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HEAT EQUATION WITH A GEOMETRIC ROUGH PATH POTENTIAL IN ONE SPACE DIMENSION: EXISTENCE AND REGULARITY OF SOLUTION

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Abstract. A solution of the heat equation with a distribution-valued potential is constructed by regularization. When the potential is the generalized derivative of a Hölder continuous function, regularity of the resulting solution is in line with the standard parabolic theory.

1. Introduction

Let $W = W(x)$ be a continuous function on $[0, \pi]$, and denote by $\dot{W}$ the generalized derivative of W:

$$\int_0^\pi W(x)h'(x)dx = -\int_0^\pi \dot{W}(x)h(x)dx$$

for every continuously differentiable h with compact support in $(0, \pi)$. By direct computations, $\dot{W}$ is a generalized function from the closure of $L_2((0, \pi))$ with respect to the norm

$$\|h\|_{-1}^2 = \sum_{k\geq 1} \frac{h_k^2}{k^2},\ h_k = \sqrt{\frac{2}{\pi}} \int_0^\pi h(x)\sin(kx)dx.$$

The objective of the paper is investigation of the equation

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)\dot{W}(x),\ 0 < t < T,\ 0 < x < \pi; \\ &u(0,x) = \varphi(x),\ u(t,0) = u(t,\pi) = 0. \end{aligned} \tag{1.1}$$

If $\dot{W}$ were a Hölder continuous function of order $\alpha \in (0, 1)$, then standard parabolic regularity would imply that the classical solution of (1.1) is Hölder $2 + \alpha$ in space and Hölder $1 + (\alpha/2)$ in time; cf. [5, Theorem 10.4.1]. If W is Hölder α, then it might be natural to expect for the solution of (1.1) to lose one

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derivative in space and "one-half of the derivative" in time, that is, to become Hölder $1+\alpha$ in space and Hölder $(1+\alpha)/2$ in time. The objective of the paper is to establish this result in a rigorous way.

The starting point must be interpretation of the product $u\dot{W}$, which we do using the ideas from the rough path theory. Here is an outline of the ideas.

If $h = h(x)$ is a continuously differentiable function with $h(0) = 0$, then, by the fundamental theorem of calculus,

$$\int_0^x h(s)h'(s)ds = \frac{h^2(x)}{2}; \tag{1.2}$$

for a *continuous* function $W = W(x)$, the integral $\int_0^x W(s)\dot{W}(s)ds$ is, in general, not defined. An extension of the Riemann-Stieltjes construction, due to Young, is possible when W is Hölder continuous of order bigger than $1/2$; then (1.2) continues to hold as long as $W(0) = 0$. For less regular functions W, the value of

$$\int_0^x W(s)\dot{W}(s)ds = \int_0^x \int_0^s \dot{W}(x_1)dx_1\dot{W}(s)ds$$

and possibly higher order iterated integrals

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dot{W}(s)ds\dot{W}(x_2)dx_2\dot{W}(x_1)dx_1,$$

etc. must be *postulated*, which is the subject of the *rough path theory.* The collection of all such iterated integrals, known as the *signature* of the rough path W, is encoded in the solution of the ordinary differential equation

$$\frac{dY(x)}{dx} = Y(x)\dot{W}(x),\ x > 0. \tag{1.3}$$

One particular case of the rough path construction *stipulates* that the traditional rules of calculus, such as (1.2), continue to hold. That is, given a continuous function $W = W(x)$, $x \geq 0$, the solution of the ordinary differential equation is *defined* to be

$$Y(x) = Y(0)e^{W(x)-W(0)}; \tag{1.4}$$

in what follows we refer to this as the *geometric rough path* (*GRP*) solution of (1.3). The main consequence of (1.4) is that if $W^{(\varepsilon)} = W^{(\varepsilon)}(x)$, $\varepsilon > 0$, is a sequence of continuously differentiable functions such that $\lim_{\varepsilon\to 0} \sup_{x\in(0,L)} |W(x) - W^{(\varepsilon)}(x)| = 0$, then

$$\lim_{\varepsilon\to 0} \sup_{x\in(0,L)} |Y(x) - Y^{(\varepsilon)}(x)| = 0, \quad \text{where } Y^{(\varepsilon)}(x) = Y(0)e^{W^{(\varepsilon)}(x)-W^{(\varepsilon)}(0)},$$

that is,

$$\frac{dY^{(\varepsilon)}(x)}{dx} = Y^{(\varepsilon)}(x)\frac{dW^{(\varepsilon)}(x)}{dx}.$$

