Commun. Korean Math. Soc. **34** (2019), No. 3, pp. 719–727
https://doi.org/10.4134/CKMS.c180249
pISSN: 1225-1763 / eISSN: 2234-3024

ON PARTIAL-ARMENDARIZ RINGS

Sang Bok Nam, Zhelin Piao, and Sang Jo Yun

Abstract. This article concerns a generalization of Armendariz rings that is done by restricting the degree to one. We shall call such rings, as to satisfy this property, *partial-Armendariz*. We first show that partial-Armendariz rings are between Armendariz rings and weak Armendariz rings. The basic structures of partial-Armendariz rings are investigated, and the relations between partial-Armendariz rings and near related ring properties are also studied.

1. Introduction

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring. Use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotents, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. $R[x]$ denotes the polynomial ring with x an indeterminate over R. $I(R)$ denotes the set of all idempotents in R. Let $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$) be the n by n full (resp., upper triangular) matrix ring over R, and denote by E_{ij} the matrix with (i,j)-entry 1 and elsewhere zeros. Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. We use $\mathbb{Z}$ and $\mathbb{Z}_n$ to denote the ring of integers and the ring of integers modulo n, respectively.

A ring is usually said to be *reduced* if it has no nonzero nilpotent elements. For a reduced ring R, Armendariz [4, Lemma 1] proved that $a_ib_j = 0$ for all i, j whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$. From this result, Rege and Chhawchharia [17] called a ring (not necessarily reduced) *Armendariz* if it satisfies this result. So reduced rings are Armendariz obviously. The class of Armendariz rings is clearly closed under subrings. A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by [11, Corollary 8] (or the proof of [1, Theorem 6]).

Received June 4, 2018; Revised September 11, 2018; Accepted November 23, 2018.
2010 *Mathematics Subject Classification.* 16S36, 16S50, 16N40.
Key words and phrases. partial-Armendariz ring, Armendariz ring, weak Armendariz ring, polynomial ring, reduced ring, matrix ring.

Remark 1.1. Given a ring R we claim that the following are equivalent:

(1) R is Armendariz;

(2) $a_0b_j = 0$ for all $j \in \{0, 1, \ldots, n\}$ whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$.

For the proof, it suffices to show the sufficiency. Assume the necessity. Let $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$. Then $a_0g(x) = 0$, entailing $(\sum_{i=1}^{m} a_ix^{i-1})g(x) = 0$. Then we get $a_1g(x) = 0$ by the necessity, and $(\sum_{i=2}^{m} a_ix^{i-2})g(x) = 0$ follows. Inductively we obtain $a_ib_j = 0$ for all i, j. Thus R is Armendariz.

Based on Remark 1.1, we next consider a generalization of Armendariz rings.

Definition 1.2. A ring R is said to be *partial-Armendariz* provided that $a_0b_1 = 0$ whenever $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$.

Armendariz rings are clearly partial-Armendariz. But the converse need not hold by the following. $char(R)$ means the characteristic of given a ring R.

Example 1.3. There exists a partial-Armendariz ring that is not Armendariz. We use the construction and argument in [3, Lemmas 2-5 and Example 4]. Let K be a field with $char(K) \neq 2$. Let R be the K-algebra with commuting generators $A = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ and with the following relations: (1) $a_0b_0 = a_2b_2 = 0$; (2) $a_0b_1 + a_1b_0 = 0$; (3) $a_1b_2 + a_2b_1 = 0$; (4) $a_0b_2 + a_1b_1 + a_2b_0 = 0$; (5) $x_1x_2x_3 = 0$, where $x_1, x_2, x_3 \in A$. Then R is partial-Armendariz by the argument in [3, Lemmas 2-5]. However R is not Armendariz by [3, Example 4].

