Commun. Korean Math. Soc. **34** (2019), No. 3, pp. 701–718
https://doi.org/10.4134/CKMS.c180226
pISSN: 1225-1763 / eISSN: 2234-3024

THE U-PROJECTIVE RESOLUTION OF MODULES OVER PATH ALGEBRAS OF TYPES A_n AND $\tilde{A}_n$

Karin Baur, Yudi Mahatma, and Intan Muchtadi-Alamsyah

Abstract. In this paper we compute the U-projective resolution of kQ-modules where Q is quiver of type A_n and $\tilde{A}_n$. The behavior of the sequence can be seen through its geometric representation.

1. Introduction

Projective resolution of modules plays an important role in homological algebra. In 2002, Davvaz and Shabani-Solt introduced the notions of U-complex, U-homology, etc. (see [5]) to generalize certain concepts in homological algebra. Inspired by this, the second and the third authors introduced the notions of U-projective resolutions and U-extension modules in 2016 (see [7]).

We also found that every short exact sequence of modules and module homomorphisms over hereditary algebra can always be extended into a long exact sequence of U-homologies (the modified homologies) consisting of U-extension modules. This encouraged us to further research on U-extension modules over hereditary algebra. The algebra of our interest is a path algebra generated by a finite acyclic quiver. In this paper we will discuss U-projective resolutions of modules over such an algebra.

A sequence $\cdots \xrightarrow{d_{k+2}} M_{k+1} \xrightarrow{d_{k+1}} M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \cdots$ of modules and module homomorphisms is said to be *exact* if $\mathrm{Im} d_{k+1} = \ker d_k$ for every k. In [4] Davvaz and Parnian-Garamaleky introduced the notion of U-exact sequence, which is a generalization of an exact sequence. The idea is to replace ker d_k with $d_k^{-1}(U_{k-1})$, for every k, where U_k is a submodule of M_k for each k. Davvaz and Shabani-Solt then redefined this concept in [5] using the so-called U-complex approach which further assumes that Im d_{k+1} must contain U_k for every k. Thus, a U-exact sequence is a sequence $\cdots \xrightarrow{d_{k+2}} M_{k+1} \xrightarrow{d_{k+1}} M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \cdots$ of modules and module homomorphisms such that $U_k \subseteq \mathrm{Im} d_{k+1} = d_k^{-1}(U_{k-1})$ where U_k is a submodule of M_k, for each integer k.

Received May 25, 2018; Revised October 12, 2018; Accepted October 24, 2018.
2010 *Mathematics Subject Classification.* 16G06, 13D06, 13C10.
Key words and phrases. projective resolution, module, path algebra, quiver.

Mahatma and Muchtadi-Alamsyah extended these result by proposing in [7] a way to define U-projective resolutions and U-extension modules. The aim of this paper is to state a formula for U-projective resolutions of kQ-modules where Q is quiver of type A_n and $\tilde{A}_n$. For an introduction to quivers and path algebras see [1, Chapter II].

2. The U-projective resolution of a module

Given any module M, there always exists a projective resolution of M, i.e., an exact sequence $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ where P_i's are projective for all $i = 1, 2, 3, \ldots$. If we replace exactness with U-exactness, then we have what we call a U-projective resolution. The complete algorithm to construct a U-projective resolution of an A-module M for any nonzero submodule U of M was given in [7]. Here, we recall the construction:

Given any module M there always exists a projective module P_0 that can map onto M. Hence we have a sequence $P_0 \xrightarrow{d_0} M \to 0$. By the same reason, we can extend the sequence into $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ for a projective module P_1 so that the new sequence is U-exact at P_0, i.e., $d_1(P_1) = d_0^{-1}(U)$. Now, set $U_0 := d_0^{-1}(U)$ and extend the last sequence into $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ for a suitable projective module P_2 so that the new sequence is U_0-exact at P_1, i.e., $d_2(P_2) = d_1^{-1}(U_0)$. Now set $U_1 := d_1^{-1}(\ker d_0)$ and continue the process. By doing this, we obtain a sequence $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ where all P_i's are projective and which is U_{i-1}-exact at P_i, i.e., $d_{i+1}(P_{i+1}) = d_i^{-1}(U_{i-1})$ for $i = 1, 2, 3, \ldots$. We denote this resolution by $\mathbf{P} : P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$. It has been shown in [7] that if both $\mathbf{P} : P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$ and $\mathbf{Q} : Q_\bullet(V_\bullet) \xrightarrow{e_\bullet} M(U)$ are U-projective resolutions of M, then there exist two U-complex morphisms $\mathbf{f} : \mathbf{P} \to \mathbf{Q}$ and $\mathbf{g} : \mathbf{Q} \to \mathbf{P}$ such that $\mathbf{gf} \simeq 1_P$ and $\mathbf{fg} \simeq 1_Q$. This implies that the U-extension modules generated by U-projective resolutions are unique up to isomorphism. Hence the projective modules P_i may be chosen arbitrarily.

