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MODULE AMENABILITY AND MODULE ARENS REGULARITY OF WEIGHTED SEMIGROUP ALGEBRAS

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Abstract. For every inverse semigroup S with subsemigroup E of idempotents, necessary and sufficient conditions are obtained for the weighted semigroup algebra $l^1(S,\omega)$ and its second dual to be $l^1(E)$-module amenable. Some results for the module Arens regularity of $l^1(S,\omega)$ (as an $l^1(E)$-module) are found. If S is either of the bicyclic inverse semigroup or the Brandt inverse semigroup, it is shown that $l^1(S,\omega)$ is module amenable but not amenable for any weight ω.

1. Introduction

The Beurling (weighted) algebra $L^1(G,\omega)$ of a locally compact group G was studied extensively; for instance [13]. The basic idea is to consider a continuous weight ω and change the norm of $L^1(G)$ by a factor of ω. For Beurling algebras, Grønbæk [17] showed that $L^1(G,\omega)$ is amenable if and only if G is amenable and the function $\omega(t)\omega(t^{-1})$ is bounded on G. It is proved in [19, Theorem 3.2] that the algebra $l^1(G,\omega)$ is amenable if and only if there exists a positive left invariant mean on $l^\infty(G,\omega^{-1})$. Furthermore, weak amenability of commutative Beurling algebras has been studied by Zhang in [30]. In [12], Craw and Young investigated the Arens regularity of weighted group algebras. They proved that a locally compact group G has a weight ω such that $L^1(G,\omega)$ is Arens regular if and only if G is discrete and countable. They also characterized the Arens regularity of weighted group algebras with respect to one feature of the (group) weight, called 0-clusterness. After that, Rejali in [24] showed that $L^1(G,\omega)$ is Arens regular if and only if G is finite or G is discrete and 0-cluster. However, it is known that every countable semigroup admits a weight ω for which the semigroup algebra $l^1(S,\omega)$ is Arens regular and no uncountable group admits such a weight [12].

The concept of module amenability for a class of Banach algebras which is in fact a generalization of the classical amenability has been developed by

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Amini in [2]. He showed that for every inverse semigroup S with subsemigroup E of idempotents, the $l^1(E)$-module amenability of $l^1(S)$ is equivalent to the amenability of S. In other words, that is the celebrated Johnson's theorem [19] (in the discrete case) asserts that a discrete group G is amenable if and only if the Banach algebra $l^1(G)$ is amenable. Generalized module amenability related to the homomorphisms was investigated by the second author in [6]. Other notions of module amenability for Banach algebras can be found in [7], [8], [9], [10], [21] and [23]. The notion of Arens regularity for a Banach algebra $\mathcal{A}$ is developed to module version in [25]. In that paper, the authors proved that when S is an inverse semigroup with totally ordered subsemigroup E of idempotents, then $l^1(S)$ is module Arens regular if and only if an appropriate group homomorphic image of S is finite. Later, Aghababa [21] showed that the hypothesis on E being totally ordered is redundant and it can be eliminated. Bodaghi and Jabbari investigated module amenability, permanent weak module amenability and module Arens regularity of the triangular Banach algebras in [11]. In addition, module Arens regularity for module actions and duals of Banach algebras are studied in [4].

In this paper, for an inverse semigroup S equipped to the corresponding weight ω with the set of idempotents E, we show that when $l^1(E)$ acts on $l^1(S,\omega)$ trivially from left and by multiplication from right, the weighted semigroup algebra $l^1(S,\omega)$ is $l^1(E)$-module amenable if and only if S is amenable and sup $\{\Omega(s) : s \in S\} < \infty$, where $\Omega(s) = \omega(s)\omega(s^*)$ for all $s \in S$. Furthermore, we show that $l^1(S,\omega)^{**}$ is module amenable if and only if the maximal group homomorphic image $\mathcal{G}_S = S/\approx$ is finite, where $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set $\{\delta_{set} - \delta_{st} : s,t \in S,\ e \in E\}$. This could be regarded as the module version (for inverse semigroups) of a result of Ghahramani and Zabandan [16] which asserts that for any locally compact group G, $L^1(G,\omega)^{**}$ is amenable if and only if G is finite. Finally, we investigate the module Arens regularity of $l^1(S,\omega)$.

2. Module amenability

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ (ab) \cdot \alpha = a(b \cdot \alpha)$$

for all $a, b \in \mathcal{A}$, $\alpha \in \mathfrak{A}$. Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x,\ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,\ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$$

for all $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$, $x \in X$ and similarly for the right and two-sided actions. Then, we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach

$\mathcal{A}$-$\mathfrak{A}$-module. Indeed, if $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A ($\mathfrak{A}$-)*module derivation* is a bounded $\mathfrak{A}$-module map $D : \mathcal{A} \longrightarrow X$ satisfying

$$D(ab) = D(a)\cdot b + a\cdot D(b),\ \ D(\alpha\cdot a) = \alpha\cdot D(a),\ \ D(a\cdot\alpha) = D(a)\cdot\alpha,\ \ (a, b \in \mathcal{A}).$$

One should note that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \le M\|a\|$ for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When X is commutative, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

These are called *inner* $\mathfrak{A}$-module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [2]. Note that if $\mathfrak{A} = \mathbb{C}$, then the module amenability will absolutely overlap with Johnson's amenability [19] for a Banach algebra.

Consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Then, J is an $\mathcal{A}$-submodule and an $\mathfrak{A}$-submodule of $\mathcal{A}$. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with the compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.

A (continuous) function ω from a semigroup S to $(0, \infty)$ is called a weight function if $\omega(st) \le \omega(s)\omega(t)$ for all $s, t \in S$. Let us consider the spaces

$$l^1(S, \omega) = \left\{ f = \sum_{s\in S} \lambda_s\delta_s : \|f\|_{\omega,1} = \sum_{s\in S} |\lambda_s|\omega(s) < \infty \right\},$$

where δ_s is the characteristic function of the singleton $\{s\}$. The dual space of $l^1(S, \omega)$ under the canonical pairing is

$$l^\infty(S, \omega^{-1}) = \left\{ f = \sum_{s\in S} \lambda_s\delta_s : \|f\|_{\omega^{-1},\infty} = \sup_{s\in S} \frac{|f(s)|}{\omega(s)} < \infty \right\}.$$

For a discrete group G, we have the same definition for the weight function with further condition that $\omega(e) = 1$, where e is the identity element of G.

An inverse semigroup is a semigroup S so that, for each $s \in S$, there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The element s^* is termed the inverse of s. The set $E(S)$ (or briefly, E) of idempotents of S is a commutative subsemigroup; it is ordered by $e \le f$ if and only if $ef = e$. With this ordering $E(S)$ is a meet semilattice with the meet given by the product; see [18, Theorem 5.1.1]. We recall that a semigroup S is a *semilattice* if S is commutative and $E = S$. The order on E extends to S as the so-called natural partial order by putting $s \le t$ if $s = et$ for some idempotent e (or equivalently $s = tf$ for some idempotent f). This is equivalent to $s = ts^*s$ or $s = ss^*t$. If $e \in E$, then the set $G_e = \{s \in S : ss^* = e = s^*s\}$ is a group, called the *maximal subgroup* of S at e.

Let ω be a weight on S. Then, for each $e \in E$, we have $\omega(e) = \omega(e^2) \leq \omega(e)^2$ and so $\omega(e) \geq 1$. Throughout this paper, we assume that ω is a weight on the discrete inverse semigroup S such that $\omega(e) = 1$ for all $e \in E$ unless otherwise stated explicitly. So, the Banach algebra $l^1(E, \omega)$ coincides with $l^1(E)$.

Here, let us show by an example that the weight ω on S satisfying $\omega(e) = 1$ for all $e \in E$ exists.

Example 2.1. Let $\mathcal{S}$ be the bicyclic inverse semigroup generated by p and q, that is

$$\mathcal{S} = \{p^a q^b : a, b \geq 0\}, \quad (p^a q^b)^* = p^b q^a.$$

The multiplication operation is defined by

$$(p^a q^b)(p^c q^d) = p^{a-b+\max\{b,c\}} q^{d-c+\max\{b,c\}}.$$

The set of idempotents of $\mathcal{S}$ is $E_{\mathcal{S}} = \{p^a q^a : a = 0, 1, \ldots\}$ which is also totally ordered with the following order

$$p^a q^a \leq p^b q^b \Longleftrightarrow a \leq b.$$

Define $\omega_\alpha(p^a q^b) = (1 + |b - a|)^\alpha$, where $\alpha \geq 0$. Then, ω_α is a weight on $\mathcal{S}$ which $\omega_\alpha(e) = 1$ for all $e \in E_{\mathcal{S}}$. Indeed, $\omega_\alpha(p^a q^a) = 1$ and

$$\begin{aligned}
\omega_\alpha((p^a q^b)(p^c q^d)) &= (1 + |b - a + d - c|)^\alpha \\
&\leq (1 + |b - a| + |d - c|)^\alpha \\
&\leq (1 + |b - a| + |d - c| + |b - a||d - c|)^\alpha \\
&\leq \omega_\alpha(p^a q^b)\omega_\alpha(p^c q^d).
\end{aligned}$$

If also we consider $\omega_\beta(p^a q^b) = \mathrm{Exp}(|a - b|^\beta)$ where $0 \leq \beta \leq 1$ and Exp is the exponential function, then it is easy to check that ω_β is a weight on $\mathcal{S}$ that $\omega_\beta(p^a q^a) = 1$.

Let S be a (discrete) inverse semigroup with the set of idempotents E. We recall that the subsemigroup E of S is a semilattice, and so $l^1(E)$ could be regarded as a commutative subalgebra of $l^1(S, \omega)$. Thus, $l^1(S, \omega)$ is a Banach algebra and a Banach $l^1(E)$-module with compatible actions [2]. We consider the following actions of $l^1(E)$ on $l^1(S, \omega)$:

$$\delta_e \cdot \delta_s = \delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, \ e \in E). \tag{2.1}$$

In this case, the ideal J is the closed linear span of $\{\delta_{set} - \delta_{st} : s, t \in S, \ e \in E\}$.

