B\left(f^2(x), n\right).$$

Now f is bijective, so for $x \in \mathcal{X}$ and n in $\mathbb{N} \setminus \{1\}$, we have

$$(2.2) \quad f(C_x(n, n+1]) = C_{f(x)}(n, n+1], \; f\left(C_{f(x)}(n, n+1]\right) = C_{f^2(x)}(n, n+1].$$

In order to show that (2.2) also holds for $n = 1$, we fix $x \in \mathcal{X}$ and choose any $z \in C_x(1, 2]$. Then $f(z) \in B(f(x), 2)$. If $u = z + (z - x)/\|z - x\|$, then

$$\|u - z\| = \left\| \frac{z - x}{\|z - x\|} \right\| = \frac{\|z - x\|}{\|z - x\|} = 1.$$

So, by SDOPP, we have $\|f(u) - f(z)\| = 1$ and also $\|f^2(u) - f^2(z)\| = 1$. Since

$$\|u - x\| \leq \|u - z\| + \|z - x\| \leq 3.$$

Hence $u = z + (z - x)/\|z - x\|$ is contained in $C_x(2, 3]$. By (2.2), we get $f(u) \in C_{f(x)}(2, 3]$, so that

$$\|f(x) - f(u)\| > 2. \tag{2.3}$$

If $\|f(z) - f(x)\| \leq 1$, then

$$\|f(x) - f(u)\| \leq \|f(x) - f(z)\| + \|f(z) - f(u)\| \leq 2$$

which contradicts (2.3). Thus we have $f\left(C_x(1, 2]\right) \subset C_{f(x)}(1, 2]$. Similarly, we obtain $f(C_{f(x)}(1, 2]) \subset C_{f^2(x)}(1, 2]$. Since the similar result will hold for f^{-1}, it follows that

$$f\left(C_x(1, 2]\right) = C_{f(x)}(1, 2], \quad f\left(C_{f(x)}(1, 2]\right) = C_{f^2(x)}(1, 2], \tag{2.4}$$

and we conclude that (2.2) is true for all n in $\mathbb{N}$. Now we will prove that

$$f\left(B^\circ(x, 1)\right) = B^\circ\left(f(x), 1\right), \quad f\left(B^\circ(f(x), 1)\right) = B^\circ\left(f^2(x), 1\right). \tag{2.5}$$

As above, it is sufficient to prove

$$f\left(B^\circ(x, 1)\right) \subset B^\circ\left(f(x), 1\right), \quad f\left(B^\circ(f(x), 1)\right) \subset B^\circ\left(f^2(x), 1\right), \tag{2.6}$$

since the similar result will then hold for f^{-1}. If (2.6) is not true, then for some integer $n \geq 1$ we would have $f(z) \in C_{f(x)}(n, n+1]$ for some z in $B^\circ(x, 1)$. Since $C_{f(x)}(n, n+1] = f\left(C_x(n, n+1]\right)$, then $f(z) \in f\left(C_x(n, n+1]\right)$ and $z \in C_x(n, n+1]$ for some $n \geq 1$, which is a contradiction. Thus (2.6) holds and we have (2.5).

The fact that (2.2) holds for all positive integers n, together with (2.5) implies that (2.1) is true. Indeed, given x and y in X, let $n + 1$ be the integral part of $\|x - y\|$, so that $y \in C_x(n, n + 1]$ if $n \geq 1$, while if $n = 0$, then either $y \in B^\circ(x, 1)$ or else $\|x - y\| = 1$ and (2.1) becomes trivial. In the non-trivial cases we find by (2.2) or (2.5) that

$$n^2 < \|f^2(x) - f^2(y)\|^2 \leq (n + 1)^2, \qquad n^2 < \|x - y\|^2 \leq (n + 1)^2,$$

$$-2(n + 1)^2 \leq -2\|f(x) - f(y)\|^2 < -2n^2,$$

from which (2.1) follows.