We can see basic properties of partial-Armendariz rings in the following. Following Lee and Wong [16], a ring R is called *weak Armendariz* provided that $a_0b_1 = 0 = a_1b_0$ whenever $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x$, $g(x) = b_0 + b_1x \in R[x]$. Weak Armendariz rings are Abelian by [16, Lemma 3.4]. Use $\lfloor - \rfloor$ to denote the *floor function*, i.e., $\lfloor r \rfloor$ is the largest integer less than or equal to r for any real number r.

According to Bell [5], a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, called an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. IFP rings are easily shown to be Abelian.

Lemma 1.4. (1) *The class of partial-Armendariz rings is closed under subrings.*

(2) *Partial-Armendariz rings are weak Armendariz.*
(3) *Partial-Armendariz rings are Abelian.*
(4) *Let R be a partial-Armendariz ring. If $ab = 0$ for $a, b \in R$, then $aN(R)b = 0$. Moreover $N(R)$ forms a subring of R.*
(5) *The concepts of partial-Armendariz rings and IFP rings are independent of each other.*
(6) *$D_m(R)$ is partial-Armendariz over a reduced ring R for $m \leq 3$. $D_n(A)$ is not weak Armendariz over any ring A for $n \geq 4$.*

Proof. (1) is obvious.

(2) Let R be a partial-Armendariz ring. Suppose that $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$. Then $a_0b_1 = 0$ since R is partial-Armendariz, and this yields $a_1b_0 = 0$ because $a_0b_1 + a_1b_0 = 0$. Thus R is weak Armendariz.

(3) is shown by (2) and [16, Lemma 3.4]; but we give another proof. Let R be a partial-Armendariz ring and assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Consider polynomials

$$f(x) = e + ea(1-e)x \text{ and } g(x) = (1-e) + ((1-e) - ea(1-e))x - ea(1-e)x^2$$

in $R[x]$. Then $f(x)g(x) = 0$. But $e[(1-e) - ea(1-e)] = -ea(1-e) \neq 0$, contrary to R being partial-Armendariz. Thus $er(1-e) = 0$ for all $e \in I(R)$ and $r \in R$. So R is Abelian.

(4) is proved by (2) and [3, Lemma 1], but we take another proof here. Let R be a partial-Armendariz ring and suppose that $ab = 0$ for $a, b \in R$. Take $c \in N(R)$ with $c^n = 0$. Then

$$0 = a(1-cx)(1+cx+\cdots+c^{n-1}x^{n-1})b = (a-acx)(b+cbx+\cdots+c^{n-1}bx^{n-1}).$$

Since R is partial-Armendariz, we have $acb = 0$. This implies $aN(R)b = 0$. Next $N(R)$ forms a subring of R by (2) and [3, Corollary 1]. To obtain this result, we can use the following argument that is a little different proof from one of [3, Corollary 1].

Let $a, b \in N(R)$ with $a^m = 0$ and $b^n = 0$. We apply the proof of [9, Proposition 3.9]. Consider $f(x) = 1 - bx$ and $g(x) = 1 + bx + \cdots + b^{n-1}x^{n-1}$ in $R[x]$. Then $f(x)g(x) = 1$ since $b^n = 0$. Multiplying $f(x)g(x)$ on the left and right by $a^{\lfloor \frac{m+1}{2} \rfloor}$ and $a^{\lfloor \frac{m+1}{2} \rfloor}b$ respectively, we obtain

$$\begin{aligned} 0 &= a^{\lfloor \frac{m+1}{2} \rfloor}a^{\lfloor \frac{m+1}{2} \rfloor}b = a^{\lfloor \frac{m+1}{2} \rfloor}f(x)g(x)a^{\lfloor \frac{m+1}{2} \rfloor}b \\ &= a^{\lfloor \frac{m+1}{2} \rfloor}(1-bx)(1+bx+\cdots+b^{n-1}x^{n-1})a^{\lfloor \frac{m+1}{2} \rfloor}b \\ &= (a^{\lfloor \frac{m+1}{2} \rfloor} - a^{\lfloor \frac{m+1}{2} \rfloor}bx)(a^{\lfloor \frac{m+1}{2} \rfloor}b + ba^{\lfloor \frac{m+1}{2} \rfloor}bx + \cdots + b^{n-1}a^{\lfloor \frac{m+1}{2} \rfloor}bx^{n-1}), \end{aligned}$$