In this paper we will consider the U-projective resolution only for case $U \neq 0$. One might suggest that the U-projective resolution of M can be substituted by the projective resolution of M/U. In fact, they are not identical as the extension modules generated from the resolutions can be not isomorphic. For example, if both M and U are projective, then $\mathrm{Ext}^1(M[U], U) = 0$ (see [7]) while the module $\mathrm{Ext}^1(M/U, U)$ is not necessarily zero. (See Chapter 7 of [8] for the fundamental concept of extension modules.)

Since all algebras discussed in this paper are hereditary, the last nonzero module in any U-projective resolution will be either P_2 or P_3. We will say that the *length* of the resolution is 2 or 3, respectively, as we now explain.

Clearly, if M is projective, then we may set $P_0 := M$ and $d_0 := 1_m$. As a consequence, $U_0 = d_0^{-1}(U) = U$. Now since the algebra is hereditary and

M is projective, then $d_0^{-1}(U) = U \subseteq M$ must be projective and hence we may set $P_1 := U$ and d_1 as an inclusion from U to M. This implies that $U_1 = d_1^{-1}(\ker d_0) = d_1^{-1}(0) = 0$ and hence we may set $P_3 := 0$. Since we have reached the zero module then we can set all the next modules and maps as zero. Hence the resolution will be as follows:

$$\begin{array}{ccccccccccc} 0 & \to & U & \to & U & \to & M & \to & M & \to & 0 \\ & & & & & & \uparrow & & \uparrow & & \\ & & & & & & U & & U & & \end{array} \tag{2.1}$$

where the objects written in the bottom row are the U_i's. Note that when U_i is not written, we mean that $U_i = 0$.

Now, for the non projective case over hereditary algebra, we start with a map $P \xrightarrow{d} M \to 0$ with P projective. Set $P_0 := P$ and $d_0 := d$. Since the algebra is hereditary and P is projective then $d^{-1}(U) \subseteq P$ must be projective. Hence we may set $P_1 := d^{-1}(U)$ and d_1 as the inclusion from $d^{-1}(U)$ to P. As a consequence, we have $U_1 = d_1^{-1}(\ker d) = \ker d$. Now, since $d_1^{-1}(U_0) = d_1^{-1}\left(d^{-1}(U)\right) = d^{-1}(U)$ is projective then we may set $P_2 := d^{-1}(U)$ and $d_2 := 1_{d^{-1}(U)}$. This implies that $U_2 = d_2^{-1}(\ker d_1) = d_2^{-1}(0) = 0$. Next, since $d_2^{-1}(U_1) = d_2^{-1}(\ker d) = \ker d \subseteq d^{-1}(U)$ is projective then we may set $P_3 := \ker d$ and d_3 as the inclusion from $\ker d$ to $d^{-1}(U)$. This implies that $U_3 = d_3^{-1}(\ker d_2) = d_2^{-1}(0) = 0$. Now, $d_3^{-1}(U_2) = d_3^{-1}(0) = 0$ and hence we may set $P_4 := 0$ and the process can be stopped. Thus, the resolution will be as follows:

$$\begin{array}{ccccccccccc} 0 & \to & \ker d & \to & d^{-1}(U) & \to & d^{-1}(U) & \to & P & \xrightarrow{d} & M & \to & 0 \\ & & & & & & \uparrow & & \uparrow & & \uparrow & & \\ & & & & & & \ker d & & d^{-1}(U) & & U & & \end{array} \tag{2.2}$$

Theorem 1. *An algebra A is hereditary if and only if every U-projective resolution of an A-module has length 2 or 3.*

Proof. We only need to prove the sufficiency. Suppose that A is an algebra with every U-projective resolution of an A-module has length 2 or 3. Suppose that there is an ideal I of A which is not projective as an A-module. Consider the surjection $d : P \to I$ with P projective. The A/I-projective resolution of A/I will be as follows:

$$\begin{array}{ccccccccccccc} X & \to & P & \xrightarrow{d} & A & \xrightarrow{1_A} & A & \xrightarrow{1_A} & A & \xrightarrow{\pi} & A/I & \to & 0 \\ & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\ & & \ker d & & & & I & & A & & A/I & & \end{array} \tag{2.3}$$