We consider an equivalence relation on S such that $s \approx t$ if and only if $\delta_s - \delta_t \in J$ for $s, t \in S$. It is shown in [21] that the quotient $S/\approx$ is a discrete group (see also [3]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image $\mathcal{G}_S$ [20] of S [22]. Moreover, S is amenable if and only if $\mathcal{G}_S = S/\approx$ is amenable ([14], [20]). Suppose that ω is a weight on inverse semigroup S such that $\inf_{s \in S} \omega(s) > 0$. Define the function $\overline{\omega} : \mathcal{G}_S \longrightarrow (0, \infty)$ by $\overline{\omega}([s]) :=$

$\inf\{\omega(t) : t \approx s\}$, where $[s]$ is the equivalence class of s. Then, $\overline{\omega}$ is a weight on $\mathcal{G}_S$. Indeed, for any $s, t \in S$, we have

$$\{x \in S : x = s't' \text{ for some } s' \approx s \text{ and } t' \approx t\} \subseteq \{x \in S : x \approx st\}$$

which implies that

$$\overline{\omega}([s][t]) = \overline{\omega}([st]) = \inf_{x \approx st}\omega(x) \leq \inf_{t' \approx t}\omega(t')\inf_{s' \approx s}\omega(s') = \overline{\omega}([s])\overline{\omega}([t]).$$

From now on, by a weight on $\mathcal{G}_S$, we mean that it is $\overline{\omega}$ which is induced by the weight ω on S as the above. In light of the above discussions, we consider the following weighted group algebra.

$$l^1(\mathcal{G}_S, \overline{\omega}) = \{\overline{f} = \sum_{s \in S} \lambda_s \delta_{[s]} : \|\overline{f}\|_{\overline{\omega},1} = \sum_{s \in S} |\lambda_s| \overline{\omega}([s]) < \infty\}.$$

Here, $l^1(S, \omega)/J$ is a commutative $l^1(E)$-bimodule with the following actions:

$$\delta_e \cdot \delta_{[s]} = \delta_{[s]}, \; \delta_{[s]} \cdot \delta_e = \delta_{[se]} \quad (s \in S, e \in E). \tag{2.2}$$

Similar to Theorem 3.3 from [25], we have the following result.

Proposition 2.2. *With the above notations, $l^1(\mathcal{G}_S, \overline{\omega})$ is a quotient of $l^1(S, \omega)$ and so the actions* (2.1) *of $l^1(E)$ on $l^1(S, \omega)$ lifts to the actions* (2.2) *of $l^1(E)$ on $l^1(\mathcal{G}_S, \overline{\omega})$, making it a Banach $l^1(E)$-module. In particular $l^1(S, \omega)/J \cong l^1(\mathcal{G}_S, \overline{\omega})$.*

Proof. Since $\overline{\omega}([s]) = \inf_{s \approx t} \omega(t)$ for all $s \in S$, we have $\|\overline{f}\|_{\overline{\omega},1} \leq \|f\|_{\omega,1}$. So, the quotient map $\pi : S \longrightarrow \mathcal{G}_S$, extends linearly to a continuous epimorphism $\pi : l^1(S, \omega) \longrightarrow l^1(\mathcal{G}_S, \overline{\omega})$ of Banach algebras with kernel J. □

To achieve our purpose in this section, we need the following results which are proved in [21, Lemma 2.8] and [3, Proposition 3.2], respectively.

Lemma 2.3. *The Banach algebra $\mathcal{A}$ is* ($\mathfrak{A}$-)*module amenable if and only if $\mathcal{A}/J$ is* ($\mathfrak{A}$-)*module amenable.*

Proposition 2.4. *Let $\mathcal{A}$ be module amenable as an $\mathfrak{A}$-module with trivial left action, and let J_0 be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has an identity, then $\mathcal{A}/J_0$ is amenable.*

We know that $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I$, where I is the closed ideal generated by elements of the form $a \cdot \alpha \otimes b - a \otimes \alpha \cdot b$, for $\alpha \in \mathfrak{A}$, $a, b \in \mathcal{A}$. We define $\widetilde{\pi} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I \longrightarrow \mathcal{A}/J$ by

$$\widetilde{\pi}(a \otimes b + I) = ab + J \quad (a, b \in \mathcal{A}).$$

Recall from [2] that a bounded net $\{\xi_j\}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a *module approximate diagonal* if $\widetilde{\pi}(\xi_j)$ is a bounded approximate identity of $\mathcal{A}/J$ and

$$\lim_j \|\xi_j \cdot a - a \cdot \xi_j\| = 0 \quad (a \in \mathcal{A}).$$

It is shown in [2, Theorem 2.1] that $\mathcal{A}$ has a module approximate diagonal if and only if $\mathcal{A}$ is module amenable and $\mathcal{A}/J$ has a bounded approximate identity (see also [21, Theorem 2.1]). We use from this fact in the proof of next theorem.