noting $a^m = 0$ and $f(x)g(x) = 1$. Since R is partial-Armendariz,

$$a^{\lfloor \frac{m+1}{2} \rfloor}ba^{\lfloor \frac{m+1}{2} \rfloor}b = 0.$$

Let $m_1 = \lfloor \frac{m+1}{2} \rfloor$ and note $m_1 < m$. Next consider

$$a^{\lfloor \frac{m_1+1}{2} \rfloor}a^{\lfloor \frac{m_1+1}{2} \rfloor}ba^{\lfloor \frac{m_1+1}{2} \rfloor}a^{\lfloor \frac{m_1+1}{2} \rfloor}b = 0.$$

Then the preceding argument provides us with

$$a^{\lfloor \frac{m_1+1}{2} \rfloor}ba^{\lfloor \frac{m_1+1}{2} \rfloor}ba^{\lfloor \frac{m_1+1}{2} b \rfloor}ba^{\lfloor \frac{m_1+1}{2} \rfloor}b = 0.$$

Proceeding inductively, we eventually obtain $(ab)^{m_2} = 0$ for some $m \leq m_2$. This process also gives us $(a+b)^l = 0$ for some $l \geq m+n$. Therefore $N(R)$ forms a subring of R.

(5) Consider $D_2(\mathbb{Z}_8)$ that is commutative (hence IFP). Then $D_2(\mathbb{Z}_8)$ is not partial-Armendariz by [17, Example 3.2], where Rege and Chhawchharia consider the polynomial $f(x) = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right) x \in D_2(\mathbb{Z}_8)[x]$ with $f(x)^2 = 0$ and $\left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. There exists an Armendariz (hence partial-Armendariz) ring that is not IFP by [11, Example 14].

(6) Let R be a reduced ring. Then $D_3(R)$ is Armendariz (hence partial-Armendariz) by [14, Proposition 2]. This implies that $D_2(R)$ is also partial-Armendariz by (1) because $D_2(R)$ is a subring of $D_3(R)$.

The proof of the remainder is done by [14, Example 3], but we use different matrices from there. Let A be any ring and consider matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in $D_4(A)$; and let $f(x) = A + Bx$, $g(x) = C + Dx$. Then $f(x)g(x) = 0$, but $AC = E_{14} \neq 0$. So $D_4(A)$ is not weak Armendariz. This implies $D_n(A)$ cannot be weak Armendariz by help of (1) because $D_4(A)$ is a subring of $D_n(A)$, for all $n \geq 5$. □

The converses of Lemma 1.4(2), (3) need not hold by the following.

Example 1.5. (1) There exists a weak Armendariz ring that is not partial-Armendariz. We use the construction and argument in [16, Example 3.2]. Let $\mathbb{Z}_3[a, b]$ be the polynomial ring with commuting indeterminates a, b over $\mathbb{Z}_3$, and I be the ideal of $\mathbb{Z}_3[a, b]$ generated by a^3, a^2b^2, b^3. Next set $R = \mathbb{Z}_3[a, b]/I$. Then R is weak Armendariz by [16, Example 3.2]. Identify a, b with their images in R for simplicity. Consider the equality $(a+bx)^3 = (a+bx)(a^2+2abx+b^2x^2) = 0$ for $a+bx, a^2+2abx+b^2x^2 \in R[x]$. But $a(2ab) = 2a^2b \neq 0$, so R is not partial-Armendariz.