Since I is non projective then d is not an inclusion and hence $X \neq 0$ contrary to the assumption that the length must not exceed 3. Thus, every ideal of A must be projective as an A-module which means A is hereditary. $\square$

We restrict the discussion in this paper only for resolutions of indecomposable modules. For decomposable case, the problem can be reduced by the following fact: given any finite collection of modules $M_1, M_2, \dots, M_n$, if we have constructed the U_α-projective resolution of M_α for each α, then we can obtain the $U_1 \oplus U_2 \oplus \cdots \oplus U_n$-projective resolution of $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ by just taking the direct sum of the corresponding components of the sequences. However, this has not covered all the situations since it is possible for a decomposable module M to have submodule which is not direct sum of submodules of direct summands of M.

3. Diagonal representation of a kQ-module of type A_n

In this section we restrict the discussion to path algebras generated by the quiver Q of the form $n \to (n-1) \to (n-2) \to \cdots \to 1$ which is a special case of quiver of type A_n. All the indecomposable modules over such an algebra will be represented through a geometric model: every indecomposable module can be viewed as a diagonal in an $(n+3)$-gon. The algebra kQ itself is given by the following triangulation of an $(n+3)$-gon.

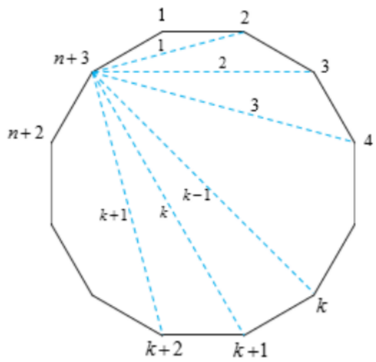

Figure 1. Triangulation for quiver $Q : n \to (n-1) \to (n-2) \to \cdots \to 1$.

Each vertex of Q corresponds to one of the diagonals in Figure 1. The arrows correspond to clockwise rotations to the nearest neighbor (so-called minimal rotations). See Subsection 3.1.3 of [9] for details. The indecomposable kQ-modules can be described as diagonals in this polygon intersecting the diagonals of the chosen triangulations. The simple modules intersect exactly one diagonal and inductively we get all the indecomposable modules from this: consider the module $0 \to \cdots \to 0 \to K^{(b)} \to K^{(b-1)} \to \cdots \to K^{(a)} \to 0 \to \cdots \to 0$ for some $n \geq b \geq a \geq 1$, i.e., the representation of Q supported at vertices $b, b-1, \dots,$ and a. The geometric representation of such a module is given by the diagonal which intersects the diagonals $b, b-1, \dots,$ and a, i.e., the diagonal whose endpoints are a and $b+2$. Thus, as notation for this module we choose $(a, b+2)$. Further, a module (i, j) is: projective if and only if $i = 1$; injective if and only if $j = n+2$; and simple if and only if $j = i+2$ ([1, Section III.2]). Also, a module (a, b) is a submodule of (i, j) if and only if $a = i$ and $i + 2 \leq b \leq j$.

We shall denote the projective module $(1, k+2)$ by $P(k)$, the injective module $(k, n + 2)$ by $I(k)$, and the simple module $(k, k + 2)$ by $S(k)$. Clearly, $P(1) = S(1)$, $P(n) = I(1)$, and $I(n) = S(n)$.

The AR-quiver of this type of an algebra is given as follows ([1, Section IV.4], see also [6]).

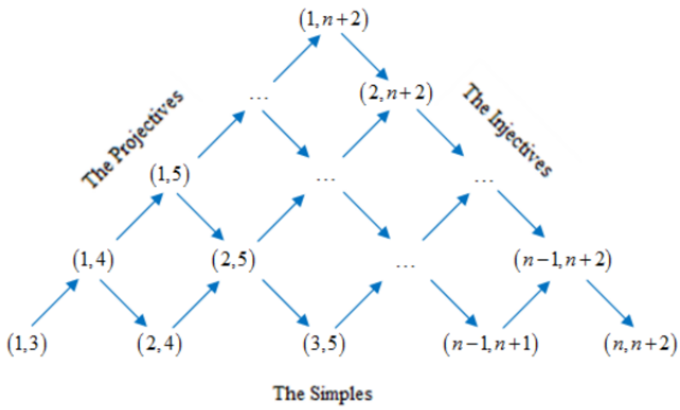

FIGURE 2. Auslander-Reiten quiver for path algebra of quiver $n \to (n-1) \to (n-2) \to \cdots \to 1$.