It is therefore natural to define a GRP solution of (1.1) as a suitable limit of the solutions of

$$
(1.5)\quad \begin{aligned} &\frac{\partial u^{(\varepsilon)}(t,x)}{\partial t} = \frac{\partial^2 u^{(\varepsilon)}(t,x)}{\partial x^2} + u^{(\varepsilon)}(t,x)\dot{W}^{(\varepsilon)}(x),\ 0 < t < T,\ x \in (0,\pi),\\ &u^{(\varepsilon)}(t,0) = u^{(\varepsilon)}(t,\pi) = 0,\ u^{(\varepsilon)}(0,x) = \varphi(x), \end{aligned}
$$

where $\varphi \in L_2((0,\pi))$ and $W^{(\varepsilon)}$ are absolutely continuous approximations of W so that $W^{(\varepsilon)}(x) = \int_0^x \dot{W}^{(\varepsilon)}(s)ds$ and $\dot{W}_\varepsilon \in L_\infty((0,\pi))$.

Without further regularity assumptions on φ and $\dot{W}^{(\varepsilon)}$, equation (1.5) can only be interpreted in variational form, leading to a generalized solution; under additional assumptions about φ and $\dot{W}^{(\varepsilon)}$, equation (1.5) can have a classical solution. Accordingly, the paper defines and investigates the two possible GRP solutions of (1.1), classical (Section 2) and generalized (Section 3). The last section discusses extensions of the results to more general equations.

In physics and related applications, equations of the type (1.1) can represent either the Schrödinger equation (after making time imaginary) or the (continuum) parabolic Anderson model. In both cases, the potential $\dot{W}$ is often random, to model complexity and/or incomplete knowledge of the corresponding Hamiltonian. When the time variable is present in W, the product $u\dot{W}$ can often be interpreted as the Itô stochastic integral; with only space parameter, such an interpretation is not possible and the Wick-Itô-Skorokhod integral becomes one of the available options.

The method of regularization, that is, using an approximation of W with smooth functions, is a different approach. In the stochastic setting, it corresponds to the Stratonovich (or Fisk-Stratonovich) integral; cf. [8]. In a more recent theory of rough paths, it corresponds to the geometric rough path formulation of the equation; cf. [1, Section 2.2].

Paper [3] presents a comprehensive study of (1.1) in the whole space for a large class of random potentials $\dot{W}$, both with and without time parameter and under various interpretations of $u\dot{W}$, including Itô, Wick-Itô-Skorokhod, and Stratonovich. This generality leads to less-than-optimal Hölder regularity (namely, $(1+\alpha)/2$ in time and $\alpha/2$ in space) in the particular case of one space dimension. With the Wick-Itô-Skorokhod interpretation, the optimal regularity for (1.1) and similar equations was established in [4].

To simplify the presentation, the current paper considers (1.1) on a bounded interval with zero boundary conditions; Section 4 outlines possible generalizations. Both classical and generalized solutions of (1.1) are constructed, as well as the fundamental solution. The limiting procedure via approximations of $\dot{W}$ by regular (as opposed to generalized) functions provides a clear connection between various constructions. The special structure of the equation leads to additional regularity results if W is Hölder continuous: the classical solution is infinitely differentiable in time for $t > 0$, and, with a particular choice of the initial condition it is also possible to ensure that the solution is Hölder $1+(\alpha/2)$ near $t=0$, that is, better than what is guaranteed by the general theory.

In two or higher space dimensions the method of regularization does not work directly because many of the limits do not exist. The difficulty is resolved with the help of additional constructions using renormalization techniques; cf. [2]. As a result, any comparison with one-dimensional setting is not especially informative.

Here is the summary of the main notations used in the paper: $h' = h'(x)$ denotes the usual derivative; $\dot{W}$ denotes the generalized derivative; u_t, u_x, u_{xx} denote the corresponding partial derivatives (classical or generalized); to shorten the notations, the interval $(0,\pi)$ is sometimes denoted by G.

2. The classical geometric rough path solution

We start with an overview of the parabolic Hölder space.