It remains to prove that f and f^{-1} both preserve the distance n for each $n \in \mathbb{N}$. Make the induction assumption that f preserves the distance n. For $n = 1$ this is true by hypothesis. Let x and z satisfy $\|x - z\| = n + 1$, so that

$z \in C_x(n, n+1]$. Hence, $f(z) \in C_{f(x)}(n, n+1]$ so that $\|f(x) - f(z)\| \leq n+1$. Put

$$u = f(x) + \frac{f(z) - f(x)}{\|f(z) - f(x)\|}, \qquad v = f^{-1}(u).$$

Since $\|u - f(x)\| = 1$, we have $\|v - x\| = 1$. Now if $\|u - f(z)\| < n$, we would have $\|v - z\| < n$, and since $\|v - x\| = 1$ it would follow that $\|x - z\| < n+1$, which is a contradiction. Hence,$\|u - f(z)\| \geq n$, so that

$$\begin{aligned} n \leq \|u - f(z)\| &= \left\| f(x) - f(z) + \frac{f(z) - f(x)}{\|f(z) - f(x)\|} \right\| \\ &= \|f(z) - f(x)\| \left(1 - \|f(z) - f(x)\|^{-1}\right) \\ &= \|f(z) - f(x)\| - 1. \end{aligned}$$

Note that $\|f(z) - f(x)\| > 1$, for otherwise, since $f\left(B(x, 1)\right) = B\left(f(x), 1\right)$ we would have $\|x - z\| \leq 1$, a contradiction. Therefore $\|f(x) - f(z)\| \geq n+1$, and it follows that $\|f(x) - f(z)\| = n+1$. This completes the induction proof. The case for f^{-1} is proved similarly.

(ii) First we show that f is a two-isometry. From the above, f is bijective and both f and f^{-1} preserve the distance n for all positive integers n. Let

$$\|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 \neq 0.$$

Then, since f is a two-expansive mapping, we have by (1.2) that

$$\|f^2(x) - f^2(y)\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 < 0. \tag{2.7}$$

Since f is bijective, it follows that

$$\|f^{-2}(x) - f^{-2}(y)\|^2 < 2\|f^{-1}(x) - f^{-1}(y)\|^2 - \|x - y\|^2. \tag{2.8}$$

Replacing x and y in (2.8) by $f^{-1}(x)$ and $f^{-1}(y)$, respectively, and then using (2.8), we obtain

$$\|f^{-3}(x) - f^{-3}(y)\|^2 < 3\|f^{-1}(x) - f^{-1}(y)\|^2 - 2\|x - y\|^2.$$

By adopting the method used above, we have

$$\|f^{-k}(x) - f^{-k}(y)\|^2 < k\|f^{-1}(x) - f^{-1}(y)\|^2 - (k-1)\|x - y\|^2,$$

where $k \in \mathbb{N} \setminus \{1\}$. Therefore,

$$\frac{k-1}{k}\|x - y\|^2 < \|f^{-1}(x) - f^{-1}(y)\|^2.$$

Allowing k tending to infinity, it is easy to see that

$$\|x - y\| < \|f^{-1}(x) - f^{-1}(y)\|.$$

Hence we obtain

$$\|x - y\| > \|f(x) - f(y)\|. \tag{2.9}$$

Given two distinct x and y in X, take $m \in \mathbb{N}$ with $\|x - y\| < m$. Set $z = x + m(y - x)/\|y - x\|$, so that $\|z - x\| = m$ and $\|z - y\| = m - \|y - x\|$. Hence

$$\|f(z) - f(x)\| = m,$$

and by (2.9) we have

$$\|f(z)-f(y)\| < \|z-y\| = m - \|y-x\|.$$

But again using (2.9),

$$\begin{aligned}\|f(z)-f(x)\| &\leq \|f(z)-f(y)\| + \|f(y)-f(x)\| \\ &< m - \|y-x\| + \|y-x\| = m,\end{aligned}$$

which is a contradiction. Thus, (2.7) is false and we have (1.1). Therefore, f is a two-isometry.