Let ω and $\overline{\omega}$ be as in the above. We define two notations as follows:

$$\Omega(s) := \omega(s)\omega(s^*), \quad \overline{\Omega}([s]) := \overline{\omega}([s])\overline{\omega}([s^*]) \qquad (s \in S).$$

Theorem 2.5. *Let S be an inverse semigroup. Consider the following assertions.*

(i) S *is amenable and* $\sup\{\Omega(s) : s \in S\} < \infty$;

(ii) $\mathcal{G}_S$ *is amenable and* $\sup\{\overline{\Omega}([s]) : s \in S\} < \infty$;

(iii) $l^1(\mathcal{G}_S, \overline{\omega})$ *is amenable*;

(iv) $l^1(\mathcal{G}_S, \overline{\omega})$ *is module amenable* (*as a* $l^1(E)$*-module*) *with trivial left action*;

(v) $l^1(S, \omega)$ *is module amenable* (*as a* $l^1(E)$*-module*) *with actions* (2.1).

Then, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). *If $\omega(s) = \omega(t)$ for each $s, t \in S$ such that $s \approx t$, then all assertions are equivalent.*

Proof. For any $s \in S$, we have $\overline{\Omega}([s]) = \overline{\omega}([s])\overline{\omega}([s^*]) = \inf_{s\approx t}\omega(t)\inf_{s^*\approx t}\omega(t) \leq \omega(s)\omega(s^*) = \Omega(s)$. Now, the implication (i) $\Rightarrow$ (ii) follows from [14, Theorem 1] and [22, Theorem 1.3].

(ii) $\Leftrightarrow$ (iii) Since $\mathcal{G}_S$ is a group, that is trivial by Theorem 0 of [17].

(iii) $\Leftrightarrow$ (iv) It follows from (2.2) that $l^1(E)$ acts trivially on $l^1(\mathcal{G}_S, \overline{\omega})$ from the both sides. This shows that

$$l^1(\mathcal{G}_S, \overline{\omega})\widehat{\otimes}_{l^1(E)}l^1(\mathcal{G}_S, \overline{\omega}) \cong l^1(\mathcal{G}_S, \overline{\omega})\widehat{\otimes}l^1(\mathcal{G}_S, \overline{\omega}),$$

and so every module approximate diagonal is an approximate diagonal and vice versa. So the result follows from by [21, Theorem 2.1] and [28, Theorem 2.2.4].

(iv) $\Leftrightarrow$ (v) This part is a direct consequence of Lemma 2.3 when $\mathcal{A} = l^1(S, \omega)$ and $\mathfrak{A} = l^1(E)$.

If $\omega(s) = \omega(t)$ for each $s, t \in S$ such that $s \approx t$, then (ii) implies (i) by [14, Theorem 1]. $\square$

In the following examples, we show that the statements of Theorem 2.5 can be happen for an arbitrary weight ω.

Example 2.6. Let $\mathcal{S}$ be the bicyclic inverse semigroup introduced in Example 2.1. It is not hard to show that

$$p^aq^b \approx p^{a'}q^{b'} \Longleftrightarrow b - a = b' - a'.$$

Consider the weight $\omega_\beta(p^aq^b) = \mathrm{Exp}(|a-b|^\beta)$ on $\mathcal{S}$ where $0 \leq \beta \leq 1$. It is easy to check that for any $s, t \in \mathcal{S}$, $\omega_\beta(s) = \omega_\beta(t)$ and $\omega_\beta(s^*) = \omega_\beta(t^*)$ if $s \approx t$. Indeed, $\mathcal{G}_\mathcal{S}$ is isomorphic to the group of integers $\mathbb{Z}$ [3]. In other words, $\approx$ is a congruence on S. So, if $p^aq^b \approx p^{a'}q^{b'}$ and $p^cq^d \approx p^{c'}q^{d'}$, then $(p^aq^b)(p^cq^d) \approx (p^{a'}q^{b'})(p^{c'}q^{d'})$. Define the function $\overline{\omega}_\beta : \mathcal{G}_\mathcal{S} \longrightarrow (0, \infty)$ by $\overline{\omega}_\beta([s]) = \omega_\beta(s)$. By the above statements, $\overline{\omega}_\beta$ is well-defined and a weight on $\mathcal{G}_\mathcal{S}$. Since $l^1(\mathcal{G}_\mathcal{S}, \overline{\omega}_\beta)$ is amenable by the celebrated Johnson's theorem and this fact that commutative groups are amenable, $l^1(\mathcal{S}, \omega_\beta)$ is module amenable. On the other hand, $E(\mathcal{S})$ is infinite and thus $l^1(\mathcal{S}, \omega)$ is not amenable for any weight ω [15, Corollary 1]. Similarly, one can show that $l^1(\mathcal{S},\omega_\alpha)$ is module amenable where ω_α is defined in Example 2.1.

