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# ON PRIME SUBMODULES OF A FINITELY GENERATED PROJECTIVE MODULE OVER A COMMUTATIVE RING

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Abstract. In this paper we give a full characterization of prime submodules of a finitely generated projective module $M$ over a commutative ring $R$ with identity. Also we study the existence of primary decomposition of a submodule of a finitely generated projective module and characterize the minimal primary decomposition of this submodule. Finally, we characterize the radical of an arbitrary submodule of a finitely generated projective module $M$ and study submodules of $M$ which satisfy the radical formula.

## 0. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. We denote a unique factorization domain by UFD and a principal ideal domain by PID. Note that in a UFD, a greatest common divisor (GCD) of any collection of elements always exists. A proper submodule $P$ of an $R$-module $M$ is called $p$-prime if $rm \in P$ for $r \in R$ and $m \in M$ implies $m \in P$ or $r \in p = (P : M)$, where $(P : M) = \{r \in R \,|\, rM \subseteq P\}$. Let $N$ be a submodule of $M$ and $N = \bigcap_{i=1}^{k} N_i$ be a minimal primary decomposition of $N$ with $\sqrt{(N_i : M)} = p_i$. Then $Ass(N) = \{p_1, \ldots, p_k\}$. The radical of a submodule $N$ in an $R$-module $M$, $Rad_M N$, is defined to be the intersection of all prime submodules of $M$ containing $N$. If there is no prime submodule containing $N$, then $Rad_M N$ is defined to be $M$. In particular, $Rad_M M = M$. Let $M$ be an $R$-module and $N$ be a submodule of $M$. The envelope of $N$ in $M$ is defined to be the set $E_M(N) = \{rm \,|\, r \in R, m \in M; r^n m \in N \text{ for some } n \in \mathbb{N}\}$. We say that the submodule $N$ of an $R$-module $M$ satisfies the radical formula in $M$ ($N$ s.t.r.f. in $M$) if $Rad_M N = \langle E_M(N) \rangle$. An $R$-module $M$ is said to satisfy the radical formula if every submodule of $M$ satisfies the radical formula. Prime and primary submodules of a finitely generated free module over a PID were studied in [2, 3]. The authors in [2] described prime submodules of a finitely

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generated free module over a UFD and characterized the prime submodules of a free module of finite rank over a PID. In [5], the authors have given a full characterization of prime submodules of a finitely generated free $R$-module $F$, where $R$ is an arbitrary commutative ring with identity and they have extended some results obtained in [2], to a Dedekind domain. Also they studied the existence of primary decomposition of a submodule of $F$, where $R$ is an integral domain, and characterized its minimal primary decomposition and they used their results in a Dedekind domain. In [6], the authors characterized the radical of an arbitrary submodule of a finitely generated free $R$-module $F$ and study submodules of $F$ which satisfy the radical formula. In this paper we give a full characterization of prime submodules of a finitely generated projective module $M$ over a commutative ring $R$ with identity. Also we study the existence of primary decomposition of a submodule of a finitely generated projective module $M$ and characterize the minimal primary decomposition of this submodule. Finally, we characterize the radical of an arbitrary submodule $N$ of a finitely generated projective module $M$ and study submodules of $M$ which satisfy the radical formula.

## 1. Prime submodules of a finitely generated projective module

Let $X$ be a subset of an $R$-module $M$. We denote the submodule of $M$ that $X$ generates, by $\langle X\rangle$ or $RX$. We use the notation $R^n$ for $\underbrace{R\oplus\cdots\oplus R}_{n\text{-times}}$. Let $m$ and $n$ be positive integers, $A\in M_{m\times n}(R)$ and $F$ be the free $R$-module $R^n$. We shall use the notation $\langle A\rangle := \langle A_1,\ldots,A_m\rangle$ for the submodule $N$ of $F$ generated by the rows $A_1,\ldots,A_m$ of the matrix $A$ and the notation $(r_1,\ldots,r_m)A$, $r_i\in R$, for any element of $N$. Let $B\in M_{m\times m}(R)$. We denote the adjoint matrix of $B$ by $B'$, so that $BB' = B'B = (\det B)I_m$, where $I_m$ is the $m\times m$ identity matrix.

**Lemma 1.1.** *Let $R$ be a commutative ring with identity and $M$ be a finitely generated projective $R$-module. Then there exist $n\in\mathbb{N}$ and a matrix $A\in M_{n\times n}(R)$ such that $M\simeq\langle A\rangle$.*

*Proof.* Let $M=\langle x_1,\ldots,x_n\rangle$. There exists an epimorphism $\Phi : R^n\to M$ with $\Phi(e_i)=x_i$, where $e_i=(0,\ldots,0,1,0,\ldots,0)\in R^n$ with 1 as the ith component. Projectivity of $M$ gives a monomorphism $\Psi : M\to R^n$ with $\Phi\Psi = 1_M$. Now there exists a unique expression $\Psi(x_i)=\sum_{j=1}^n r_{ij}e_j$ for each $x_i$ $(1\le i\le n)$. Let $A=[r_{ij}]\in M_{n\times n}(R)$. Since for every $t_i\in R$ $(1\le i\le n)$, $\Psi(\sum_{i=1}^n t_ix_i)=(t_1,\ldots,t_n)A$, we get $\Psi(M)=\langle A\rangle$ and hence $M\simeq\langle A\rangle$. □

In the rest of this paper we use the following notations.

i) Let $T_i = (t_{i1}, \ldots, t_{in}) \in F = R^n$ for some $t_{ij} \in R$, $1 \le i \le m$, $1 \le j \le n$. We put

$$B = [T_1 \cdots T_m] := \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix} \in M_{m\times n}(R).$$

Thus the $j$th row of the matrix $[T_1 \cdots T_m]$ consists of the components of element $T_j$ in $F$. We use $W$ to be a non-zero submodule of $F$ with generating set $\xi = \{T_i = (t_{i1}, \ldots, t_{in}) \in F \,|\, i \in \Omega\}$, where $\Omega(\subseteq \mathbb{N})$ is an index set with $|\ \Omega\ | < \infty$. When $|\ \Omega\ | \geqslant n$, we define $\Re_\xi = \sum_{i_1,\ldots,i_n \in \Omega} RD_{i_1\cdots i_n}$, where $D_{i_1\cdots i_n} = \det[T_{i_1} \cdots T_{i_n}]$.

For example, let $R = \mathbb{Z}$, $F = R^2$ and $\xi = \{T_1 = (1,1),\ T_2 = (2,0),\ T_3 = (2,6)\}$. Then $D_{12} = \det[T_1T_2] = \det\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right) = -2$, $D_{13} = \det[T_1T_3] = \det\left(\begin{smallmatrix} 1 & 1 \\ 2 & 6 \end{smallmatrix}\right) = 4$ and $D_{23} = \det[T_2T_3] = \det\left(\begin{smallmatrix} 2 & 0 \\ 2 & 6 \end{smallmatrix}\right) = 12$. Now we have $\Re_\xi = \langle -2, 4, 12\rangle = 2\mathbb{Z}$.