Next we show that f is an isometry. From the above, f is a bijective two-isometry. Then we see that f^ℓ ($\ell = 1, -1$) satisfy (1.1). Thus, by (1.1), we get

$$\|f^{2\ell}(x)-f^{2\ell}(y)\|^2 - \|f^\ell(x)-f^\ell(y)\|^2 = \|f^\ell(x)-f^\ell(y)\|^2 - \|x-y\|^2. \tag{2.10}$$

Setting $x = f^\ell(x)$ and $y = f^\ell(y)$ in (2.10), and using (2.10), we get

$$\|f^{3\ell}(x)-f^{3\ell}(y)\|^2 - \|f^{2\ell}(x)-f^{2\ell}(y)\|^2 = \|f^\ell(x)-f^\ell(y)\|^2 - \|x-y\|^2.$$

Then, for any integer $j \in \mathbb{N}$,

$$\|f^{\ell j}(x)-f^{\ell j}(y)\|^2 - \|f^{\ell(j-1)}(x)-f^{\ell(j-1)}(y)\|^2 = \|f^\ell(x)-f^\ell(y)\|^2 - \|x-y\|^2.$$

Hence

$$\begin{aligned}0 &\leq \|f^{\ell k}(x)-f^{\ell k}(y)\|^2 \\ &= \sum_{j=1}^{k}\Big(\|f^{\ell j}(x)-f^{\ell j}(y)\|^2 - \|f^{\ell(j-1)}(x)-f^{\ell(j-1)}(y)\|^2\Big) + \|x-y\|^2 \\ &= k\left(\|f^\ell(x)-f^\ell(y)\|^2 - \|x-y\|^2\right) + \|x-y\|^2 \\ &= k\|f^\ell(x)-f^\ell(y)\|^2 - (k-1)\|x-y\|^2,\end{aligned}$$

which yields

$$\frac{k-1}{k}\|x-y\|^2 \leq \|f^\ell(x)-f^\ell(y)\|^2.$$

Allowing k tending to infinity, it is easy to see that

$$\|x-y\| \leq \|f^\ell(x)-f^\ell(y)\|,$$

which proves f is an isometry. Finally, applying Mazur-Ulam theorem, we conclude that f is a linear isometry up to translation. □

Corollary 2.2. *Let $\mathcal{X}$ be a normed space and f be a surjective two-isometry of $\mathcal{X}$ onto itself. Then:*

(i) *f^{-1} is a two-isometry.*

(ii) *f is an isometry.*

(iii) *f is a linear mapping up to translation.*

Now, we consider the Mazur-Ulam theorem for two-isometries without the surjectivity condition.

The notion of rotundity plays an important role in the studies of the geometry of Banach spaces [14]. A normed space $\mathcal{X}$ is said to be rotund or strictly convex if in its closed unit ball $B_{\mathcal{X}}$ every boundary point in $B_{\mathcal{X}}$ is an extreme point, i.e., any one of the following equivalent conditions holds [14, pp. 425–441]:

(a) The unit sphere $S_{\mathcal{X}}$ contains no line segments;

(b) Every supporting hyperplane intersects $S_{\mathcal{X}}$ in at most one point;

(c) Distinct boundary points have distinct supporting planes;

(d) If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\left\|\frac{x+y}{2}\right\| < 1$;

(e) If $\|x + y\| = \|x\| + \|y\|$ and $y \neq 0$, then $x = \lambda y$ for some $\lambda \geq 0$;

(f) If x, y in $\mathcal{X}$ are linearly independent, then $\|x + y\| < \|x\| + \|y\|$.

The main result of [3] asserts that if f is an isometry from a normed space $\mathcal{X}$ into a rotund normed space $\mathcal{Y}$ (not necessarily onto), then f is a linear mapping up to translation. The proof was based on the following lemma [3].

Lemma 2.3. *If $\mathcal{X}$ is a strictly convex normed space, $x, y, z \in \mathcal{X}$ and $\|z - x\| = \|z - y\| = \frac{1}{2}\|x - y\|$, then $z = \frac{x+y}{2}$.*

Theorem 2.4. *Suppose $\mathcal{X}$ is a strictly convex normed space. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$\left\|f^2\left(\frac{x+y}{2}\right) - f^2(x)\right\| = \frac{1}{2}\left\|f^2(x) - f^2(y)\right\| \tag{2.11}$$

for all $x, y \in \mathcal{X}$. Then

(i) *$f^2 - f^2(0)$ is additive.*

(ii) *If f is a continuous two-isometry satisfying* (2.11), *then f is a linear mapping up to translation.*