4. Geometric representation of a kQ-module of type $\tilde{A}_n$

For algebras of type $\tilde{A}_n$, we will use the geometric model given by Warkentin in his diploma thesis ([10, Chapter 4]). Further details can also be found if [2] and [3]. In this paper we restrict the discussion to quiver of type $\tilde{A}_n$ of the form

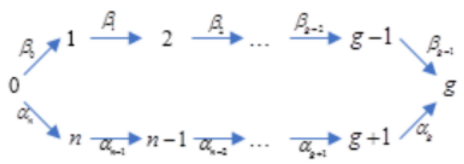

FIGURE 3. Quiver of type $\tilde{A}_n$.

Clearly, there are g clockwise arrows in the quiver described in Figure 3. Now if we denote the number of anti-clockwise arrows by h, then we have $n = g+h-1$. We shall denote this quiver by $Q_{g,h}$. The geometric representation of the algebra will be based on following triangulation in an annulus.

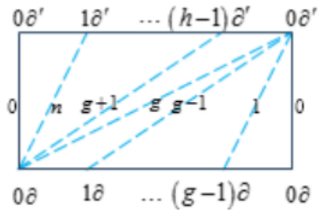

FIGURE 4. Triangulation for $Q_{g,h}$.

Indecomposable modules correspond to oriented arcs crossing these diagonals in a similar way as for type A_n. First we state the convention that for post-projective modules we will orient the arcs from the base to the top, and for pre-injective modules in the opposite direction. If both endpoints are in the same boundary, then we orient the arc from left to right. As an example, we consider the geometric representation for the module $P(i)$, $0 < i \leq g$, whose representation is as follows:

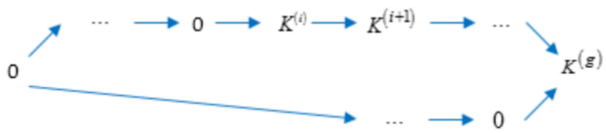

FIGURE 5. Representation of module $P(i)$, $0 < i \leq g$.

We must represent this module with an arc intersecting the edges $i, i+1, \dots, g$ in the fixed triangulation of the annulus. Hence we represent the module by the following oriented arc:

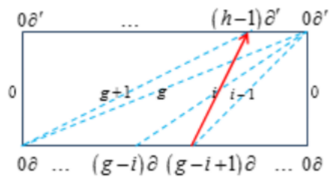

FIGURE 6. Geometric representation of module P_i, $0 < i \leq g$.

We will use ∂ to denote the lower boundary and ∂' for the upper boundary. So we can denote the module as $((g-i+1)\partial, (h-1)\partial')$ or we will just write it as $\left(\underline{g-i+1}, \overline{h-1}\right)$ to drop ∂ and ∂'. Note that, in the figure, the module $P(1)$ has starting point the right hand copy of 0∂. Hence the appropriate notation for $P(1)$ is $(\underline{0}, \overline{h-1})$ instead of $(\underline{g}, \overline{h-1})$. To make the unambiguous, we use $\left(\underline{(g-1+1) \mod g}, \overline{h-1}\right)$ as notation for $P(i)$, $0 < i \leq g$.

Next, consider the module $P(0)$ whose representation is as follows:

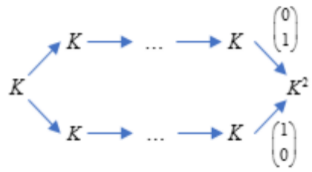

FIGURE 7. Representation of module $P(0)$.

Clearly, the geometric representation of $P(0)$ is

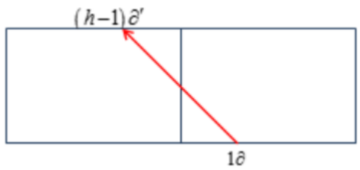

FIGURE 8. Geometric representation of P_0.

which can be shortened as

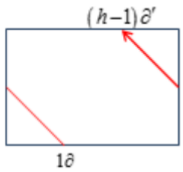

FIGURE 9. Geometric representation of P_0 (shortened).

Hence our notation for $P(0)$ is $(\underline{1}, \overline{h-1})$.

Now, we point out that there are many other modules whose arcs start at 1∂ and end at $(h-1)\partial'$. For example, consider the following arc:

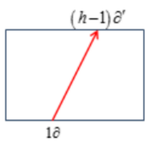

FIGURE 10. Geometric representation of $P(g) = S(g)$.