(2) There exists an Abelian ring that is not partial-Armendariz. We refer to the argument in [14, Example 3]. Let S be an Abelian ring and $R = D_4(S)$. Then R is Abelian by [10, Lemma 2]. Let $f(x) = E_{12} + (E_{12} - E_{13})x, g(x) = E_{34} + (E_{24} + E_{34})x \in R[x]$ as in [14, Example 3]. Then $f(x)g(x) = 0$ but $E_{12}(E_{24} + E_{34}) = E_{14} \neq 0$ and so R is not partial-Armendariz.

The classes of IFP rings and (weak) Armendariz rings are independent of each other by [17, Example 3.2] and [11, Example 14]. However for a semiprime right Goldie ring R, R is Armendariz if and only if R is IFP by [11, Corollary 13]. Given an IFP ring R, a simple computation gives us $W(R) = N(R) =$

$N_*(R) = N^*(R)$. So whenever a ring R is semiprime, R is reduced if and only if R is IFP.

Summarzing the results above, we have the following diagram:

$$\begin{array}{ccccc}
\bullet\ \{\text{Armendariz rings}\} & \Rightarrow & \{\text{partial-Armendariz rings}\} & \Rightarrow & \{\text{weak Armendariz rings}\} \\
 & & & & \Downarrow \\
\Uparrow & & & & \{\text{Abelian rings}\} \\
 & & & & \Uparrow \\
\{\text{reduced rings}\} & & \Longrightarrow & & \{\text{IFP rings}\}
\end{array}$$

2. Near related ring properties to partial-Armendariz

In this section we observe relations between partial-Armendariz rings and other ring properties which are nearly related.

We next observe a kind of algebraic structure in which the power-Armendariz and (weak) Armendariz are equivalent. Let K be a field and R_1, R_2 be K-algebras. Use $R_1 *_K R_2$ to denote the ring coproduct (free product) of R_1 and R_2 (see Antoine [2] and Bergman [6, 7] for details).

Theorem 2.1. *Let K be a field and A be a K-algebra. Let $C = K[b]$ be the polynomial ring with an indeterminate b over K, and I be the ideal of C generated by b^n for some $n \geq 2$. Set $B = C/I$ and $R = A *_K B$. Then the following conditions are equivalent*:

(1) *R is Armendariz*;
(2) *R is partial-Armendariz*;
(3) *R is weak Armendariz*;
(4) *$N(R)$ forms a subring of R*;
(5) *$N(R)$ is multiplicatively closed.*

Proof. The conditions (1), (3), and (5) are equivalent by [13, Theorem 1]. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) are obvious. (2) $\Rightarrow$ (4) is proved by Lemma 1.4(4). □

Following [8], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Every von Neumann regular ring R is clearly semiprimitive (i.e., $J(R) = 0$).

Proposition 2.2. *For a von Neumann regular ring R, the following conditions are equivalent*:

(1) *R is reduced*;
(2) *R is Armendariz*;
(3) *R is partial-Armendariz*;
(4) *R is weak Armendariz*;
(5) *R is IFP*;
(6) *R is Abelian.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (5), and (5) $\Rightarrow$ (6) are obvious. (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6) are proved by Lemma 1.4(2) and Lemma 1.4(3), respectively.

(6) $\Rightarrow$ (1) is shown by [8, Theorem 3.2]. (4) $\Rightarrow$ (1) is proved by [1, Theorem 6]. $\square$

In the following we see a condition under which Armendariz, partial-Armendariz, and IFP are related.