Example 2.7. Let G be a group with identity e, and let I be a non-empty set. Then, the Brandt inverse semigroup corresponding to G and I, denoted by $S=\mathcal{M}(G,I)$, is the collection of all $I\times I$ matrices $(g)_{ij}$ with $g\in G$ in the $(i,j)^{\text{th}}$ place and 0 (zero) elsewhere and the $I\times I$ zero matrix 0. Multiplication in S is given by the formula

$$(g)_{ij}(h)_{kl}=\begin{cases}(gh)_{il} & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}\qquad (g,h\in G,\ i,j,k,l\in I),$$

and $(g)_{ij}^*=(g^{-1})_{ji}$ and $0^*=0$. The set of all idempotents is $E_S=\{(e)_{ii}: i\in I\}\bigcup\{0\}$. Assume that ω is an arbitrary weight on S. It is shown in [21, Example 3.2] that $(g)_{ij}\approx 0$ for all $g\in G$ and $i,j\in J$ and hence $\mathcal{G}_S$ is the trivial group. This means that $\overline{\omega}$ is a weight on $\mathcal{G}_S$ and in fact it is a constant function. Since $l^1(\mathcal{G}_S,\overline{\omega})$ is amenable (because $\mathcal{G}_S$ is amenable), $l^1(S,\omega)$ is module amenable by Theorem 2.5. But if the index set I is infinite, then E_S is infinite and thus $l^1(S,\omega)$ is not amenable for any weight ω [15, Corollary 1].

A (continuous) function ϕ from a semigroup S to $(0,\infty)$ is called a *character* if $\phi(st)=\phi(s)\phi(t)$ for all $s,t\in S$. A character ϕ on an inverse semigroup S is called $*$-*character* if

$$\phi((st)^*)=\phi(t^*s^*)\qquad (s,t\in S). \tag{2.3}$$

Clearly, for $*$-character ϕ, $\phi(e)=1$ for all $e\in E$. We note that the function $\omega_\gamma(p^aq^b)=\operatorname{Exp}(a-b)$ defined on the bicyclic inverse semigroup $\mathcal{S}$ is a $*$-character.

In the upcoming example, we start with a $*$-character on S and make a weight on the group homomorphic image $G_S\cong S/\propto$ where $\propto$ is a congruence on S.

Example 2.8. Let S be an inverse semigroup and ϕ be a $*$-character on S. We consider an equivalence relation on S as follows:

$$s\propto t \text{ if and only if } \delta_s-\delta_t\in J \text{ and } \phi(s)=\phi(t),\ \phi(s^*)=\phi(t^*).$$

For each $s\in S$ if $e=s^*s$, then $s=se$, and so $\phi(se)=\phi(s)$. Similarly, $\phi(es)=\phi(s)$ where $e=ss^*$. Since E is a semilattice, for given $e,f\in E$, we have $ef\in E$ and $ef\leq e,f$, and then the argument in [25, Lemma 3.2] shows that $e\propto ef\propto f$ and so $e\propto f$ by transitivity. Let $s_1\propto t_1$ and $s_2\propto t_2$. Then

$$\delta_{s_1}-\delta_{t_1}\in J,\ \delta_{s_2}-\delta_{t_2}\in J.$$

Since J is ideal, we have

$$\delta_{s_1s_2}-\delta_{s_1t_2}\in J,\ \delta_{s_1t_2}-\delta_{t_1t_2}\in J.$$

So, $\delta_{s_1s_2}-\delta_{t_1t_2}\in J$. It is easy to check that $\phi(s_1s_2)=\phi(t_1t_2)$ and $\phi(s_1s_2)^*)=\phi((t_1t_2)^*)$. This implies that $G_S=S/\propto$ is a group. Define the function $\overline{\phi}$:

$G_S \longrightarrow (0,\infty)$ through $\overline{\phi}([s]) = \phi(s) + \phi(s^*)$, where $[s]$ denotes the equivalence class of s in G_S. Clearly, $\overline{\phi}$ is well-defined. Furthermore,

$$\begin{aligned}\overline{\phi}([s][t]) &= \overline{\phi}([st]) \\ &= \phi(st) + \phi((st)^*) \\ &= \phi(s)\phi(t) + \phi(s^*)\phi(t^*) \\ &\leq (\phi(s) + \phi(s^*))(\phi(t) + \phi(t^*)) \\ &\leq \overline{\phi}([s])\overline{\phi}([t]).\end{aligned}$$

Therefore, $\overline{\phi}$ is a weight on discrete group G_S. Obviously, $\omega_0(s) := \overline{\phi}([s])\,(s \in S)$ is well-defined and a weight on S. Similar to Proposition 2.2, one can show that $l^1(S,\omega_0)/J \cong l^1(G_S,\overline{\phi})$. Now, all hypotheses of Theorem 2.5 holds. Therefore, the following assertions are equivalent:

(i) S is amenable and $\sup\,\{\Omega_0(s) : s \in S\} < \infty$ where $\Omega_0(s) := \omega_0(s)\omega_0(s^*)$;
(ii) $l^1(S,\omega)$ is module amenable (as a $l^1(E)$-module).