Also $B(j_1, \ldots, j_k) \in M_{m\times k}(R)$ denotes a submatrix of $B \in M_{m\times n}(R)$ consisting of the columns $j_1, \ldots, j_k \in \{1, \ldots, n\}$ of $B$.

ii) Let $M = \langle x_1, \ldots, x_n\rangle$ be a projective $R$-module and $F = R^n$. By Lemma 1.1, there exist an $R$-module monomorphism $\Psi : M \to R^n$ and a unique matrix $A \in M_{n\times n}(R)$ such that $\Psi(M) = \langle A\rangle$ and $M \simeq \langle A\rangle$. Put $\alpha = \{x_1, x_2, \ldots, x_n\}$ and $\beta = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots,\ 0) \in F$ with 1 as the ith component and $\Psi(x_i) = \sum_{j=1}^n r_{ij}e_j$. We will use the notation $[\Psi]_\alpha^\beta := A$. Let $M = \langle x'_1, \ldots, x'_m\rangle$, $\alpha' = \{x'_1, \ldots, x'_m\}$ and $\beta' = \{e_1, \ldots, e_m\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^m$ with 1 as the ith component. Put $A' = [\Psi']_{\alpha'}^{\beta'}$, where $\Psi' : M \to R^m$. Since $M \simeq \langle A\rangle$ and $M \simeq \langle A'\rangle$, we have $\langle A\rangle \simeq \langle A'\rangle$.

iii) With the same notations as in parts (i) and (ii), let $\eta := \{y_i \in M \,|\, i \in \Omega\}$, where $y_i = \sum_{j=1}^n t_{ij}x_j$ and $N := \langle \eta\rangle$ be a submodule of $M$. We put $\xi(A) := \{T_iA \in \langle A\rangle \,|\, i \in \Omega\}$.

Let $R$ be a commutative ring with identity, $a \in R$ and $I$ be an ideal of $R$. We put $(I : a) = \{r \in R \,|\, ra \in I\}$. Clearly, $(I : a)$ is an ideal of $R$.

**Lemma 1.2.** *Let $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta\rangle$. Then*

i) $\Re_{\xi(A)} \subseteq (N : M) \subseteq \sqrt{(\Re_{\xi(A)} : \det A)}$.

ii) *If $N$ is a prime submodule of $M$, then*

$$\sqrt{\Re_{\xi(A)}} \subseteq (N : M) \subseteq \sqrt{(\Re_{\xi(A)} : \det A)}.$$

iii) *Let $R$ be a domain, $N = \langle y_1, \ldots, y_m\rangle$ $(m < n)$, and* $\det A \neq 0$. *Then* $(N : M) = 0$.

*Proof.* i) Let $T = \langle \xi(A)\rangle$ be a submodule of $F = R^n$. Since $\Psi$ is a monomorphism, $N \simeq T$ and hence $(T : \langle A\rangle) = (N : M)$. Now by [5, Lemma 1.1], we have $\Re_{\xi(A)} \subseteq (T : F)$. So $\Re_{\xi(A)} \subseteq (T : F) \subseteq (T : \langle A\rangle) = (N : M)$.

Suppose that $r \in (N : M)$. Since $M = \langle x_1, \dots, x_n\rangle$ and $N = \langle \eta\rangle$, where $\eta = \{y_i \in M \,|\, i \in \Omega\}$, $rx_i \in N$ for every $i (1 \leqslant i \leqslant n)$. So for every $i(1 \leqslant i \leqslant n)$, there exist $s_i \in \mathbb{N}$, $k_{il} \in R$ and $y_{il} \in \eta (1 \le l \le s_i)$ such that $rx_i = \sum_{l=1}^{s_i} k_{il} y_{il}$. Since for every $il(1 \leqslant l \leqslant s_i)$, $y_{il} \in M$ then there exists $t_{ilm} \in R(1 \leqslant m \leqslant n)$ such that $y_{il} = \sum_{m=1}^{n} t_{ilm} x_m$. Now we have $rx_i = \sum_{l=1}^{s_i} k_{il} y_{il} = \sum_{l=1}^{s_i} k_{il}(\sum_{m=1}^{n} t_{ilm} x_m) = \sum_{m=1}^{n}\sum_{l=1}^{s_i} k_{il} t_{ilm} x_m$. Now for every $i(1 \leqslant i \leqslant n)$ and for every $m(1 \leqslant m \leqslant n)$, we put $b_{im} = \sum_{l=1}^{s_i} k_{il} t_{ilm}$. So $rx_i = \sum_{i=1}^{n} b_{im} x_m$. Since $\Psi$ is a monomorphism, $(0, \dots, 0, r, 0, \dots, 0)A = \Psi(rx_i) = \Psi(\sum_{m=1}^{n} b_{im} x_m) = (b_{i1}, \dots, b_{in})A$. Put $T_{il} = (t_{il1}, \dots, t_{iln})(1 \leqslant l \leqslant s_i)$ and let $A_1, \dots, A_n$ be the rows of matrix $A$. Then for every $i(1 \leqslant i \leqslant n)$ we have $rA_i = b_{i1}A_1 + \cdots + b_{in}A_n = k_{i1}T_{i1}A + \cdots + k_{is_i}T_{is_i}A$. Then

$$\begin{aligned} r^n \det A &= \det \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & r \end{pmatrix} \det A \\ &= \det \begin{pmatrix} k_{11}T_{11}A+ & \cdots & +k_{1s_1}T_{1s_1}A \\ \vdots & & \vdots \\ k_{n1}T_{n1}A+ & \cdots & +k_{ns_n}T_{ns_n}A \end{pmatrix}. \end{aligned}$$

Now we have $r^n \det A \in \Re_{\xi(A)}$ and hence $r \in \sqrt{(\Re_{\xi(A)} : \det A)}$. Therefore, $\Re_{\xi(A)} \subseteq (N : M) \subseteq \sqrt{(\Re_{\xi(A)} : \det A)}$.

ii) Since $N$ is a prime submodule of $M$, $(N : M)$ is a prime ideal of $R$. Thus by part (i), we have $\sqrt{\Re_{\xi(A)}} \subseteq \sqrt{(N : M)} = (N : M) \subseteq \sqrt{(\Re_{\xi(A)} : \det A)}$.