Which correspond to the module $P(g) = S(g)$, and next consider the following arc:

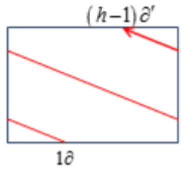

Figure 11. Another geometric representative whose endpoints are $\underline{1}$ and $\overline{h-1}$.

We see from these examples that we need more information to distinguish between these three modules. Let us add an integer in the notation that shows how many times the arc crosses the edge 0 from right to left. If the arc crosses in the other direction, then we put negative sign on the number. As a consequence, the notation for $P(0)$, $P(g) = S(g)$, and the module in Figure 11 are $(\underline{1}, \overline{h-1}, 1)$, $(\underline{1}, \overline{h-1}, 0)$, and $(\underline{1}, \overline{h-1}, 2)$ respectively. Note also that the notation for the module $P(i), 0 < i \leq g$, is $\Big(\underline{(g-i+1) \mod g}, \overline{h-1}, 0\Big)$.

Given a notation of a module whose arc contains an endpoint 0, we should not be confused by which copy of 0 should be the endpoint of the arc in the geometric representation since we require that the arc must cross at least one edge. As an example, consider the module denoted by $(\underline{0}, \overline{h-1}, 0)$. In this case, the arc must be started from the right hand copy of 0∂ for if it was started from the left hand copy of 0∂, then the whole arc will coincide with the edge $g+1$ (see Figure 4 for reference) and this is not allowed. Hence the module corresponds with this notation is unique.

Here are the descriptions of the projective, simple, and injective $kQ_{g,h}$-modules. Not first that $S(0) = I(0)$.

Table 1. Geometric notation of projective, simple, and injective $kQ_{g,h}$-modules.

Module	Notation
$P(i), 0 < i \leq g$	$\Big(\underline{(g-i+1) \mod g}, \overline{h-1}, 0\Big)$
$P(i), g < i \leq n$	$(\underline{1}, \overline{n-i}, 0)$
$S(i), 0 < i < g$	$\Big(\underline{g-i-1}, \underline{(g-i+1) \mod g}, 0\Big)$
$S(i), g < i \leq n$	$\Big(\overline{n-i}, \overline{(n-i+2) \mod h}, 0\Big)$
$I(i), 0 \leq i < g$	$(\overline{1}, \underline{g-i-1}, 1)$
$I(i), g < i \leq n$	$\Big(\overline{(n-i+2) \mod h}, \underline{g-1}, 1\Big)$
$I(g)$	$(\overline{1}, \underline{g-1}, 2)$

5. General results

In this section we describe U-projective resolutions for indecomposable modules over the algebras from Section 3 and Section 4.

5.1. Type A_n

We begin this section with the results for an algebra of type A_n with linear orientation. We will compute resolutions for non projective indecomposable modules.

Clearly, for every module $(i,j), 1 \leq i \leq j-2$, we have a surjective map $(1,j) \xrightarrow{d} (i,j) \to 0$. Thus, we can take $(1,j)$ to be the P_0.

Now consider the submodule (i,m) of (i,j) which is given by the module $0 \to \cdots \to 0 \to K^{(m-2)} \to \cdots \to K^{(i)} \to 0 \to \cdots \to 0$. To find $d^{-1}((i,m))$, note that $d = \left(d_t : K^{(t)} \to K^{(t)}\right)$ is described as follows:

$$\begin{array}{ccccccccccccccc} 0 & \to & \cdots & \to & 0 & \to & K^{(j-2)} & \to & \cdots & \to & K^{(i)} & \to & K^{(i-1)} & \to & \cdots & \to & K^{(1)} \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ 0 & \to & \cdots & \to & 0 & \to & K^{(j-2)} & \to & \cdots & \to & K^{(i)} & \to & 0 & \to & \cdots & \to & 0 \end{array} \tag{5.1}$$

Clearly, we have $d_t = 1_k$ if $i \leq t \leq j-2$ and 0 otherwise. Hence, $d^{-1}((i,m))$ is given by the module $0 \to \cdots \to 0 \to K^{(m-2)} \to \cdots \to K^{(1)}$ whose diagonal notation is $(1,m)$. From this we have $P_1 = P_2 = (1,m) = P(m-2)$.

Now, from (5.1) again we can easily verify that $\ker d$ is given by the module $0 \to \cdots \to 0 \to K^{(i-1)} \to \cdots \to K^{(1)}$ whose diagonal notation is $(1,i+1)$. From this we have $P_3 = (1,i+1) = P(i-1)$. We state the result in the following lemma.