Suppose that $l^1(E)$ acts on $l^1(S)$ by the canonical actions, that is

$$\delta_e \bullet \delta_s = \delta_{es}, \quad \delta_s \bullet \delta_e = \delta_{se} \qquad (s \in S, e \in E). \tag{2.4}$$

It is clear that $l^1(S)$ is an $l^1(E)$-bimodule with compatible actions (2.4). In addition, consider $\mathcal{J}_c$ as the closed ideal of $l^1(S)$ generated by $\{\delta_{se} - \delta_{es} : s \in S, e \in E\}$. We consider an equivalence relation corresponding to $\sim$ on S as follows:

$$s \sim t \ \text{ if and only if } \ \delta_s - \delta_t \in \mathcal{J}_c.$$

It follows from [18, Theorem 5.1.4] that $S/\sim$ is an inverse semigroup and by definition of $\mathcal{J}$, $S/\sim$ is a clifford inverse semigroup. Recall that S is a Clifford semigroup if it is an inverse semigroup in which every idempotent is central, that is $es = se$ for all $s \in S$ and $e \in E$. Similar to the proof of [29, Proposition 3.4], we can prove the next result.

Proposition 2.9. *Let S be an inverse semigroup and ω be a weight on S. If $l^1(S,\omega)$ is $l^1(E)$-module amenable with canonical actions* (2.4)*, then it is $l^1(E)$-module amenable with trivial left action. In particular, S is amenable.*

Let M be a subspace of a Banach space X. Then, its *annihilator*

$$M^\perp := \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}$$

is a w^*-closed subspace of X^*, where the w^*-topology of X^* is the topology induced by the pointwise convergence of nets on X. Similarly, we have

$$M^{\perp\perp} := \{\lambda \in X^{**} : \lambda(f) = 0 \text{ for all } f \in M^\perp\}.$$

Let M be a closed subspace of X. Then

$$X^*/M^\perp \cong M^* \quad \text{and} \quad (X/M)^* \cong M^\perp,$$

and so $(X/M)^{**} \cong X^{**}/M^{\perp\perp}$ (for the proofs refer to Theorems 4.7 and 4.9 of [27]).

We are now going back to the case that $X = l^1(S,\omega)$ and $M = J$. Then

$$J^\perp = \{h \in l^\infty(S,\omega) : h(set) = h(st) \quad (s,t \in S,\ e \in E)\}.$$

Theorem 2.10. *Let S be an inverse semigroup with the idempotents set E. Then, $l^1(S,\omega)^{**}$ is module amenable (as an $l^1(E)$-module with actions (2.1)) if and only if $\mathcal{G}_S$ is finite.*

Proof. Let $l^1(S,\omega)^{**}$ be module amenable. Let $\mathcal{N}$ be the closed ideal of $l^1(S,\omega)^{**}$ generated by $(F \cdot \delta_e)\Box G - F\Box(\delta_e \cdot G)$ for $F, G \in l^1(S,\omega)^{**}$ and $e \in E$. Obviously, $J \subseteq \mathcal{N}$. Similar to the proof of [3, Theorem 3.4], one can show that $\mathcal{N} \subseteq J^{\perp\perp}$. Since $l^1(S,\omega)^{**}/J^{\perp\perp} \cong l^1(\mathcal{G}_S,\overline{\omega})^{**}$ has an identity, $l^1(\mathcal{G}_S,\overline{\omega})^{**}$ is amenable by Proposition 2.4 when $\mathcal{A} = l^1(S,\omega)^{**}$, $J = \mathcal{N}$ and $J_0 = J^{\perp\perp}$. Now, it follows from [16, Theorem 2.4] that $\mathcal{G}_S$ is finite.

Conversely, is $\mathcal{G}_S$ is finite, then $l^1(\mathcal{G}_S,\overline{\omega})^{**}$ is amenable by [16, Theorem 2.4]. Suppose that X is a Banach $l^1(S,\omega)^{**}$-$l^1(E)$-module and $D : l^1(S,\omega)^{**} \longrightarrow X^*$ is a module derivation. We have $X \cdot \mathcal{N} = \mathcal{N} \cdot X = \{0\}$ and the map $\lambda : l^1(S,\omega)^{**}/\mathcal{N} \longrightarrow l^1(S,\omega)^{**}/J^{\perp\perp}; F+\mathcal{N} \mapsto F+J^{\perp\perp}$ is a module epimorphism. Since $\mathcal{N}$ is w^*-dense in $J^{\perp\perp}$, the module actions of $l^1(E)$ on $l^1(S,\omega)^{**}$ are w^*-w^*-continuous. Thus $X \cdot J^{\perp\perp} = J^{\perp\perp} \cdot X = \{0\}$. Hence X is a Banach $l^1(S,\omega)^{**}/J^{\perp\perp}$-bimodule, and the module derivation $\tilde{D} : l^1(S,\omega)^{**}/J^{\perp\perp} \longrightarrow X^*$ defined via $\tilde{D}(F+J^{\perp\perp}) := D(F)$ is well-defined. Since the left action $l^1(E)$ on $l^1(S,\omega)^{**}/J^{\perp\perp}$ is trivial, it follows from [8, Lemma 3.13] that $\tilde{D}$ is $\mathbb{C}$-linear and thus it is inner. In other words, D is an inner module derivation. $\Box$

3. Module Arens regularity of $l^1(S,\omega)$

In this section, we study the module Arens regularity of $l^1(S,\omega)$. Here, we bring the definition of module Arens regular Banach algebras which is introduced in [25] and [26].