iii) Let $r \in (N : M)$ and suppose that $rx_i = \sum_{l=1}^{m} k_{i_l} y_l$ for some $k_{i_l} \in R$ $(1 \le i \le n)$. Then

$$\begin{aligned} r^n \det A &= \det \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} \det A \\ &= \det \begin{pmatrix} k_{11}T_1A+ & \cdots & +k_{1m}T_mA \\ \vdots & & \vdots \\ k_{n1}T_1A+ & \cdots & +k_{nm}T_mA \end{pmatrix}. \end{aligned}$$

Since $m < n$, the right side of equality above is zero and hence $r^n \det A = 0$. But $\det A \neq 0$ and $R$ is a domain, thus $r = 0$. Therefore, $(N : M) = 0$. $\square$

**Lemma 1.3.** *Let $R$ be a commutative ring with identity and $I$ be an ideal of $R$.*

i) *If $U$ is an $R$-module and $V$ is a direct summand of $U$, then $IU \cap V = IV$.*

ii) *If $A = [\Psi]_{\alpha}^{\beta}$, then $IF \cap \langle A\rangle = I\langle A\rangle$, where $F = R^n$.*

*Proof.* i) There exists a submodule $V'$ of $U$ such that $U = V + V'$. Then $IU = IV + IV'$. By the modular law, $IU \cap V = V \cap (IV + IV') = IV + (V \cap IV') = IV$.

ii) By the notations in the proof of Lemma 1.1, since $M$ is projective,

$$0 \longrightarrow Ker\,\Phi \longrightarrow R^n \xrightarrow{\Phi} M \to 0$$

splits. There is an $R$-homomorphism $\Psi : M \to R^n$ such that $\Phi.\Psi = id_M$. It follows that $F = R^n = Ker\Phi \oplus \Psi(\Phi(R^n)) = Ker\Phi \oplus \Psi(M) = Ker\Phi \oplus \langle A \rangle$. Hence $\langle A \rangle$ is a direct summand of $F$. By (i), we have the result. $\square$

Let $A = [\Psi]_\alpha^\beta$, $N = \langle y_1, \ldots, y_k \rangle$ $(k < n)$, and $p$ be a prime ideal of $R$. Let $B = [T_1 \cdots T_k] \in M_{k\times n}(R)$ and $C = BA$. Put $T_p(B) = \{T = (t_1, \ldots, t_n) \in F \mid \det \beta(i_1, \ldots, i_{k+1}) \in p$ for every $i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\}$, where $\beta = [T\ T_1 \cdots T_k] \in M_{(k+1)\times n}(R)$. Let $S_p(N) = \{y = \sum_{i=1}^n t_i x_i \in M \mid (t_1, \ldots, t_n)A \in T_p(C)\}$. Now by [5, Lemma 1.5(i)], $T_p(C)$ is a submodule of $F = R^n$ and since $S_p(N) = \Psi^{-1}(T_p(C))$, hence $S_p(N)$ is a submodule of $M$. Also if the determinant of every submatrix $k \times k$ of $C$ is in $p$, by [5, Lemma 1.5(iii)], $T_p(C) = F$ and hence $S_p(N) = \Psi^{-1}(F) = M$.

**Lemma 1.4.** *Let $A = [\Psi]_\alpha^\beta$, $N = \langle y_1, \ldots, y_k \rangle$ $(k < n)$, and $p$ be a prime ideal of $R$. Let $B = [T_1 \cdots T_k] \in M_{k\times n}(R)$ and $C = BA$.*

i) *If $y \in S_p(N)$, then $\det C(j_1, \ldots, j_k)y \in pM + N$ for all submatrices $C(j_1, \ldots, j_k)$ of $C$.*

ii) *If there exists a submatrix $C(j_1, \ldots, j_k) \in M_{k\times k}(R)$ of $C$ such that $\det C(j_1, \ldots, j_k) \notin p$ and $\langle A \rangle \nsubseteq T_p(C)$, then $S_p(N)$ is a $p'$-prime submodule of $M$ such that $p \subseteq p'$, where $(S_p(N) : M) = p'$.*

*Proof.* i) Let $y = \sum_{i=1}^n t_i x_i \in S_p(N)$ and $C(j_1, \ldots, j_k) \in M_{k\times k}(R)$ be a submatrix of $C$. Then $(t_1, \ldots, t_n)A \in T_p(C)$ and by [5, Lemma 1.5(ii)], $X = \det C(j_1, \ldots, j_k)(t_1, \ldots, t_n)A \in pF + \langle C \rangle$. So there exist $X' \in pF$ and $Y' \in \langle C \rangle$ such that $X = X' + Y'$. By Lemma 1.3, $X' = X - Y' \in pF \cap \langle A \rangle = p\langle A \rangle$ and hence there exist $v_i \in p$ $(1 \le i \le n)$ and $z_i \in R$ $(1 \le i \le k)$ such that $X = (v_1, \ldots, v_n)A + (z_1, \ldots, z_k)C$. Since $C = BA$ and $X = \det C(j_1, \ldots, j_k)(t_1, \ldots, t_n)A$, we have $\Psi(\det C(j_1, \ldots, j_k)\sum_{i=1}^n t_i x_i) = X = \Psi(\sum_{i=1}^n v_i x_i + \sum_{i=1}^k z_i y_i)$. Since $\Psi$ is a monomorphism, we have

$$\det C(j_1, \ldots, j_k)y = \sum_{i=1}^n v_i x_i + \sum_{i=1}^k z_i y_i \in pM + N.$$

ii) Let $C(j_1, \ldots, j_k) \in M_{k\times k}(R)$ be a submatrix of $C$ such that $\det C(j_1, \ldots, j_k) \notin p$. Let $S_p(N) = M$. So $R = (S_p(N) : M) = (T_p(C) : \langle A \rangle)$ and hence $\langle A \rangle \subseteq T_p(C)$, which is a contradiction. Thus $S_p(N) \neq M$. By [5, Lemma 1.5(iv)], $T_p(C)$ is a $p$-prime submodule of $F$ and hence $S_p(N) = \Psi^{-1}(T_p(C))$ is a prime submodule of $M$. Let $(S_p(N) : M) = p'$. Now we have $p = (T_p(C) : F) \subseteq (T_p(C) : \langle A \rangle) = (S_p(N) : M) = p'$. $\square$