Lemma 1. *The (i,m)-projective resolution of (i,j) is given by:*

$$\begin{array}{ccccccccccc} 0 & \to & (1,i+1) & \to & (1,m) & \to & (1,m) & \to & (1,j) & \to & (i,j) & \to & 0 \\ & & & & & & \uparrow & & \uparrow & & \uparrow & & \\ & & & & & & (1,i+1) & & (1,m) & & (i,m) & & \end{array} \tag{5.2}$$

In the AR-quiver, the (i,m)-projective resolution of (i,j) has the following form:

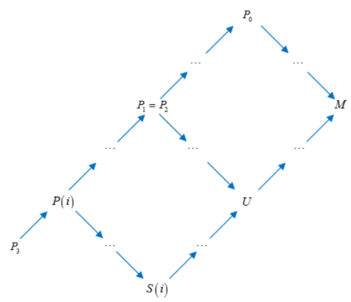

FIGURE 12. The U-projective resolution of $M = (i,j)$ viewed in the AR-quiver.

Note again that every module (i,j) has exactly $j-i-1$ nonzero submodules. Hence for a chosen i we can consider $1+2+\cdots+(n-i+1)$ different U-projective resolutions with $U \neq 0$. Since we consider only the resolution of indecomposable non projective modules, where i runs from 2 to n, then there are in total of

$$\begin{aligned}
\sum_{i=2}^{n}(1+2+\cdots+(n-i+1)) &= \sum_{i=2}^{n}\frac{(n-i+1)(n-i+2)}{2} \\
&= \sum_{l=1}^{n-1}\frac{l(l+1)}{2} \\
&= \frac{1}{2}\left(\sum_{l=1}^{n-1} l^2 + \sum_{l=1}^{n-1} l\right) \qquad (5.3)\\
&= \frac{1}{2}\left(\frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}\right) \\
&= \frac{n^3-n}{6}
\end{aligned}$$

possible U-projective resolutions we have to consider for linear oriented type A_n.

For example, there are 20 different U-projective resolutions with $U \neq 0$ for indecomposable non projective modules over path algebra A_5. We give here the geometric representation of 4 of them, which are the resolutions of $I(2)$. Recall that $I(2)$ has exactly 4 nonzero submodules: $I(2) = (2,7), (2,6), (2,5)$, and $(2,4) = S(2)$. The dotted lines in Figure 13 represent the U_i's.

5.2. Type $\tilde{A}_n$

We restrict the computation of the U-projective resolutions only for the simples and for the injectives. We distinguish 2 cases: $U = M$ and $U \neq M$ (always assuming $U \neq 0$). For the latter case, we will restrict to the situation $M = I(i)$ while $U = S(i)$. Recall that the sequence will have the form (2.2). Hence from now we will not write down the U_i's on the sequence since U_0 appears as P_1 and P_2, and U_1 appears as P_3. Further, on the case $U = M$ we have $P_0 = P_1 = P_2$.

Lemma 2. *For $U \neq 0$, the U-projective resolution of $S(i)$ is given by:*

(1) $0 \to P(i+1) \to P(i) \to P(i) \to P(i) \to S(i) \to 0$ *if* $0 < i < g$, *and*
(2) $0 \to P(i-1) \to P(i) \to P(i) \to P(i) \to S(i) \to 0$ *if* $g < i \leq n$.

Proof. Since $S(i)$ is simple, the only nonzero submodule of $S(i)$ is $S(i)$. Now consider the map $d : P(i) \to S(i)$ defined by

$$d = (d') := \begin{cases} 1_K & \text{if } t = i, \\ 0 & \text{if } t \neq i. \end{cases}$$