Definition 3.1. $\mathcal{A}$ is called *module Arens regular* (as an $\mathfrak{A}$-module) if $\mathfrak{A}$-module homomorphisms R_λ $(R_\lambda : \mathcal{A} \longrightarrow \mathcal{A}^*; a \mapsto a \cdot \lambda)$ are weakly compact for any $\lambda \in J^\perp$ satisfying $\lambda(\alpha \cdot ab) = \lambda(ab \cdot \alpha)$, $\lambda(c(\alpha \cdot ab)) = \lambda(c(ab \cdot \alpha))$, $\lambda((\alpha \cdot ab)d) = \lambda((ab \cdot \alpha)d)$, and $\lambda(c(\alpha \cdot ab)d) = \lambda(c(ab \cdot \alpha)d)$ for $\alpha \in \mathfrak{A}$ and $a,b,c,d \in \mathcal{A}$.

Let X, Y be sets and $f : X \times Y \longrightarrow \mathbb{R}$ be a bounded function. We say that f is *cluster* on $X \times Y$ if for each pair of sequences $(x_n), (y_n)$ of distinct elements of X, Y, respectively,

$$\lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m) \tag{3.1}$$

whenever both sides of (3.1) exist. If f is cluster and both sides of (3.1) are zero in all cases, then f is 0-cluster on $X \times Y$ [5]. Note that if f is bounded, then for each pair of sequence $(a_n), (b_m)$ in $X \times Y$ we can find subsequence $(x_n), (y_m)$ respectively so that both sides of equation (3.1) exist.

Let $\mathcal{A}$ be an $\mathfrak{A}$-module. We say that $\lambda \in J^{\perp}$ is *module cluster* if for bounded sequences $(a_n), (b_m)$ in $\mathcal{A}$, we have

$$\lim_n \lim_m \lambda(a_n b_m) = \lim_m \lim_n \lambda(a_n b_m) \tag{3.2}$$

whenever both sides of (3.2) exist. Put $\mathcal{A}_1 = \{a \in \mathcal{A} : \|a\| = 1\}$, the unit ball of $\mathcal{A}$. It is easily verified that $\lambda \in J^{\perp}$ is module cluster if and only if for the sequence $(a_n), (b_m)$ in $\mathcal{A}_1$, the unit ball of $\mathcal{A}$, the equality (3.2) holds. From this fact and [25, Theorem 2.3], we have the following result.

Theorem 3.2. *A Banach algebra $\mathcal{A}$ is module Arens regular (as an $\mathfrak{A}$-module) if and only if for the sequences $(a_n), (b_m)$ in $\mathcal{A}_1$ and all $\lambda \in J^{\perp}$, (3.2) holds.*

Let ω be a weight on an inverse semigroup S. Here and subsequently, we denote by Γ for the map $\Gamma : S \times S \longrightarrow (0, 1]$ defined via $\Gamma(s,t) = \frac{\omega(st)}{\omega(s)\omega(t)}$ $(s, t \in S)$. We also use s for δ_s and vice versa if there is no ambiguity. Note that if $l^1(E)$ acts on $l^1(S, \omega)$ by actions (2.1), similar to [25, Theorem 3.3], it is easily verified that $\left(l^1(S, \omega)/J\right)^* = l^{\infty}(\mathcal{G}_S, \overline{\omega}^{-1})$ is a commutative $l^1(E)$-module.

Proposition 3.3. *Let Γ be as in the above. Then, the following statements are equivalent:*

(i) *$l^1(S, \omega)$ is module Arens regular as $l^1(E)$-module;*
(ii) *The map $(a, b) \mapsto \lambda(ab)\Gamma(a, b)$ is cluster on $S \times S$ for all $\lambda \in J^{\perp}$.*

Proof. We firstly note that J is a closed ideal in $l^1(S)$, $l^1(S, \omega)$ and $J^{\perp}$ is the same for both Banach algebras.

(i)$\Rightarrow$ (ii) Let $\lambda \in J^{\perp} = l^{\infty}(\mathcal{G}_S)$. Then, $\Lambda = \lambda\omega \in J^{\perp} = l^{\infty}(\mathcal{G}_S, \overline{\omega}^{-1})$. For any two sequences $(s_m), (t_n) \subseteq S$, we have $\|\frac{1}{\omega(s_m)}\delta_{s_m}\|_{\omega,1} = \|\frac{1}{\omega(t_n)}\delta_{t_n}\|_{\omega,1} = 1$. By assumption, we have

$$\begin{aligned}
\lim_n \lim_m \lambda(s_m t_n)\Gamma(s_m, t_n) &= \lim_n \lim_m \Lambda\left(\frac{1}{\omega(s_m)\omega(t_n)}\delta_{s_m t_n}\right)\\
&= \lim_n \lim_m \Lambda\left(\frac{1}{\omega(s_m)}\delta_{s_m} * \frac{1}{\omega(t_n)}\delta_{t_n}\right)\\
&= \lim_m \lim_n \Lambda\left(\frac{1}{\omega(s_m)}\delta_{s_m} * \frac{1}{\omega(t_n)}\delta_{t_n}\right)\\
&= \lim_m \lim_n \lambda(s_m t_n)\Gamma(s_m, t_n).
\end{aligned}$$