**Theorem 1.5.** *Let $A = [\Psi]_\alpha^\beta$, $N = \langle \eta \rangle$ and $p$ be a prime ideal of $R$. Then*

i) *$N$ is a $p$-prime submodule of $M$ if and only if $(N : M) = p$ and $N = pM$ or there exists a positive integer $k < n$, $y_i \in \eta$ $(1 \leq i \leq k)$ such that $N = S_p(L)$, where $L = \langle y_1, \ldots, y_k \rangle$.*

ii) *Let $N$ be a $p$-prime submodule of $M$, $N \neq pM$ and $k$ be a positive integer in part* (i). *Suppose that for every submodule $H = \langle z_1, \ldots, z_k \rangle$, $z_i \in N$ $(1 \leq i \leq k)$, $z_i = \sum_{j=1}^{n} s_{ij} x_j$, $D = [s_{ij}]_{k\times n}$, $E = DA$ such that $\langle A \rangle \nsubseteq T_p(E)$ and $\det E(j_1, \ldots, j_k) \notin p$ for some $j_1, \ldots, j_k \in \{1, \ldots, n\}$. Then $N = S_p(H)$.*

iii) *Let $N$ and $N'$ be $p$-prime submodules of $M$ and $N, N' \neq pM$. Suppose that $N' \subsetneqq N$ and $k_N$, $k_{N'}$ are positive integers for $N$ and $N'$ in part* (i). *Then $k_{N'} < k_N$.*

*Proof.* i) Suppose that $N$ is a $p$-prime submodule of $M$ and $N \neq pM$. Let $\theta$ be the collection of all positive integers $m$ such that there exists a submodule $L = \langle y_1, \ldots, y_m \rangle$ for some $y_i \in \eta$ $(1 \leq i \leq m)$, such that $\det C(j_1, \ldots, j_m) \notin p$ for some $j_1, \ldots, j_m \in \{1, \ldots, n\}$, where $B = [T_1 \cdots T_m]$ and $C = BA$. Since $N \neq pM$, $1 \in \theta$ and hence $\theta \neq \emptyset$. By the proof of Lemma 1.2, every element of $\theta$ is less than $n$. In particular, $\max(\theta) < n$. Now let $k = \max(\theta)$. Now there exists a submodule $L = \langle y_1, \ldots, y_k \rangle$ for some $y_i \in \eta$ $(1 \leq i \leq k)$, such that $\det C(j_1, \ldots, j_k) \notin p$ for some $j_1, \ldots, j_k \in \{1, \ldots, n\}$, where $B = [T_1 \cdots T_k]$ and $C = BA$. Now we show that $N = S_p(L)$. Let $y \in S_p(L)$. Then by Lemma 1.4(i), $\det C(j_1, \ldots, j_k) y \in pM + L \subseteq N$. Since $\det C(j_1, \ldots, j_k) \notin p$ and $N$ is a $p$-prime submodule of $M$, we have $y \in N$. Thus $S_p(L) \subseteq N$. If $\langle A \rangle \subseteq T_p(C)$, then $M \subseteq S_p(L) \subseteq N$ and so $N = M$, which is a contradiction. So $\langle A \rangle \nsubseteq T_p(C)$ and by Lemma 1.4(ii), $S_p(L)$ is a prime submodule of $M$. Now since $k = \max(\theta)$, $\eta \subseteq S_p(L)$ and hence $N \subseteq S_p(L)$. Thus $N = S_p(L)$. Conversely, suppose that $N = pM$. By [1, Corollary 2.3], $pM$ is a $p$-prime submodule of $M$. Assume that there exist positive integer $k < n$ and $L = \langle y_1, \ldots, y_k \rangle$, $B = [T_1 \cdots T_k]$, $C = BA$ such that $N = S_p(L)$. If $\det C(i_1, \ldots, i_k) \in p$ for every $i_1, \ldots, i_k \in \{1, \ldots, n\}$, then by the statement just prior to Lemma 1.4, $N = M$. So $(N : M) = R$, which is a contradiction. Therefore there exists a submatrix $C(j_1, \ldots, j_k)$ of $C$ such that $\det C(j_1, \ldots, j_k) \notin p$. On the other hand, since $N \neq M$, $\langle A \rangle \nsubseteq T_p(C)$ and by Lemma 1.4(ii), $N$ is a $p$-prime submodule of $M$.

ii) By part (i), there exist $y_i \in \eta$ $(1 \leq i \leq k)$, and $L = \langle y_1, \ldots, y_k \rangle$ such that $N = S_p(L)$. Let $B = [T_1 \cdots T_k]$ and $C = BA$. By [5, Proposition 1.7], $T_p(E) = T_p(C)$. So $N = S_p(L) = \Psi^{-1}(T_p(C)) = \Psi^{-1}(T_p(E)) = S_p(H)$.

iii) By the proofs of parts (i) and (ii), there exist matrices $C$ and $C'$ such that $N = \Psi^{-1}(T_p(C))$ and $N' = \Psi^{-1}(T_p(C'))$. Let $P = T_p(C)$ and $P' = T_p(C')$. We have $k_N = k_P$ and $k_{N'} = k_{P'}$. Now the proof follows by [5, Proposition 1.8]. □

**Corollary 1.6.** *Let $R$ be a domain, $A = [\Psi]_\alpha^\beta$, $N = \langle y_1, \ldots, y_k \rangle$ $(k < n)$ and $\det A \neq 0$. Suppose that $B = [T_1 \cdots T_k] \in M_{k\times n}(R)$ and $C = BA$ such that $rankC = k$ and $\langle A \rangle \nsubseteq T_{(0)}(C)$. Then $N$ is a prime submodule of $M$ if and only if $N = S_{(0)}(N)$.*

*Proof.* By Lemma 1.2(iii), $(N : M) = 0$. Since $rankC = k$, there exists a submatrix $C(j_1, \dots, j_k) \in M_{k\times k}(R)$ such that $\det C(j_1, \dots, j_k) \neq 0$. Now the proof follows from Theorem 1.5(i). $\square$

Let $N$ be a $p$-prime submodule of an $R$-module $M$. We recall that the $p$-height of $N$ is equal to $n$ and denoted by $p$-$ht(N)$, if there exists a chain $N_0 \subsetneqq N_1 \subsetneqq \cdots \subsetneqq N_n = N$ of $p$-prime submodules of $M$ with maximal length.

**Proposition 1.7.** *Let $A = [\Psi]_\alpha^\beta$ and $N = \langle\eta\rangle$. If $N$ is a $p$-prime submodule of $M$ and $k_N$ is the positive integer in Theorem 1.5*(i)*, then $p$-$ht(N) = k_N$.*