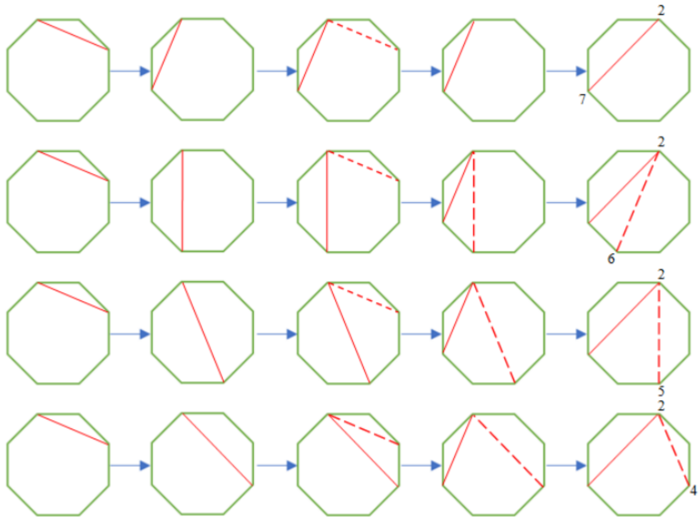

FIGURE 13. Geometric representation for U-projective resolution of $I(2)$ in type A_5. From top to bottom: $U = I(2)$, $U = (2, 6)$, $U = (2, 5)$, $U = S(2)$.

Clearly, d is a surjection whose kernel is $P(i+1)$ if $0 < i < g$ and $P(i-1)$ if $g < i \leq n$. The result then follows. $\square$

Lemma 3. *The $I(i)$-projective resolution of $I(i)$ is given by:*

(1a) $0 \to P(i+1) \oplus P(n) \to P(0) \to P(0) \to P(0) \to I(i) \to 0$ *if* $0 \leq i < g$, *and*

(1b) $0 \to P(1) \oplus P(i-1) \to P(0) \to P(0) \to P(0) \to I(i) \to 0$ *if* $g < i \leq n$.

The $S(i)$-projective resolution of $I(i)$ is given by:

(1a) $0 \to P(i+1) \oplus P(n) \to P(i) \oplus P(n) \to P(i) \oplus P(n) \to P(0) \to I(i) \to 0$ *if* $0 < i < g$, *and*

(1b) $0 \to P(1) \oplus P(i-1) \to P(1) \oplus P(i) \to P(1) \oplus P(i) \to P(0) \to I(i) \to 0$ *if* $g < i \leq n$.

Proof. We will show this only for the case $0 \leq i < g$. Consider the map $d : P(0) \to I(i)$ defined by

$$d = (d') := \begin{cases} 1_k, & \text{if } 0 \leq t \leq i, \\ 0, & \text{if } i < t \leq n, \ t \neq g, \\ 0_{2\times 1}, & \text{if } t = g. \end{cases}$$

Clearly, d is a surjection whose kernel is $P(i+1) \oplus P(n)$. Hence we have (1a). Now, for $i \neq 0$ we have $d^{-1}(S(i)) = P(i) \oplus P(n)$. Hence (1b) holds. The other parts can be handled similarly. $\square$

5.3. Geometric notation for type $\tilde{A}_n$

Now we give the geometric notation of the results obtained in previous subsections.

CASE 1: $U = M$

TABLE 2. Geometric notation for the simple case, $U = M$

	$S(i)$	
	$0 < i < g$	$g < i \leq n$
$M = U$	$\big(\underline{g-i-1}, \underline{(g-i+1) \mod g}, 0\big)$	$\big(\overline{n-i}, \overline{(n-i+2) \mod h}, 0\big)$
$P_0 = P_1 = P_2 = U_0$	$\big(\underline{(g-i+1) \mod g}, \overline{h-1}, 0\big)$	$(\underline{1}, \overline{n-i}, 0)$
$P_3 = U_1$	$(\underline{g-i}, \overline{h-1}, 0)$	$\underline{1}, (\overline{n-i+1}, 0)$

TABLE 3. Geometric notation for the injective case, $U = M$

	$S(i)$	
	$0 < i < g$	$g < i \leq n$
$M = U$	$(\overline{1}, \underline{g-i-1}, 1)$	$\big(\overline{(n-i+2) \mod h}, \underline{g-1}, 1\big)$
$P_0 = P_1 = P_2 = U_0$	$(\underline{1}, \overline{h-1}, 1)$	
$P_3 = U_1$	$(\underline{g-i}, \overline{h-1}, 0) \cup (\underline{1}, \overline{0}, 0)$	$(\underline{0}, \overline{h-1}, 0) \cup (\underline{g-i+2}, \overline{h-1}, 0)$

CASE 2: $U \neq M$

TABLE 4. Geometric notation for the injective case taking the simple submodule as the submodule U in each case.