Proposition 2.3. *Given a ring R the following conditions are equivalent*:
(1) *R is a reduced ring*;
(2) *$D_3(R)$ is Armendariz*;
(3) *$D_3(R)$ is partial-Armendariz*;
(4) *$D_3(R)$ is weak Armendariz*;
(5) *$D_3(R)$ is IFP*;
(6) *$D_2(R)$ is weak Armendariz.*

Proof. The conditions (1), (2), and (5) are equivalent by [12, Proposition 2.8]. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (6) are obvious. (3) $\Rightarrow$ (4) is proved by Lemma 1.4(2). So it suffices to prove (6) $\Rightarrow$ (1). The proof is done by the proof of [16, Theorem 2.3], but we take here another computation. Let $D_2(R)$ be weak Armendariz, and assume on the contrary that there exist $0 \neq a \in R$ with $a^2 = 0$. Consider two polynomials

$$f(x) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \text{ and } g(x) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$$

in $D_2(R)[x]$. Then $f(x)g(x) = 0$ but $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. Hence $D_2(R)$ is not weak Armendariz, a contradiction. Therefore R is reduced. $\square$

The following illuminates Proposition 2.3.

Remark 2.4. (1) The proof of (4) $\Rightarrow$ (1) in Proposition 2.3 is done by [12, Proposition 2.8], but one can use different matrices from there as follows. Let $D_3(R)$ be weak Armendariz, and assume on the contrary that there exists $0 \neq a \in R$ with $a^2 = 0$. Consider two polynomials

$$f(x) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \text{ and } g(x) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & a \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x$$

in $D_3(R)[x]$. Then $f(x)g(x) = 0$ but $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Hence $D_3(R)$ is not weak Armendariz, a contradiction. Therefore R is reduced.

(2) From the equivalence of R being reduced and $D_3(R)$ being IFP in Proposition 2.3, we obtain that if R is reduced, then $D_2(R)$ is IFP. So one may ask whether R is reduced when $D_2(R)$ is IFP. But the answer is negative as can seen by $D_2(R)$ is commutative (hence IFP) over any commutative (possibly non-reduced) ring R.

(3) Observe that for any ring A we have $N(A) \neq 0$ if and only if $N_2(A) \neq 0$, where $N_2(A) = \{r \in N(A) \mid r^2 = 0\}$. Suppose that $D_n(R)$ is weak Armendariz for $n \geq 2$. Then we claim that R is reduced. Assume on the contrary that there exists $0 \neq a \in R$ with $a^2 = 0$. Consider $f(x) = aI_n + E_{1n}x$ and $g(x) = aI_n - E_{1n}x$ in $D_n(R)[x]$, where I_n is the n by n identity matrix. Then $f(x)g(x) = 0$ but $aI_nE_{1n} = aE_{1n} \neq 0$. So $D_n(R)$ is not weak Armendariz, contrary to the assumption. So R must be reduced when $D_n(R)$ is weak Armendariz for $n \geq 2$. However $D_n(A)$ is not weak Armendariz over any ring A when $n \geq 4$, by Lemma 1.4(6). This argument illuminates the proof of Proposition 2.3.

Let R be a ring and $n \geq 2$. Recall the subring

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}$$

of $D_n(R)$. Observe that $V_n(R)$ is isomorphic to $R[x]/(x^n)$, where (x^n) is the ideal of $R[x]$ generated by x^n. One may compare the following with Proposition 2.3, checking the difference between $D_n(R)$ and $V_n(R)$.

Proposition 2.5. *For a ring R and $n \geq 2$, the following are equivalent*:
(1) *R is reduced*;
(2) *$V_n(R)$ is Armendariz*;
(3) *$V_n(R)$ is partial Armendariz*;
(4) *$V_n(R)$ is weak Armendariz.*

Proof. Observe first that $V_n(R)$ is isomorphic to $R[x]/(x^n)$. (1) $\Rightarrow$ (2) is proved by [1, Theorem 5]. (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (4) is shown by Lemma 1.4(2). (4) $\Rightarrow$ (1) is proved by [16, Theorem 3.1]. □

By help of [15, Theorem 2.5], we have that if R is a reduced ring, then $R[x]/(x^n)$ (hence $V_n(R)$) is IFP. But the converse need not hold because $R[x]/(x^n)$ is commutative (hence IFP) over a commutative (possibly non-reduced) ring R.