*Proof.* Note that if $N = pM$, then $p$-$ht(N) = 0$. We define $k_N = 0$. Thus $p$-$ht(N) = k_N$. Now assume that $k_N \geq 1$. We shall use induction on $k_N$ to prove the proposition. Let $k_N = 1$. Suppose that $L$ is a $p$-prime submodule of $M$ with $L \subsetneqq N$. If $L \neq pM$, since $pM \subsetneqq L \subsetneqq N$, then $0 < k_L < 1$, which is a contradiction. Thus $L = pM$ and hence $p$-$ht(N) = 1$. Assume that the assertion is true for any $k_N$, $1 \leq k_N \leq m-1$. Suppose that $k_N = m$. Then there exists a submodule $L = \langle y_1, \dots, y_m\rangle$, $y_i \in \eta$ $(1 \leq i \leq m)$ such that $N = S_p(L)$ and $\det C(j_1, \dots, j_m) \notin p$ for some $j_1, \dots, j_m \in \{1, \dots, n\}$, $B = [T_1 \cdots T_m]$ and $C = BA$. Let $L'' = \langle y_1, \dots, y_{m-1}\rangle$, $B'' = [T_1 \cdots T_{m-1}]$ and $C'' = B''A$. Since $\det C(j_1, \dots, j_m) \notin p$, there exists an $(m-1) \times (m-1)$ submatrix, $C''(s_1, \dots, s_{m-1})$ of $C(j_1, \dots, j_m)$ with $\det C''(s_1, \dots, s_{m-1}) \notin p$. Now by Lemma 1.4(ii), $S = S_p(L'')$ is a $p'$-prime submodule of $M$. Since $S \subseteq N$, $p = p'$. By the induction hypothesis, $p$-$ht(S) = m-1$. So there exists a chain of $p$-prime submodules $N_0 = pM \subsetneqq N_1 \subsetneqq \cdots \subsetneqq N_{m-1} = S \subsetneqq N_m = N$ of length $m$ and hence $p$-$ht(N) \geq m$. Now let $N_0 = pM \subsetneqq N_1 \subsetneqq \cdots \subsetneqq N_{l-1} \subsetneqq N_l = N$ be a chain of $p$-prime submodules of $M$. Then $k_{N_{l-1}} < k_N = m$ and by the induction hypothesis, $l - 1 \leq k_{N_{l-1}} < m$. Thus $l \leq m$ and hence $p$-$ht(N) = m$. Therefore $p$-$ht(N) = k_N$. $\square$

In the rest of this section we describe the structure of prime submodules of a finitely generated projective module over a UFD and a Dedekind domain.

**Theorem 1.8.** *Let $R$ be a* UFD *(respectively, Dedekind domain), $A = [\Psi]_\alpha^\beta$ and $N = \langle y_1, \dots, y_k\rangle$ $(k \leq n)$. Suppose that $B = [T_1 \cdots T_k] \in M_{k\times n}(R)$, $C = BA$ and $rankC = k$. We have*

i) *If $k < n$, then $N$ is a prime submodule of $M$ if and only if a GCD of (respectively, the ideal generated by) the determinants of all $k \times k$ submatrices of $C$, is* 1 *(respectively, $R$).*
ii) *If $k = n$, then $N$ is a prime submodule of $M$ if and only if there exist an irreducible element $p \in R$ (respectively, a prime ideal $p$ of $R$), a unit $u \in R$ and a positive integer $\alpha \leq n$ such that $\langle\det C\rangle = up^\alpha$ and a GCD of (respectively, the ideal generated by) entries of $C'$ is $p^{\alpha-1}$.*

*Proof.* Let $T = \langle\xi(A)\rangle$ be a submodule of $F = R^n$. Since $\Psi$ is a monomorphism, $N \simeq T$. Now the proof (i) and (ii), follows by [2, Theorem 2.5] (respectively, [5, Theorem 2.2]). $\square$

**Example 1.9.** Let $R = \mathbb{Z}[\sqrt{10}]$. We know that $R$ is a Dedekind domain but it is not a $UFD$. Let $M = R(\frac{1}{2}) + R(\frac{1}{3})$. By [4, Theorem VIII, 6.8], $M$ is a projective $R$-module. We define $\Phi : R^2 \to M$ by $\Phi((1,0)) = \frac{1}{2}$ and $\Phi((0,1)) = \frac{1}{3}$. We define $\Psi : M \to R^2$ by $\Psi(\frac{1}{2}) = (3,-3)$ and $\Psi(\frac{1}{3}) = (2,-2)$. So by Lemma 1.1, $M \simeq \langle A \rangle$, where $A = \left(\begin{smallmatrix} 3 & -3 \\ 2 & -2 \end{smallmatrix}\right) \in M_{2\times 2}(R)$. Let $N = \langle \frac{1}{2}t_1 + \frac{1}{3}t_2 \rangle$, where $t_1 = x_1 + x_2\sqrt{10}$ and $t_2 = y_1 + y_2\sqrt{10}$, $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. We have, $B = [t_1\ t_2]_{1\times 2}$ and $C = BA = [3t_1 + 2t_2\ \ -3t_1 - 2t_2]_{1\times 2}$. Let $J = R(3t_1 + 2t_2)$. Then by Theorem 1.8, $N$ is prime if and only if $J = R$ if and only if $3t_1 + 2t_2$ is a unit element of $R$. For example, let $t_1 = 3 + \sqrt{10}$ and $t_2 = -3 - \sqrt{10}$. Then $3t_1 + 2t_2 = 3 + \sqrt{10}$ and $(3+\sqrt{10})(-3+\sqrt{10}) = 1$. So $3t_1 + 2t_2$ is a unit element of $R$ and hence $N = \langle \frac{1}{2}(3+\sqrt{10}) + \frac{1}{3}(-3-\sqrt{10}) \rangle = \langle \frac{1}{2} + \frac{1}{6}\sqrt{10} \rangle$ is a prime submodule of $M$.

## 2. Primary decomposition of submodules of a finitely generated projective module

In this section we describe a primary decomposition of a submodule of a finitely generated projective module over a domain.

Let $A = [\Psi]_\alpha^\beta$, $N = \langle y_1, \ldots, y_k \rangle$ $(k < n)$, and $Q$ be a $p$-primary ideal of $R$ containing $\Re_{\xi(A)}$. Let $B = [T_1 \cdots T_k]$ and $C = BA$.

Put $T_Q(B) = \{T = (t_1, \ldots, t_n) \in F \mid \det \beta(i_1, \ldots, i_{k+1}) \in Q$ for every $i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\}$, where $\beta = [T\ T_1 \cdots T_k] \in M_{(k+1)\times n}(R)$ and $S_Q(N) = \{y = \sum_{i=1}^n t_i x_i \in M \mid (t_1, \ldots, t_n)A \in T_Q(C)\}$. As an observations before Lemma 1.4, $S_Q(N)$ is a submodule of $M$. Also if the determinant of every $k \times k$ submatrix of $C$ is in $Q$, $S_Q(N) = M$.