Proof. (i) Interchanging x and y in (2.11), we get

$$\left\|f^2\left(\frac{x+y}{2}\right) - f^2(y)\right\| = \frac{1}{2}\left\|f^2(x) - f^2(y)\right\| \tag{2.12}$$

for all $x, y \in \mathcal{X}$. It follows from (2.11), (2.12) and Lemma 2.3 that

$$f^2\left(\frac{x+y}{2}\right) = \frac{f^2(x) + f^2(y)}{2} \tag{2.13}$$

for all $x, y \in \mathcal{X}$. Setting $y = 0$ in (2.13), we obtain $f^2\left(\frac{x}{2}\right) = \frac{1}{2}(f^2(x) + f^2(0))$ for all $x \in \mathcal{X}$. Combining the last equation with (2.13) yields $\frac{1}{2}(f^2(x+y) + f^2(0)) = \frac{1}{2}(f^2(x) + f^2(y))$, which is

$$f^2(x + y) + f^2(0) = f^2(x) + f^2(y) \tag{2.14}$$

for all $x, y \in \mathcal{X}$. Define $F : \mathcal{X} \to \mathcal{X}$ by $F(x) = f^2(x) - f^2(0)$. Hence from (2.14), we see that $F(x + y) = F(x) + F(y)$ for all $x, y \in \mathcal{X}$; i.e., F is additive.

(ii) Since f is a two-isometry, it follows from (1.1) that

$$\begin{aligned} &2\left\|f\left(\frac{x+y}{2}\right)-f(x)\right\|^2-\left\|f^2\left(\frac{x+y}{2}\right)-f^2(x)\right\|^2 \\ &=\left\|\frac{x+y}{2}-x\right\|^2 \\ &=\frac{1}{4}\|x-y\|^2 \\ &=\frac{1}{4}\left(2\|f(x)-f(y)\|^2-\left\|f^2(x)-f^2(y)\right\|^2\right) \end{aligned} \tag{2.15}$$

for all $x, y \in \mathcal{X}$. Similarly,

$$\begin{aligned} &2\|f\left(\frac{x+y}{2}\right)-f(y)\|^2-\|f^2\left(\frac{x+y}{2}\right)-f^2(y)\|^2 \\ &=\frac{1}{4}\left(2\|f(x)-f(y)\|^2-\left\|f^2(x)-f^2(y)\right\|^2\right) \end{aligned} \tag{2.16}$$

for all $x, y \in \mathcal{X}$. It follows from (2.11), (2.12), (2.15) and (2.16) that

$$\|f\left(\frac{x+y}{2}\right)-f(x)\|=\|f\left(\frac{x+y}{2}\right)-f(y)\|=\frac{1}{2}\|f(x)-f(y)\|$$

for all $x, y \in \mathcal{X}$. Using Lemma 2.3, we have

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

for all $x, y \in \mathcal{X}$. The argument above implies that $f - f(0)$ is additive. By the continuity of f, we obtain that f is a linear two-isometry up to translation. $\square$

In the next example we show that there exists a continuous surjective mapping having the SDOPP which is not a two-isometry.

Example 2.5. Suppose that $f : \left(\mathbb{R}^2, \|\cdot\|_\infty\right) \longrightarrow \left(\mathbb{R}^2, \|\cdot\|_\infty\right)$ is a mapping defined by $f(x, y) = \left([x] + (x - [x])^2, y\right)$. Then it is easy to see that f is a continuous surjective mapping satisfying the SDOPP, but f is not a two-isometry.

The following example shows that the surjectivity condition of Corollary 2.2 and the hypothesis (2.11) of Theorem 2.4 cannot be omitted in general.