Given a locally compact metric space $\mathbb{X}$ with the distance function ρ, denote by $\mathcal{C}(\mathbb{X})$ the space of real-valued continuous functions on $\mathbb{X}$. For $\alpha \in (0,1)$, denote by $\mathcal{C}^{\alpha}(\mathbb{X})$ the space of Hölder continuous real-valued functions on $\mathbb{X}$, that is, functions F satisfying

$$\sup_{a,b:\rho(a,b)>0} \frac{|F(a)-F(b)|}{\rho^{\alpha}(a,b)} < \infty;$$

$\mathcal{C}^{\alpha}(\mathbb{X})$ is a Banach algebra with norm

$$\|F\|_{\mathcal{C}^{\alpha}(\mathbb{X})} = \sup_{a} |F(a)| + \sup_{a,b:\rho(a,b)>0} \frac{|F(a)-F(b)|}{\rho^{\alpha}(a,b)}.$$

For a positive integer k, the space $\mathcal{C}^{k+\alpha}(\mathbb{X})$ consists of real-valued functions that are k times continuously differentiable, and the derivative of order k is in $\mathcal{C}^{\alpha}(\mathbb{X})$.

Of special interest is $\mathbb{X} = (0,T)\times G$, $G \subseteq \mathbb{R}^{\mathrm{d}}$, with *parabolic* distance

$$\rho\big((t,x),(s,y)\big) = \sqrt{|t-s|} + |x-y|.$$

To emphasize the presence of both time and space variables, the corresponding notations become $\mathcal{C}^{\alpha/2,\alpha}\big((0,T)\times G\big)$ and $\mathcal{C}^{k+(\alpha/2),2k+\alpha}\big((0,T)\times G\big)$. Also,

$$\begin{aligned}\|v\|_{\mathcal{C}^{(1+\alpha)/2,1+\alpha}\left((0,T)\times G\right)} = {} & \sup_{t,x} |v(t,x)| + \|v_x\|_{\mathcal{C}^{\alpha/2,\alpha}\left((0,T)\times G\right)} \\ & + \sup_{t\neq s,x} \frac{|v(t,x)-v(s,x)|}{|t-s|^{(1+\alpha)/2}}.\end{aligned}$$

Definition. The classical solution of equation

$$\begin{gathered} U_t(t,x) = U_{xx}(t,x) + b(t,x)U_x(t,x) + c(t,x)U(t,x) + f(t,x), \\ x\in(0,\pi),\ t\in(0,T),\ U(0,x)=\varphi(x),\ U(t,0)=U(t,\pi)=0, \end{gathered} \tag{2.1}$$

is a function $U = U(t,x)$ with the following properties:

(1) for every $t \in (0,T)$ and $x \in (0,\pi)$, the function U is continuously differentiable in t, twice continuously differentiable in x, and (2.1) holds;
(2) for every $t \in (0,T)$, $U(t,0) = U(t,\pi) = 0$;
(3) for every $x \in [0,\pi]$, $\lim_{t\to 0+} U(t,x) = \varphi(x)$.

The following result is well known; cf. [5, Theorem 10.4.1].

Proposition 2.1. *If $\varphi \in \mathcal{C}((0,\pi))$, $\varphi(0)=\varphi(\pi)=0$, and $b,c,f \in \mathcal{C}^{\beta/2,\beta}\big((0,T) \times(0,\pi)\big)$ for some $\beta \in (0,1)$, then equation (2.1) has a unique classical solution. If in addition $\varphi \in \mathcal{C}^{2+\beta}((0,\pi))$, then $U \in \mathcal{C}^{1+\beta/2,2+\beta}\big((0,T)\times(0,\pi)\big)$.*

The next result makes it possible to define the classical geometric rough path solution of equation (1.1).

Theorem 2.2. *Assume that, for some $\alpha,\beta \in (0,1)$, $W^{(\varepsilon)} \in \mathcal{C}^{1+\beta}((0,\pi))$ for all $\varepsilon > 0$, $W \in \mathcal{C}^{\alpha}((0,\pi))$, and $\varphi \in \mathcal{C}^{1+\alpha}((0,\pi))$. If*

$$\lim_{\varepsilon\to 0} \|W - W^{(\varepsilon)}\|_{L_\infty((0,\pi))} = 0,$$

then there exists a function $u \in \mathcal{C}^{(1+\alpha)/2,1+\alpha}\big((0,T)\times(0,\pi)\big)$ such that

$$\lim_{\varepsilon\to 0} \|u - u^{(\varepsilon)}\|_{L_\infty\big((0,T)\times(0,\pi)\big)} = 0.$$

Note that there is no connection between α and β in the conditions of the theorem.

Definition. The function u from Theorem 2.2 is called the classical GRP (geometric rough path) solution of equation (1.1).