**Lemma 2.1.** *Let $A = [\Psi]_\alpha^\beta$, $N = \langle y_1, \ldots, y_k \rangle$ $(k < n)$, and $Q$ be a $p$-primary ideal of $R$ containing $\Re_{\xi(A)}$. Let $B = [T_1 \cdots T_k]$ and $C = BA$. We have*

i) *If $y = \sum_{i=1}^n t_i x_i \in S_Q(N)$, then for every submatrix $C(j_1, \ldots, j_k)$ of $C$, $\det C(j_1, \ldots, j_k) y \in QM + N$.*
ii) *If $\langle A \rangle \not\subseteq S_Q(N)$ and there exists a submatrix $C(j_1, \ldots, j_k)$ of $C$ such that $\det C(j_1, \ldots, j_k) \notin Q$, then $S_Q(N)$ is $p'$-primary submodule of $M$ such that $p \subseteq p'$.*

*Proof.* The proof is similar to the proof of Lemma 1.4. □

*Remark.* Let $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. Suppose that $N$ is a $p$-primary submodule of $M$ with $(N : M) = Q$ and $\eta \not\subset QM$. Let $\theta$ be the collection of all positive integers $m$ such that there exist a submodule $L = \langle y_1, \ldots, y_m \rangle$ and a submatrix $C(j_1, \ldots, j_m)$ such that $\det C(j_1, \ldots, j_m) \notin Q$ for some $j_1, \ldots, j_m \in \{1, \ldots, n\}$. Since $\eta \not\subset QM$, then $1 \in \theta$. Let $k = \max \theta$. Since $\Re_{\xi(A)} \subseteq Q$, $k < n$. Assume that $L = \langle y_1, \ldots, y_k \rangle$ with $\det C(j_1, \ldots, j_k) \notin Q$ for some $j_1, \ldots, j_k \in \{1, \ldots, n\}$. If $y \in S_Q$, then by Lemma 2.1(i), $\det C(s_1, \ldots, s_k) y \in QM + N \subset N$ for all $s_1, \ldots, s_k \in \{1, \ldots, n\}$. So, if $\det C(s_1, \ldots, s_k) \notin p$ for some $s_1, \ldots, s_k \in$

$\{1, \ldots, n\}$, then $S_Q \subseteq N$. Now by Lemma 2.1(ii), $S_Q$ is a $p'$-primary submodule of $M$ with $N \subseteq S_Q$. Thus $N = S_Q$.

**Theorem 2.2.** *Let $R$ be a domain, $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. Suppose that $N$ is a proper submodule of $M$ with $|\Omega| \geq n$. Let $\Re_{\xi(A)}$ be a nonzero ideal of $R$ such that $\Re_{\xi(A)} = RE_{j_1 \cdots j_n}$ for some $j_1, \ldots, j_n \in \Omega$, where $E_{j_1 \cdots j_n} = \det[T_{i_1} \cdots T_{i_n}]A$. Suppose that $\Re_{\xi(A)} = \bigcap_{i=1}^m Q_i$ is a minimal primary decomposition of $\Re_{\xi(A)}$ with $Ass(\Re_{\xi(A)}) = \{p_i\}_{i=1}^m$ and $S_{Q_i}$ as above is a submodule of $M$ with $\sqrt{(S_{Q_i} : M)} = p'_i$. Let $\{q_i\}_{i=1}^t = \{p'_i \mid \langle A \rangle \nsubseteq T_{Q_i}\}$. Then*

a) *$\bigcap_{i=1}^t S_{Q_i}$ is a primary decomposition of $N$.*

b) *If $\{q_i\}_{i=1}^t$ has no embedded prime ideal, then $\bigcap_{i=1}^t S_{Q_i}$ is a minimal primary decomposition of $N$ with $Ass(N) = \{q_i\}_{i=1}^t$.*

c) *Let $i \in \{1, \ldots, m\}$. There exist submodules $L_i = \langle y_1(i), \ldots, y_{k_i}(i) \rangle$ of $N$ such that $y_l(i) \in \eta$ $(1 \leq l \leq k_i)$, $\det C_i(j_{i1}, \ldots, j_{ik_i}) \notin Q_i$, for some $j_{i1}, \ldots, j_{ik_i} \in \{1, \ldots, n\}$ and $S_{Q_i} = S_{Q_i}(L_i)$. If there exists $C_i(s_{i1}, \ldots, s_{ik_i})$ for some $s_{i1}, \ldots, s_{ik_i} \in \{1, \ldots, n\}$ such that*

$$\det C_i(s_{i1}, \ldots, s_{ik_i}) \notin p_i$$

*for all $i \in \{1, \ldots, m\}$, and for every $i, j \in \{1, \ldots, t\}$, $q_i \neq q_j$, then $\bigcap_{i=1}^t S_{Q_i}$ is a minimal primary decomposition of $N$ with $Ass(N) = \{q_i\}_{i=1}^t$.*

*Proof.* a) By [5, Theorem 3.2(a)], $\langle C \rangle = \bigcap_{i=1}^m T_{Q_i}$. Then

$$N = \Psi^{-1}(\langle C \rangle) = \bigcap_{i=1}^m \Psi^{-1}(T_{Q_i}) = \bigcap_{i=1}^m S_{Q_i} = \bigcap_{i=1}^t S_{Q_i}.$$

b) Suppose that $\bigcap_{1=i\neq j}^t S_{Q_i} \subseteq S_{Q_j}$ for some $j$ $(1 \leq j \leq m)$. Then $\sqrt{(\bigcap_{1=i\neq j}^t S_{Q_i} : M)} \subseteq \sqrt{(S_{Q_j} : M)}$ and hence $\bigcap_{1=i\neq j}^t q_i \subseteq q_j$. It follows that $q_i \subseteq q_j$ for some $i$ $(1 \leq i \leq m)$, $i \neq j$, which is a contradiction.

c) Suppose that $\bigcap_{1=i\neq j}^t S_{Q_i} \subset S_{Q_j}$ for some $j$ $(1 \leq j \leq t)$. Then $\bigcap_{i=1}^m T_{Q_i} \subset \bigcap_{i=1}^t T_{Q_i} \subset T_{Q_j}$ and by [5, Theorem 3.2(c)], it is a contradiction. □

**Corollary 2.3.** *Let $R$ be a Dedekind domain, $A = [\Psi]_\alpha^\beta$ and $N = \langle y_1, \ldots, y_n \rangle$. Let $B = [T_1 \cdots T_n]$, $C = BA$ and $rankC = n$. Then $\bigcap_{i=1}^k S_{p_i^{\alpha_i}}$ is a minimal primary decomposition of $N$ for some distinct maximal ideals $p_1, \ldots, p_k$ of $R$.*

*Proof.* Since $R$ is a Dedekind domain, by [5, Corollary 3.3], there exist distinct maximal ideals $q_1, \ldots, q_t$ of $R$ and $\alpha_i \in \mathbb{N}$ such that $\langle C \rangle = \bigcap_{i=1}^t T_{q_i^{\alpha_i}}$ is a minimal primary decomposition of $\langle C \rangle$. Since $\Psi$ is a monomorphism, $N = \Psi^{-1}(\langle C \rangle) = \Psi^{-1}(\bigcap_{i=1}^t T_{q_i^{\alpha_i}}) = \bigcap_{i=1}^t S_{q_i^{\alpha_i}}$. Let $\{p_1, \ldots, p_k\} = \{q_i \mid \langle A \rangle \nsubseteq T_{q_i^{\alpha_i}}\}$. Now by Lemma 2.1(ii) and since $R$ is a Dedekind domain, $N = \bigcap_{i=1}^k S_{p_i^{\alpha_i}}$ is a minimal primary decomposition of $N$ and $Ass(N) = \{p_1, \ldots, p_k\}$. □

## 3. Radical of a submodule of a finitely generated projective module

In this section we characterize the radical of an arbitrary submodule of a finitely generated projective module $M$ over a commutative ring $R$ with identity. Also we study submodules of $M$ which satisfy the radical formula.