Example 2.6. Let $f : \ell_2 \to \ell_2$ be defined by

$$f(x_1, x_2, x_3, \ldots) := \left(cx_1x_2, \frac{1}{c}, x_1, x_2, x_3, \ldots\right),$$

where $x_1 \neq 0$ and c is a nonzero real constant. Then

$$f^2(x_1, x_2, x_3, \ldots) := \left(cx_1x_2, \frac{1}{c}, cx_1x_2, \frac{1}{c}, x_1, x_2, x_3, \ldots\right),$$

and we obtain

$$\begin{aligned}
&\|f^2\left(\frac{x+y}{2}\right) - f^2(x)\|^2 \\
&= \frac{1}{4}\left(2\left|c(x_1+y_1)(x_2+y_2) - cx_1x_2\right|^2 + \sum_{n=1}^{\infty}|x_n - y_n|^2\right) \\
&\neq \frac{1}{4}\left(2|cx_1x_2 - cy_1y_2|^2 + \sum_{n=1}^{\infty}|x_n - y_n|^2\right) = \frac{1}{4}\left\|f^2(x) - f^2(y)\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|f^2(x) - f^2(y)\right\|^2 - 2\|f(x) - f(y)\|^2 + \|x - y\|^2 \\
= &\left(2|cx_1x_2 - cy_1y_2|^2 + \sum_{n=1}^{\infty}|x_n - y_n|^2\right) - 2\left(|cx_1x_2 - cy_1y_2|^2 + \sum_{n=1}^{\infty}|x_n - y_n|^2\right) \\
&+ \sum_{n=1}^{\infty}|x_n - y_n|^2 = 0
\end{aligned}$$

for all $x, y \in \ell_2$. Thus, f is a two-isometry. It is clear that f is continuous but f is not linear mapping up to translation.

Theorem 2.7. *Let $\mathcal{X}$ be an unital C^*-algebra and $U(\mathcal{X})$ be the set of unitary elements in $\mathcal{X}$. If $f : \mathcal{X} \to \mathcal{X}$ is a surjective two-isometry with $f(\jmath) = \jmath$, where $\jmath \in \{0, i\}$ and*

$$f(uv) = f(u)f(v) \tag{2.17}$$

for all $u, v \in U(\mathcal{X})$, then f is a C^*-isomorphism.

Proof. From Corollary 2.2 and Kadison [10], it follows that if $f : \mathcal{X} \to \mathcal{X}$ is a surjective two-isometry and $u \in U(\mathcal{X})$ (i.e., $uu^* = u^*u = 1$), then $f(u)$ is not a zero divisor and $f(u) \in U(\mathcal{X})$. Thus

$$f(u)f(u)^* = 1 \tag{2.18}$$

for all $u \in U(\mathcal{X})$. It follows from (2.17) that $f(u) = f(1)f(u)$, and so $f(1) = 1$. Replacing v by u^* in (2.17) and using $f(1) = 1$, we obtain

$$1 = f\left(uu^*\right) = f(u)f\left(u^*\right) \tag{2.19}$$

for all $u \in U(\mathcal{X})$. Now it follows from (2.18) and (2.19) that

$$f\left(u^*\right) = f(u)^* \tag{2.20}$$

for all $u \in U(\mathcal{X})$. Again applying the equality (2.17) and using $f(i) = i$, we have

$$f(iu) = if(u) \tag{2.21}$$

for all $u \in U(\mathcal{X})$. Using Corollary 2.2 and $f(0) = 0$, We observe that f is $\mathbb{R}$-linear. Since for every $c \in \mathbb{C}$ we can find $\alpha, \beta \in \mathbb{R}$ such that $c = \alpha + i\beta$, we can conclude from (2.21) and the $\mathbb{R}$-linearity of f that $f(cu) = cf(u)$ for all $c \in \mathbb{C}$ and all $u \in U(\mathcal{X})$. Also, since for every $x \in \mathcal{X}$ we can find a finite linear combination of unitary elements $\sum_{k=1}^{n} c_k u_k$ such that $x = \sum_{k=1}^{n} c_k u_k$ where $c_k \in \mathbb{C}$ and $u_k \in U(\mathcal{X})$, we can conclude from $f(cu) = cf(u)$ and the $\mathbb{R}$-linearity of f that $f(cx) = cf(x)$ for all $c \in \mathbb{C}$ and all $x \in \mathcal{X}$. Thus f is $\mathbb{C}$-linear. Utilizing (2.20) and the $\mathbb{C}$-linearity of f, we get