Proof of Theorem 2.2. Let $u^{(\varepsilon)} = u^{(\varepsilon)}(t,x)$ be the classical solution of (1.5). Define the functions

$$H_W^{(\varepsilon)}(x) = \exp\left(\int_0^x W^{(\varepsilon)}(y)dy\right),\ v^{(\varepsilon)}(t,x) = u^{(\varepsilon)}(t,x)H_W^{(\varepsilon)}(x).$$

By direct computation,

$$\begin{aligned} &\frac{\partial v^{(\varepsilon)}(t,x)}{\partial t} = \frac{\partial^2 v^{(\varepsilon)}(t,x)}{\partial x^2} - 2W^{(\varepsilon)}(x)\frac{\partial v^{(\varepsilon)}(t,x)}{\partial x} + \big(W^{(\varepsilon)}(x)\big)^2 v^{(\varepsilon)}(t,x),\\ &\qquad t>0,\ x\in(0,\pi),\\ &v^{(\varepsilon)}(t,0) = v^{(\varepsilon)}(t,\pi) = 0,\ v^{(\varepsilon)}(0,x) = \varphi(x)H_W^{(\varepsilon)}(x). \end{aligned} \tag{2.2}$$

Define

$$H_W(x) = \exp\left(\int_0^x W(y)dy\right) \tag{2.3}$$

and let $v = v(t,x)$ be the classical solution of

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} - 2W(x)\frac{\partial v(t,x)}{\partial x} + W^2(x)v(t,x),\\ &t>0,\ x\in(0,\pi),\ v(t,0)=v(t,\pi)=0,\ v(0,x)=\varphi(x)H_W(x). \end{aligned} \tag{2.4}$$

Let $V^{(\varepsilon)}(t,x) = v(t,x) - v^{(\varepsilon)}(t,x)$. We write the equation for $V^{(\varepsilon)}$ as

$$\begin{aligned} V_t^{(\varepsilon)} &= \widetilde{\mathbf{A}}_W V^{(\varepsilon)} + V^{(\varepsilon)} W^2 + F^{(\varepsilon)}, \\ V^{(\varepsilon)}(0,x) &= \varphi(x)\big(H_W(x) - H_W^{(\varepsilon)}(x)\big), \end{aligned} \tag{2.5}$$

where $\widetilde{\mathbf{A}}_W$ is the operator

$$h \mapsto h'' - 2Wh'$$

with zero Dirichlet boundary conditions on $G = (0,\pi)$, and

$$F^{(\varepsilon)}(t,x) = 2v_x^{(\varepsilon)}(t,x)\big(W(x) - W^{(\varepsilon)}(x)\big) + v^{(\varepsilon)}(t,x)\Big(W^2(x) - \big(W^{(\varepsilon)}(x)\big)^2\Big).$$

Denote by $\widetilde{\Phi}_t,\ t>0$, the semigroup generated by the operator $\widetilde{\mathbf{A}}_W$. Then (2.5) becomes

$$\begin{aligned} V^{(\varepsilon)}(t,x) = \widetilde{\Phi}_t[V^{(\varepsilon)}(0,\cdot)](x) &+ \int_0^t \widetilde{\Phi}_{t-s}[V^{(\varepsilon)}(s,\cdot)W^2(\cdot)](x)ds \\ &+ \int_0^t \widetilde{\Phi}_{t-s}[F^{(\varepsilon)}(s,\cdot)](x)ds. \end{aligned} \tag{2.6}$$

The maximum principle for the operator $\widetilde{\mathbf{A}}_W$ implies

$$\|\widetilde{\Phi}_t h\|_{L_\infty(G)} \le \|h\|_{L_\infty(G)},\ t>0; \tag{2.7}$$

similarly,

$$\sup_{0<t<T} \|(\widetilde{\Phi}_t h)_x\|_{L_\infty(G)} \le C_1\big(T, \|W\|_{L_\infty(G)}\big)\big(\|h\|_{L_\infty(G)} + \|h_x\|_{L_\infty(G)}\big); \tag{2.8}$$

cf. [5, Sections 8.2, 8.3].

By assumption, there exists a positive number C_0 such that

$$\|W\|_{L_\infty(G)} \le C_0,\ \|W^{(\varepsilon)}\|_{L_\infty(G)} \le C_0$$

for all $\varepsilon > 0$. Define

$$\Delta_W^{(\varepsilon)} = \|W^{(\varepsilon)} - W\|_{L_\infty(G)}.$$

Then (2.2), (2.7), (2.8), and Gronwall's inequality imply

$$\sup_{0<t<T} \|v_x^{(\varepsilon)}\|_{L_\infty(G)}(t) + \sup_{0<t<T} \|v^{(\varepsilon)}\|_{L_\infty(G)}(t) \le C_2 \|\varphi\|_{L_\infty(G)},$$

with C_2 depending only on C_0 and T.