(ii)$\Rightarrow$ (i) It is enough to show that for each $\Lambda \in J^{\perp} = l^{\infty}(\mathcal{G}_S, \overline{\omega}^{-1})$,

$$\lim_n \lim_m \Lambda\left(\frac{1}{\omega(s_m)}\delta_{s_m} * \frac{1}{\omega(t_n)}\delta_{t_n}\right) = \lim_m \lim_n \Lambda\left(\frac{1}{\omega(s_m)}\delta_{s_m} * \frac{1}{\omega(t_n)}\delta_{t_n}\right). \tag{3.3}$$

On the other hand, one can choose $\lambda \in J^{\perp} = l^{\infty}(\mathcal{G}_S)$ such that $\Lambda = \lambda\omega$. Therefore, $l^1(S, \omega)$ is module Arens regular by the equality (3.3) and hypothesis. $\square$

Theorem 3.4. *Let S be an inverse semigroup with idempotents E. Then, $l^1(S,\omega)$ is module Arens regular if and only if one of the following statements holds*:

(i) $\mathcal{G}_S$ *is finite*;
(ii) Γ *is* 0*-cluster if* $\omega(s)=\omega(t)$ *for each* $s,t\in S$ *such that* $s\approx t$;

Proof. We firstly note that $l^1(S,\omega)/J\cong l^1(\mathcal{G}_S,\overline{\omega})$ is a commutative Banach $l^1(E)$-bimodule. It is also proved in [25, Theorem 2.4] that $\mathcal{A}$ is module Arens regular if and only if $\mathcal{A}/J$ is Arens regular (see also [4, Theorem 3.3]). Put $\widetilde{\Gamma}([s],[t])=\frac{\overline{\omega}([s][t])}{\overline{\omega}([s])\overline{\omega}([t])}$ for all $s,t\in S$. Assume that $l^1(S,\omega)$ is module Arens regular. Then, $l^1(\mathcal{G}_S,\overline{\omega})$ is Arens regular. It follows from [24] that $\mathcal{G}_S$ is finite. Once more, [24] implies that $\widetilde{\Gamma}$ is 0-cluster, that is

$$\lim_n\lim_m\overline{\omega}([s_m][t_n])=0=\lim_m\lim_n\overline{\omega}([s_m][t_n]).$$

It now follows from the assumption that Γ is 0-cluster.

For the converse, it is seen that if $\mathcal{G}_S$ is finite, then $l^1(S,\omega)$ is module Arens regular by [24] and [25, Theorem 2.4]. If Γ is 0-cluster, then by definition of $\overline{\omega}$, we have

$$\lim_n\lim_m\overline{\omega}([s_m][t_n])\leq\lim_n\lim_m\omega(s_mt_n)=0\qquad(s_m,t_n\in S).$$

Similarly, $\lim_m\lim_n\overline{\omega}([s_m][t_n])=0$. Using the result in [24] and Theorem 2.4 of [25], it concludes that $l^1(S,\omega)$ is module Arens regular. □

We close the paper by some examples.

Example 3.5. (i) Let $\mathcal{S}$ be the bicyclic inverse semigroup generated by p and q, as described in Section 2. Since $\mathcal{G}_{\mathcal{S}}$ is countable, $l^1(\mathcal{S},\omega)$ is module Arens regular by Theorem 3.4, [25, Theorem 2.3] and using this fact that $L^1(G,\omega)$ is Arens regular if and only if G is discrete and 0-cluster [24].

(ii) Let $S=\mathcal{M}(G,I)$ be the Brandt inverse semigroup corresponding to G and I as in Example 2.7. Since $\mathcal{G}_S$ is the trivial group, $l^1(S,\omega)$ is module Arens regular by Theorem 3.4. However, if S is infinite (this can be happen when I is infinite), $l^1(S)$ is not Arens regular [1].

(iii) Let $\mathbb{N}$ be the commutative semigroup of positive integers. Consider $(\mathbb{N},\vee)$ with maximum operation $m\vee n=\max\{m,n\}$, then each element of $\mathbb{N}_\vee$ is an idempotent, hence $\mathcal{G}_{\mathbb{N}}$ is the trivial group with one element. Thus, $\mathcal{A}=\mathfrak{A}=\ell^1(\mathbb{N}_\vee,\omega)$ is module Arens regular by Theorem 3.4. We note that ω can be chosen as $\omega(n)>1$ at least for an element $n\in\mathbb{N}_\vee$. For instance $\omega_1(n)=(1+n)^\alpha$ for $\alpha>0$ or $\omega_2(n)=\mathrm{Exp}n^\beta$ for $0\leq\beta\leq1$. Since $\mathcal{G}_{\mathbb{N}}$ is amenable, $\mathcal{A}$ is module amenable while it is not amenable by [15, Corollary 1] for any weight ω.

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which is induced by a weight ω on S can not be a weight) that improved the manuscript substantially.

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