$$\begin{aligned} f(x^*) = f\left(\sum_{k=1}^{n} \bar{c}_k u_k^*\right) &= \sum_{k=1}^{n} \bar{c}_k f\left(u_k^*\right) \\ &= \sum_{k=1}^{n} \bar{c}_k f\left(u_k\right)^* = f\left(\sum_{k=1}^{n} c_k u_k\right)^* = f(x)^* \end{aligned}$$

for all $x \in \mathcal{X}$. Furthermore, if $y \in \mathcal{X}$, then there exists a finite linear combination of unitary elements $\sum_{\ell=1}^{m} d_\ell v_\ell$ such that $y = \sum_{\ell=1}^{m} d_\ell v_\ell$ where $d_\ell \in \mathbb{C}$ and $v_\ell \in U(\mathcal{X})$. Hence, we have

$$\begin{aligned} f(xy) &= f\left(\sum_{\ell=1}^{m} d_\ell x v_\ell\right) = \sum_{\ell=1}^{m} d_\ell f\left(x v_\ell\right) \\ &= \sum_{\ell=1}^{m} d_\ell f\left(\sum_{k=1}^{n} c_k u_k v_\ell\right) = \sum_{\ell=1}^{m} d_\ell \sum_{k=1}^{n} c_k f\left(u_k v_\ell\right) \\ &= \sum_{\ell=1}^{m} d_\ell \sum_{k=1}^{n} c_k f\left(u_k\right) f\left(v_\ell\right) = \sum_{\ell=1}^{m} d_\ell f\left(\sum_{k=1}^{n} c_k u_k\right) f\left(v_\ell\right) \\ &= \sum_{\ell=1}^{m} d_\ell f\left(x\right) f\left(v_\ell\right) = f\left(x\right) f\left(\sum_{\ell=1}^{m} d_\ell v_\ell\right) = f(x) f(y) \end{aligned}$$

for all $x, y \in \mathcal{X}$. That is, f is multiplicative. Therefore, f is a C^*-isomorphism. □

3. Final remarks

The results above raise the following remarks and problems about generalized two-isometries.

Remark 3.1. A mapping f (not necessarily linear) on a normed space $\mathcal{X}$ is an (m, p)-isometry ($m \geq 1$ integer and $p > 0$ real) if, for all $x, y \in \mathcal{X}$,

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\| f^{m-i}(x) - f^{m-i}(y) \right\|^p = 0$$

(see [4, 22]). In case of $m, p = 2$, especially, we call it a two-isometry (see Definition 1.1).

Problem 3.2. Let $\mathcal{X}$ be a normed space (or a strictly convex normed space). Under what conditions is a mapping preserving a distance $r > 0$ an (m, p)-isometry?

Problem 3.3. Let f on a normed space $\mathcal{X}$ be a surjective (m,p)-isometry. Is then f necessarily a linear mapping up to translation?

Another related subject is the study of ε-(m,p)-isometries on Banach spaces (see [9, 17, 18]).

Remark 3.4. Given $\varepsilon > 0$, a mapping f on a Banach space $\mathcal{X}$ is called an ε-(m,p)-isometry if, for all $x, y \in \mathcal{X}$,

$$\left|\sum_{i=0}^{m}(-1)^i \binom{m}{i} \left\|f^{m-i}(x) - f^{m-i}(y)\right\|^p\right| \leq \epsilon.$$

Problem 3.5. Let f on a Banach space $\mathcal{X}$ be an ε-(m,p)-isometry. Under what assumptions are there a constant γ and an (m,p)-isometry T on $\mathcal{X}$ such that $\|f(x) - T(x)\| \leq \gamma\varepsilon$ for all $x \in \mathcal{X}$?

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Hamid Khodaei
Faculty of Mathematical Sciences and Statistics
Malayer University
P. O. Box 65719-95863, Malayer, Iran
Email address: hkhodaei@malayeru.ac.ir; hkhodaei.math@gmail.com

Abdulqader Mohammadi
Faculty of Mathematical Sciences and Statistics
Malayer University
P. O. Box 65719-95863, Malayer, Iran
Email address: amohammadi8566@gmail.com