Also, by the intermediate value theorem,

$$\|V^{(\varepsilon)}\|_{L_\infty(G)}(0) \le \|\varphi\|_{L_\infty(G)} e^{2\pi C_0}\, \Delta_W^{(\varepsilon)}.$$

Then we deduce from (2.6) that

$$\sup_{0<t<T} \|V^{(\varepsilon)}\|_{L_\infty(G)}(t) \le C_3 \Delta_W^{(\varepsilon)},$$

with C_3 depending only on C_0, T, and φ. It remains to note that

$$u = \frac{v}{H_W}, \tag{2.9}$$

$$|u - u^{(\varepsilon)}| \le \frac{|V^{(\varepsilon)}|}{H_W} + \frac{|H_W - H_W^{(\varepsilon)}|}{H_W \cdot H_W^{(\varepsilon)}} \, |v^{(\varepsilon)}|$$

and then, with C_4 depending only on C_0, T and φ,

$$\sup_{t\in(0,T),x\in G} |u(t,x) - u^{(\varepsilon)}(t,x)| \le C_4 \|W - W^{(\varepsilon)}\|_{L_\infty(G)}, \tag{2.10}$$

completing the proof of the theorem. □

The following is the main result about the classical GRP solution of (1.1).

Theorem 2.3. *If $W \in \mathcal{C}^\alpha((0,\pi))$, $\varphi \in \mathcal{C}^{1+\alpha}((0,\pi))$, and $\varphi(0) = \varphi(\pi) = 0$, then* (1.1) *has a unique GRP solution given by*

$$u(t,x) = \frac{\Phi_t[H_W\varphi](x)}{H_W(x)},$$

where Φ_t is the semigroup of the operator

$$\mathbf{A}_W : h(x) \mapsto h''(x) - 2W(x)h'(x) + W^2(x)h(x)$$

with zero boundary conditions on $(0,\pi)$ and H_W is from (2.3). *The solution belongs to the parabolic Hölder space*

$$\mathcal{C}^{(1+\alpha)/2,1+\alpha}\big((0,T)\times(0,\pi)\big)$$

and is an infinitely differentiable function of t for $t > 0$. If, in addition, $\varphi(x) = \psi(x)/H_W(x)$ for some $\psi \in \mathcal{C}^{2+\alpha}((0,\pi))$, then $u \in \mathcal{C}^{1+(\alpha/2),1+\alpha}\big((0,T)\times(0,\pi)\big)$.

Proof. This is a direct consequence of (2.9) and Proposition 2.1. The solution is a smooth function of t for $t > 0$ because the coefficients of the operator $\mathbf{A}_W$ do not depend on time (cf. [5, Theorem 8.2.1]), whereas $\mathcal{C}^\alpha$ regularity of the coefficients of $\mathbf{A}_W$ implies that the function $\Phi_t[\varphi](x)$ cannot be better than $\mathcal{C}^{2+\alpha}$ in space. Because $H_W \in \mathcal{C}^{1+\alpha}((0,\pi))$, a typical initial condition cannot be better than $\mathcal{C}^{1+\alpha}((0,\pi))$ and similarly, the solution cannot be more regular in space than $\mathcal{C}^{1+\alpha}((0,\pi))$. If, with a special choice of the initial condition, we ensure that $\varphi H_W \in \mathcal{C}^{2+\alpha}((0,\pi))$, then, by Proposition 2.1, we get better time regularity of the solution near $t = 0$. □

Remark 2.4. If W is a sample trajectory of the standard Brownian motion, then, with probability one, $W \in \mathcal{C}^{1/2-}((0,\pi))$, that it, W is Hölder continuous of every order less than 1/2. By Theorem 2.3, with $\alpha < 1/2$, we conclude that, with probability one, a *typical* classical GRP solution of the corresponding equation (1.1) is $\mathcal{C}^{3/4-}$ in time and $\mathcal{C}^{3/2-}$ in space; with a special choice of the initial condition, it is possible to achieve $\mathcal{C}^{5/4-}$ regularity in time. Similarly, if $W = B^H$ is fractional Brownian motion with Hurst parameter $H \in (0,1)$, then a typical classical GRP solution of (1.1) is $\mathcal{C}^{(1+H)/2-}$ in time and $\mathcal{C}^{1+H-}$ in space.