Let $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. We put

$$[T_1 A \cdots T_m A]_m = \sum_{j_1,\ldots,j_m \in \{1,\ldots,n\}} R \det C(j_1,\ldots,j_m),$$

where $C = [T_1 A \cdots T_m A]$ and $\Re_t = \sum_{i_1,\ldots,i_t \in \Omega} R[T_{i_1} A \ldots T_{i_t} A]_t$ $(1 \le t \le n)$. Note that $\Re_1 \supseteq \Re_2 \supseteq \cdots \supseteq \Re_n = \Re_{\xi(A)}$.

Let $M$ be an $R$-module, $p$ be a prime ideal of $R$ and $N$ be a submodule of $M$. In [7], Pusat-Yilmaz and Smith defined the submodule $K(N,p) = \{m \in M \,|\, cm \in N + pM$ for some $c \in R \backslash p\}$. They showed that this is the smallest $p$-prime submodule of $M$ containing $N$ and so $Rad_M N = \cap\{K(N,p) \,|\, p$ is a prime ideal of $R\}$.

**Lemma 3.1.** *Let $A = [\Psi]_\alpha^\beta$, $p$ be a prime ideal of $R$ and $N = \langle \eta \rangle$. We have*

i) *If $(N : M) \nsubseteq p$, then $K(N,p) = M$.*
ii) *If $\Re_1 \subseteq p$ and $(pM : M) = p$, then $K(N,p) = pM$.*
iii) *If $p \neq 0$ is a maximal ideal of $R$, $\Re_1 \nsubseteq p$ and $K(N,p) \neq M$, then there exist a positive integer $k < n$ and a submodule $L = \langle y_1,\ldots,y_k \rangle$, $y_i \in \eta$ $(1 \le i \le k)$ such that $K(N,p) = S_p(L)$.*

*Proof.* i) Let $p$ be a prime ideal of $R$. Assume that $(N : M)$ is not contained in $p$ and $c \in (N : M) \backslash p$. So $cM \subseteq N$ and hence $M \subseteq K(N,p)$.

ii) Let $\Re_1 \subseteq p$. Since $pM$ contains $N$, by [1, Corollary 2.3], $pM$ is a $p$-prime submodule of $M$. So $K(N,p) = pM$.

iii) Let $\Re_1$ be not contained in $p$. Suppose that $\theta$ is the set of all positive integers $m$ such that there exist a submodule $L = \langle y_1,\ldots,y_m \rangle$ for some $y_i \in \eta$ $(1 \le i \le m)$, and a submatrix $C(j_1,\ldots,j_m)$ such that $\det C(j_1,\ldots,j_m) \notin p$ for some $j_1,\ldots,j_m \in \{1,\ldots,n\}$, where $B = [T_1 \ldots T_m]$ and $C = BA$. Since $\Re_1 \nsubseteq p$, $\eta \not\subset pM$ and hence $1 \in \theta$. Thus $\theta \neq \emptyset$. Let $k = \max(\theta)$. By Lemma 1.2(i), we have $k < n$. Suppose that $L = \langle y_1,\ldots,y_k \rangle$ is a submodule of $M$ such that $\det C(j_1,\ldots,j_k) \notin p$ for some $j_1,\ldots,j_k \in \{1,\ldots,n\}$, where $B = [T_1 \cdots T_k]$ and $C = BA$. By Lemma 1.4(i), $S_p(L) \subseteq K(N,p)$. So $\langle A \rangle \nsubseteq S_p(L)$ and by Lemma 1.4(ii), we have $S_p(L)$ is a prime submodule of $M$. Since $p \neq 0$ is maximal ideal, $(S_p(L) : M) = p$. Thus $K(N,p) \subseteq S_p(L)$. Therefore $K(N,p) = S_p(L)$. □

Let $F$ be the free $R$-module $R^n$ and $W = \langle \xi \rangle$. By [6, Theorem 2.4], $Rad_F W = \{T = (t_1,\ldots,t_n) \in \sqrt{\Re_1} F \,|\, [T\ T_{i_1} \cdots T_{i_{k-1}}]_k \subseteq \sqrt{\Re_k}$ for every $i_1,\ldots,i_{k-1} \in \Omega$, $2 \le k \le n\}$, where $\Re_k = \sum_{i_1,\ldots,i_k \in \Omega} R[T_{i_1} \ldots T_{i_k}]_k$ and $[T\ T_{i_1} \cdots T_{i_{k-1}}]_k = \sum_{j_1,\ldots,j_k \in \{1,\ldots,n\}} R\ \det B(j_1,\ldots,j_k)$ with $B = [T\ T_{i_1} \cdots T_{i_{k-1}}]$.

**Theorem 3.2.** *Let $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. Then $Rad_M N = \{y = \sum_{i=1}^n t_i x_i \in M \mid (t_1, \ldots, t_n)A \in Rad_F T\}$, where $T = \langle \xi(A) \rangle$.*

*Proof.* We know that $Rad_M N = \bigcap_p K(N,p)$. At first we will show that $K(T,p) \cap \langle A \rangle = \Psi(K(N,p))$ for every prime ideal $p$ of $R$. Let $y = \sum_{i=1}^n t_i x_i \in K(N,p)$. There exists $c \in R - p$ such that $cy = \sum_{i=1}^n ct_i x_i \in N + pM$. So there exist $m \in \mathbf{N}$, $k_i \in R$ $(1 \le i \le m)$ and $l_j \in p$ $(1 \le j \le n)$ such that $\sum_{i=1}^n ct_i x_i = \sum_{i=1}^m k_i y_i + \sum_{j=1}^n l_j x_j = \sum_{i=1}^m k_i (\sum_{j=1}^n t_{ij} x_j) + \sum_{j=1}^n l_j x_j = \sum_{j=1}^n \sum_{i=1}^m k_i t_{ij} x_j + \sum_{j=1}^n l_j x_j$. Now we have

$$\begin{aligned} c\Psi(y) &= \Psi(cy) \\ &= (\sum_{i=1}^m k_i t_{i1}, \ldots, \sum_{i=1}^m k_i t_{in})A + (l_1, \ldots, l_n)A \in T + p\langle A \rangle \subseteq T + pF. \end{aligned}$$