Recall that a semiprime ring R is IFP if and only if R is reduced. The following extends [15, Proposition 2.8]. It is well-known that a ring R is semiprime right Goldie if and only if the classical right quotient ring of R exists and is semisimple Artinian.

Proposition 2.6. *Let R be a semiprime right Goldie ring with Q its classical right quotient ring. Then the following conditions are equivalent*:
(1) *R is Armendariz*;
(2) *R is reduced*;
(3) *R is partial-Armendariz*;
(4) *R is weak Armendariz*;
(5) *Q is Armendariz*;
(6) *Q is reduced*;
(7) *Q is partial-Armendariz*;

(8) *Q is weak Armendariz*;
(9) *Q is a finite direct product of division rings.*

Proof. The conditions (1), (2), (5), and (6) are equivalent by [15, Proposition 2.8]. (1) ⇒ (3) and (5) ⇒ (7) are obvious. (3) ⇒ (4) and (7) ⇒ (8) are shown by Lemma 1.4(2).

(4) ⇒ (9). Let R be weak Armendariz. Then R is Abelian by [16, Lemma 3.4]; hence Q is a finite direct product of division rings because Q is semisimple Artinian. The remainder of the proof follows the condition (9). □

Acknowledgments. The authors thank the referees deeply for very careful reading of the manuscript and valuable suggestions in depth that improved the paper by much. The second named author was supported by the National Natural Science Foundation of China(11361063). The third named author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIP; Ministry of Science, ICT & Future Planning) (No. 2017R1C1B5017863).

References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272. https://doi.org/10.1080/00927879808826274
[2] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140. https://doi.org/10.1016/j.jalgebra.2008.01.019
[3] ______, *Examples of Armendariz rings*, Comm. Algebra **38** (2010), no. 11, 4130–4143. https://doi.org/10.1080/00927870903337968
[4] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.
[5] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368. https://doi.org/10.1017/S0004972700042052
[6] G. M. Bergman, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. **200** (1974), 1–32. https://doi.org/10.2307/1997246
[7] ______, *Coproducts and some universal ring constructions*, Trans. Amer. Math. Soc. **200** (1974), 33–88. https://doi.org/10.2307/1997247
[8] K. R. Goodearl, *von Neumann Regular Rings*, Monographs and Studies in Mathematics, **4**, Pitman (Advanced Publishing Program), Boston, MA, 1979.
[9] J. Han, T. K. Kwak, C. I. Lee, Y. Lee, and Y. Seo, *Generalizations of reversible and Armendariz rings*, Internat. J. Algebra Comput. **26** (2016), no. 5, 911–933. https://doi.org/10.1142/S0218196716500387
[10] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52. https://doi.org/10.1016/S0022-4049(01)00149-9
[11] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761. https://doi.org/10.1081/AGB-120013179
[12] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), no. 1, 135–146. https://doi.org/10.4134/BKMS.2009.46.1.135
[13] D. W. Jung, N. K. Kim, Y. Lee, and S. P. Yang, *Nil-Armendariz rings and upper nilradicals*, Internat. J. Algebra Comput. **22** (2012), no. 6, 1250059, 13 pp. https://doi.org/10.1142/S0218196712500592
[14] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488. https://doi.org/10.1006/jabr.1999.8017

[15] ———, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223. https://doi.org/10.1016/S0022-4049(03)00109-9

[16] T.-K. Lee and T.-L. Wong, *On Armendariz rings*, Houston J. Math. **29** (2003), no. 3, 583–593.

[17] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17. http://projecteuclid.org/euclid.pja/1195510144

Sang Bok Nam
Department of Computer Engineering
Kyungdong University
Geseong 24764, Korea
Email address: k1sbnam@kduniv.ac.kr

Zhelin Piao
Department of Mathematics
Yanbian University
Yanji 133002, P. R. China
Email address: zlpiao@ybu.edu.cn

Sang Jo Yun
Department of Mathematics
Dong-A University
Busan 49315, Korea
Email address: sjyun@dau.ac.kr