Inequality (2.10) provides the rate of convergence of the approximate solutions $u^{(\varepsilon)}$ to u; this rate depends on the particular approximation $W^{(\varepsilon)}$ of W. As an example, consider

$$W^{(\varepsilon)}(x) = \frac{1}{\varepsilon}\int_0^{\pi} \phi\left(\frac{x-y}{\varepsilon}\right) W(y)dy,$$

where

$$\phi \geq 0,\ \phi \in \mathcal{C}^{\infty}(\mathbb{R}),\ \phi(x) = 0,\ x \notin (-\pi,\pi),\ \int_0^1 \phi(x)dx = 1,$$

so that $W^{(\varepsilon)} \in \mathcal{C}^{\infty}((0,\pi))$.

By direct computation, if $W \in \mathcal{C}^{\alpha}((0,\pi))$, then

$$|W(x) - W^{(\varepsilon)}(x)| \leq \varepsilon^{\alpha} \sup_{x\neq y} \frac{|W(x)-W(y)|}{|x-y|^{\alpha}},$$

and so

$$\sup_{t,x} |u(t,x) - u^{(\varepsilon)}(t,x)| \leq \tilde{C}_4 \varepsilon^{\alpha}.$$

With some extra effort, similar arguments show that, for every $\gamma \in (0,\alpha)$,

$$\|u - u^{(\varepsilon)}\|_{\mathcal{C}^{(1+\gamma)/2,1+\gamma}\big((0,T)\times(0,\pi)\big)} \leq C_5 \varepsilon^{\alpha-\gamma}.$$

3. The generalized geometric rough path solution

The classical solution of (1.5) might not exist if the initial condition φ is not a continuous function, or if the functions $\dot{W}^{(\varepsilon)}$ are not Hölder continuous, or if condition $\varphi(0) = \varphi(\pi) = 0$ does not hold. The generalized solution is an extension of the classical solution using the idea of integration by parts.

To simplify the notations, we write

$$G = (0,\pi),\ (g,h)_0 = \int_0^{\pi} g(x)h(x)dx,\ \|h\|_0^2 = (h,h)_0.$$

Next, we need an overview of the Sobolev space on G. The space $H_0^1(G)$ is the closure of the set of smooth functions with compact support in G with respect to the norm $\|\cdot\|_1$, where

$$\|h\|_1^2 = \|h\|_0^2 + \|h'\|_0^2.$$

The space $H^{-1}(G)$, mentioned in the introduction, is a separable Hilbert space with norm $\|\cdot\|_{-1}$ and is the dual of $H_0^1(G)$ relative to the inner product $(\cdot,\cdot)_0$; the corresponding duality will be denoted by $[\cdot,\cdot]_0$.

Definition. Given bounded measurable functions a, b, c and a generalized function f from $L_2\big((0,T); H^{-1}(G)\big)$, the generalized solution of equation

$$v_t = (v_x + av)_x + bv_x + cv + f,\ t > 0,\ x \in G, \tag{3.1}$$

with initial condition $v|_{t=0} = \varphi \in L_2(G)$ and zero Dirichlet boundary conditions $v|_{x=0} = v|_{x=\pi} = 0$ is an element of $L_2\big((0,T); H_0^1(G)\big)$ such that, for every $h \in H_0^1(G)$ and $t \in (0,T)$,

$$
\begin{aligned}
\big(v,h\big)_0(t) = \big(\varphi,h\big)_0 - \int_0^t \big(v_x + av, h'\big)_0(s)ds \\
+ \int_0^t \big(bv_x + cv, h\big)_0(s)ds + \int_0^t [f,h]_0(s)ds.
\end{aligned}
\tag{3.2}
$$

The following result is well known; cf. [6, Theorem III.4.1].

Proposition 3.1. *Let $a = a(t,x),\ b = b(t,x),\ c = c(t,x)$ be bounded measurable functions,*

$$\varphi \in L_2(G),\ f \in L_2\big((0,T); H^{-1}(G)\big).$$

Then equation (3.1) has a unique generalized solution and

$$v \in L_2\big((0,T); H_0^1(G)\big) \bigcap \mathcal{C}\big((0,T); L_2(G)\big),$$

$$\sup_{t\in(0,T)} \|v\|_0^2(t) + \int_0^T \|v\|_1^2(s)ds \le C\Big(\|\varphi\|_0^2 + \int_0^T \|f\|_{-1}^2(s)ds\Big),$$

with C depending only on T and the L_∞ norms of a, b, c.