So $\Psi(y) \in K(T,p) \cap \langle A \rangle$ and hence $\Psi(K(N,p)) \subseteq K(T,p) \cap \langle A \rangle$. Conversely, let $Y \in K(T,p) \cap \langle A \rangle$. There exist $c \in R - p$ and $l_j \in R$ $(1 \le j \le n)$ such that $Y = (l_1, \ldots, l_n)A$ and $cY = (cl_1, \ldots, cl_n)A \in T + pF$. So there exist $m \in \mathbb{N}$, $k_i \in R$, $T_i A \in T$ $(1 \le i \le m)$ and $z_i \in p$ $(1 \le i \le n)$ such that $(cl_1, \ldots, cl_n)A = (k_1, \ldots, k_n)BA + (z_1, \ldots, z_n)$, where $B = [T_1 \cdots T_m]$. So by Lemma 1.3, $(z_1, \ldots, z_n) \in pF \cap \langle A \rangle = p\langle A \rangle$. Then there exists $z'_i \in p$ $(1 \le i \le n)$ such that $(z_1, \ldots, z_n) = (z'_1, \ldots, z'_n)A$ and hence $(cl_1, \ldots, cl_n)A = ((k_1, \ldots, k_n)B + (z'_1, \ldots, z'_n))A$. Thus $\Psi(\sum_{i=1}^n cl_i x_i) = \Psi(\sum_{i=1}^m k_i y_i + \sum_{i=1}^n z'_i x_i)$. Since $\Psi$ is a monomorphism, we have $\sum_{i=1}^n cl_i x_i = \sum_{i=1}^m k_i y_i + \sum_{i=1}^n z'_i x_i$ and hence $c(\sum_{i=1}^n l_i x_i) \in N + pM$. Now we have

$$\sum_{i=1}^n l_i x_i \in K(N,p).$$

So $Y = \Psi(\sum_{i=1}^n l_i x_i) \in \Psi(K(N,p))$ and hence $K(T,p) \cap \langle A \rangle = \Psi(K(N,p))$. Since for every $y = \sum_{i=1}^n t_i x_i \in M$, $\Psi(y) \in \langle A \rangle$, we have $y \in Rad_M N$ if and only if $y \in \bigcap_p K(N,p)$ if and only if $\Psi(y) \in \bigcap_p K(T,p)$ if and only if $(t_1, \ldots, t_n)A \in \bigcap_p K(T,p)$ if and only if $(t_1, \ldots, t_n)A \in Rad_F T$. $\square$

**Proposition 3.3.** *Let $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. If there exist $m$ $(1 \le m \le n-1)$, and a submodule $L = \langle y_1, \ldots, y_m \rangle$ of $N$ for some $y_i \in \eta$ $(1 \le i \le m)$ such that $C$ contains an $m \times m$ submatrix whose determinant is a unit in $R$ and $\sqrt{\Re_{m+1}} = \sqrt{(N:M)}$, where $B = [T_1 \cdots T_m]$ and $C = BA$, then $N$ s.t.r.f in $M$.*

*Proof.* Suppose that there exist a submodule $L = \langle y_1, \ldots, y_m \rangle$ of $N$ for some $y_i \in \eta$ $(1 \le i \le m)$, and a submatrix $C(j_1, \ldots, j_m)$ for some $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $\det C(j_1, \ldots, j_m)$ is unit. Let $y = \sum_{i=1}^n t_i x_i \in Rad_M N$ and $p = \sqrt{(N:M)}$. Then $[TA\; T_1 A \cdots T_m A]_{m+1} \subseteq \sqrt{\Re_{m+1}} = \sqrt{(N:M)}$. If we replace the radical $p$ in [5, Lemma 1.5(ii)] with $\sqrt{(N:M)}$, then $\det C(i_1, \ldots, i_m)$ $TA \in pF + \langle C \rangle$. Since $\det C(j_1, \ldots, j_m)$ is unit, $TA \in pF + \langle C \rangle$ and hence

$y \in pM + N$. It follows that $y \in \sqrt{(N : M)}M + N$ and hence $Rad_M N = \sqrt{(N : M)}M + N = \langle E_M(N)\rangle$. $\square$

**Corollary 3.4.** *Let $(R, m)$ be a local ring with $m$ as maximal ideal, $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. If $\Re_j = R$ and $\sqrt{\Re_{j+1}} = \sqrt{(N : M)}$ for some $j$ $(1 \leq j \leq n-1)$, then $N$ s.t.r.f in $M$.*

*Proof.* Let $\Re_j = \sum_{i_1,\ldots,i_j \in \Omega} R[T_{i_1}A \cdots T_{i_j}A]_j = R$ for some $j$ $(1 \leq j \leq n-1)$, and $\sqrt{\Re_{j+1}} = \sqrt{(N : M)}$. Since $R$ is a local ring, there exist a submodule $L = \langle y_1, \ldots, y_j \rangle$ for some $y_1, \ldots, y_j \in \eta$ and a submatrix $C(i_1, \ldots, i_j) \in M_{j\times j}(R)$ for some $i_1, \ldots, i_j \in \{1, \ldots, n\}$ such that $\det C(i_1, \ldots, i_j)$ is unit. Now by Proposition 3.3, $N$ s.t.r.f in $M$. $\square$

**Proposition 3.5.** *Let $R$ be a commutative ring with identity, $A = [\Psi]_\alpha^\beta$ and $N = \langle \eta \rangle$. If $\sqrt{\Re_1} = \sqrt{\Re_2} = \cdots = \sqrt{\Re_{n-1}} = \sqrt{(T : F)}$, where $T = \langle \xi(A) \rangle$, then $Rad_M N = \sqrt{(N : M)}M = \langle E_M(N) \rangle$.*

*Proof.* Since $\Psi$ is a monomorphism, $N \simeq T$. By [6, Proposition 2.7], we have $Rad_F T = \sqrt{(T : F)}F = \langle E_F(T) \rangle$. Let $y = \sum_{i=1}^n t_i x_i \in Rad_M N$. Then $X = (t_1, \ldots, t_n)A \in Rad_F T = \sqrt{(T : F)}F$. Suppose that $I = \sqrt{(T : F)}$. By Lemma 1.3, we have $X \in IF \cap \langle A \rangle = I\langle A \rangle$. So there exists $r_i \in I$ $(1 \leq i \leq n)$ such that $X = (r_1, \ldots, r_n)A$. Thus $y \in IM \subseteq \sqrt{(N : M)}M$ and hence $Rad_M N = \sqrt{(N : M)}M = \langle E_M(N) \rangle$. $\square$

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