If $W^{(\varepsilon)}$ is continuously differentiable and $u^{(\varepsilon)}$ is a classical solution of (1.5), then we can multiply both sides by a continuously differentiable function h with compact support in G and integrate with respect to t, x to get, after integration by parts,

$$
\begin{aligned}
\big(u^{(\varepsilon)},h\big)_0(t) = \big(\varphi,h\big)_0 - \int_0^t \big(u_x^{(\varepsilon)} + u^{(\varepsilon)}W^{(\varepsilon)}, h'\big)_0(s)ds \\
- \int_0^t \big(u_x^{(\varepsilon)}W^{(\varepsilon)}, h\big)_0(s)ds.
\end{aligned}
\tag{3.3}
$$

Comparing (3.2) and (3.3), we conclude that $u^{(\varepsilon)}$ is a generalized solution of

$$u_t^{(\varepsilon)} = (u_x^{(\varepsilon)} + u^{(\varepsilon)}W^{(\varepsilon)})_x - u_x^{(\varepsilon)}W^{(\varepsilon)}. \tag{3.4}$$

If we now pass to the limit $\varepsilon \to 0$ in (3.4) and assume that all the limits exist and all the equalities continue to hold, then we get the equation

$$u_t = (u_x + uW)_x - u_xW, \tag{3.5}$$

which is a particular case of (3.1). Before confirming that this passage to the limit is indeed justified, let us use this non-rigorous argument to define the generalized GRP solution of (1.1).

Definition. The generalized geometric rough path (GRP) solution of (1.1) is the generalized solution of (3.5).

The next result is an immediate consequence of Proposition 3.1.

Theorem 3.2. *If $W \in L_\infty((0,\pi))$ and $\varphi \in L_2((0,\pi))$, then* (1.1) *has a unique generalized GRP solution u and*

$$u \in L_2\big((0,T); H_0^1((0,\pi))\big) \bigcap \mathcal{C}\big((0,T); L_2((0,\pi))\big),$$

$$\sup_{t\in(0,T)} \|u\|_0^2 + \int_0^T \|u\|_1^2(s)ds \le C\|\varphi\|_0^2,$$

with C depending only on T and $\|W\|_{L_\infty((0,\pi))}$.

Note that generalized GRP solution may exist even when W is not continuous.

It remains to confirm that passing to the limit $\varepsilon \to 0$ in (3.4) is justified and indeed leads to (3.5).

Theorem 3.3. *Assume that $W^{(\varepsilon)} \in L_\infty(G)$ for all $\varepsilon > 0$, $W \in L_\infty(G)$, and $\varphi \in L_2(G)$. Let $u^{(\varepsilon)}$ be the generalized solution of* (3.3) *and let u be the generalized solution of* (3.5)*. If*

$$\lim_{\varepsilon\to 0} \|W - W^{(\varepsilon)}\|_{L_\infty(G)} = 0,$$

then

$$\lim_{\varepsilon\to 0}\Big(\sup_{t\in(0,T)} \|u-u^{(\varepsilon)}\|_0^2(t) + \int_0^T \|u-u^{(\varepsilon)}\|_1^2(s)ds\Big) = 0.$$

Proof. Let $U^{(\varepsilon)} = u - u^{(\varepsilon)}$. Then (3.4) and (3.5) imply that $U^{(\varepsilon)}$ is the generalized solution of

$$U_t^{(\varepsilon)} = (U_x^{(\varepsilon)} + U^{(\varepsilon)}W)_x - U_x^{(\varepsilon)}W + (u^{(\varepsilon)} - u_x^{(\varepsilon)})\cdot(W - W^{(\varepsilon)}),\ U|_{t=0} = 0.$$

Recall that C_0 denotes the common upper bound on the L_∞ norms of W and $W^{(\varepsilon)}$. By Proposition 3.1,

$$\sup_{t\in(0,T)} \|U^{(\varepsilon)}\|_0^2(t) + \int_0^T \|U^{(\varepsilon)}\|_1^2(s)ds$$
$$\le C_6(T,C_0)\|W - W^{(\varepsilon)}\|_{L_\infty(G)} \int_0^T \|u^{(\varepsilon)}\|_1^2(t)dt,$$

and

$$\int_0^T \|u^{(\varepsilon)}\|_1^2(t)dt \le C_7(T,C_0)\|\varphi\|_0^2,$$

completing the proof. □

Remark 3.4. By construction, a classical solution of (1.5) is automatically a generalized solution. Then Theorems 2.3 and 3.3 imply that a classical GRP solution of (1.1), if exists, coincides with the generalized GRP solution.

We will now construct the fundamental GRP solution of equation (1.1). Let $\mathbf{L} : H_0^1(G) \to H^{-1}(G)$ be the operator

$$h \mapsto -\big(h_x + hW\big)_x + h_x W.$$

Then (3.5) becomes

$$u_t = -\mathbf{L}u.$$

It is known [7] that, for every $W \in L_\infty((0,\pi))$,

- The operator $\mathbf{L}$ has discrete pure point spectrum;
- The eigenvalues $\lambda_k, k \geq 1$, satisfy

$$\lim_{k\to\infty} \frac{\lambda_k}{k^2} = 1;$$

- The corresponding eigenfunctions $\mathfrak{m}_k$ belong to $H_0^1((0,\pi))$;
- The collection $\{\mathfrak{m}_k,\ k \geq 1\}$ can be chosen to form an orthonormal basis in $L_2((0,\pi))$.

Then, using the functions $\mathfrak{m}_k$ in the definition of the generalized solution of (3.5), we conclude by Theorem 3.2 that

$$u(t,x) = \sum_{k=1}^{\infty} e^{-\lambda_k t} (\varphi, \mathfrak{m}_k)_0\, \mathfrak{m}_k(x). \tag{3.6}$$

Equality (3.6) suggests calling the function

$$\mathfrak{p}(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \mathfrak{m}_k(x)\mathfrak{m}_k(y)$$

the *fundamental GRP solution* of (1.1). By Theorem 2.3,

$$\mathfrak{p}(t,x,y) = e^{-\int_0^x W(s)ds}\, \mathfrak{p}_W(t,x,y)\, e^{\int_0^y W(s)ds},$$

where $\mathfrak{p}_W$ is the fundamental solution of (2.4).

4. Further directions

The following extensions are straightforward:

(1) Classical GRP solution for the equation

$$u_t = au_{xx} + bu_x + cu + f + u\dot{W} \tag{4.1}$$

$t \in (0,T)$, $x \in (L_1, L_2)$ with Hölder continuous, in t, x, functions a, b, c, f, with $\inf_{t,x} a(t,x) > 0$, and with *separated* boundary conditions

$$p_1 u(t,L_1) + p_2 u_x(t,L_1) = 0,\ \ p_3 u(t,L_2) + p_4 u_x(t,L_2) = 0,$$

because the results from [5, Chapter 10] and [6, Chapter IV] about solvability of parabolic equations with Hölder continuous coefficients still apply.

Note that the change of the unknown function

$$u(t,x) = v(t,x) \exp\left(-\int_{L_1}^x W(s)ds \right) \tag{4.2}$$

affects the boundary conditions by changing some of the coefficients p_k.

(2) Generalized GRP solution for equation (4.1) or for equation

$$u_t = \big(au_x + \tilde{a}u\big)_x + bu_x + cu + f + u\dot{W},$$

with zero Dirichlet boundary conditions.

The following extensions are most likely to be possible as well, but, because the standard parabolic regularity results (e.g. those in [5,6]) do not apply, much more effort could be necessary:

(1) Classical GRP solution for (4.1) with general boundary conditions

$$\begin{aligned} &p_{11}u(t,L_1) + p_{12}u_x(t,L_1) + p_{13}u(t,L_2) + p_{14}u_x(t,L_2) = 0,\\ &p_{21}u(t,L_1) + p_{22}u_x(t,L_1) + p_{23}u(t,L_2) + p_{24}u_x(t,L_2) = 0. \end{aligned} \tag{4.3}$$

Understanding (4.3) is necessary, for example, to study (1.1) with periodic boundary conditions

$$u(t,L_1) = u(t,L_2),\ \ u_x(t,L_1) = u_x(t,L_2),$$

after the change of the unknown function according to (4.2).

(2) Generalized GRP solution with boundary conditions other than zero Dirichlet: complications start at the integration by parts stage.

(3) Equation (1.1) on the $(-\infty,+\infty)$ or $(0,+\infty)$ without assuming that W is bounded: while Hölder regularity is essentially a local property and should be expected to hold, there are technical difficulties related to the analysis of equation (2.4) unless there is an additional assumption that W is uniformly bounded.

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