	$I(i)$	
	$0 < i < g$	$g < i \leq n$
M	$(\overline{1}, \underline{g-i-1}, 1)$	$\big(\overline{(n-i+2) \mod h}, \underline{g-1}, 1\big)$
$U = S(i)$	$\big(\underline{g-i-1}, \underline{(g-i+1) \mod g}, 0\big)$	$\big(\overline{n-i}, \overline{(n-i+2) \mod h}, 0\big)$
P_0	$(\underline{1}, \overline{h-1}, 1)$	
$P_1 = P_2 = U_0$	$(\underline{g-i+1}, \overline{h-1}, 0) \cup (\underline{1}, \overline{0}, 0)$	$(\underline{0}, \overline{h-1}, 0) \cup (\underline{g-i+1}, \overline{h-1}, 0)$
$P_3 = U_1$	$(\underline{g-i}, \overline{h-1}, 0) \cup (\underline{1}, \overline{0}, 0)$	$(\underline{0}, \overline{h-1}, 0) \cup (\underline{g-i+2}, \overline{h-1}, 0)$

5.4. Resolution for $I(g)$

The construction of U-projective resolutions for the module $I(g)$ is done in different way. Let us first describe the module $I(g)$ with Figure 3 as a reference. This will help us to understand the maps. The module $I(g)$ is given by $\left(K^{(t)}, \alpha, \beta\right)$, described as follows:

$$K^{(t)} = \begin{cases} K^2, \text{ if } t = 0, \\ K, \text{ if } t \neq 0, \end{cases}$$

$$\alpha_t = \begin{cases} (1 \quad 0), \text{ if } t = n, \\ 1_K, \text{ if } t \neq n, \end{cases}$$

$$\beta_t = \begin{cases} (0 \quad 1), \text{ if } t = 0, \\ 1_K, \text{ if } t \neq 0. \end{cases}$$

Now consider the module $P(0)^2 = \left(K^{(t)}, \alpha, \beta\right)$ where $K^{(t)}, \alpha_t$, and β_t are described as follows:

$$K^{(t)} = \begin{cases} K^4, \text{ if } t = g, \\ K^2, \text{ if } t \neq g, \end{cases}$$

$$\alpha_t = \begin{cases} \left(\frac{I_2}{0_{2\times 2}}\right), \text{ if } t = g, \\ I_2, \text{ if } t \neq g, \end{cases}$$

$$\beta_t = \begin{cases} \left(\frac{0_{2\times 2}}{I_2}\right), \text{ if } t = g-1, \\ I_2, \text{ if } t \neq g-1. \end{cases}$$

We can easily verify that the map $P(0)^2 \xrightarrow{d} I(g) \to 0$ is defined by

$$d = (d_t) = \begin{cases} I_2, \text{ if } t = 0, \\ (0 \quad 1), \text{ if } 1 \leq t < g, \\ (1 \quad 0 \quad 0 \quad 1), \text{ if } t = g, \\ (1 \quad 0), \text{ if } g < t \leq n. \end{cases}$$

Then the projective module P_0 of $I(g)$ is given by $P(0)^2$.

Now, we can easily verify that $\ker d$ is the module $\left(K^{(t)}, \alpha_t, \beta_t\right)$ given by the following properties:

$$K^{(t)} = \begin{cases} 0^2, \text{ if } t = 0, \\ K \oplus 0, \text{ if } 1 \leq t < g, \\ \langle (1,0,0,-1), (0,1,0,0), (0,0,1,0) \rangle, \text{ if } t = g, \\ 0 \oplus K, \text{ if } g < t \leq n, \end{cases}$$

$$\alpha_t = \begin{cases} 0_{2\times 2}, \text{ if } t = n, \\ E_{22}^{2\times 2}, \text{ if } g < t < n, \\ \left(\frac{E_{22}^{2\times 2}}{0_2}\right), \text{ if } t = g, \end{cases}$$

$$\beta_t = \begin{cases} 0_{2\times 2}, \text{ if } t = 0, \\ E_{11}^{2\times 2}, \text{ if } 1 \le t < g-1, \\ \left(\frac{0_2}{E_{22}^{2\times 2}}\right), \text{ if } t = g-1. \end{cases}$$

Consider the module $P(1) \oplus P(g) \oplus P(n) = \left(K^{(t)}, \alpha_t, \beta_t\right)$ with the following properties:

$$K^{(t)} = \begin{cases} 0^3, \text{ if } t = 0, \\ K \oplus 0^2, \text{ if } 1 \le t < g, \\ K^3, \text{ if } t = g, \\ 0^2 \oplus K, \text{ if } g < t \le n, \end{cases}$$

$$\alpha_t = \begin{cases} 0_{3\times 3}, \text{ if } t = n, \\ E_{33}^{(3\times 3)}, \text{ if } g \le t < n, \end{cases}$$

$$\beta_t = \begin{cases} 0_{3\times 3}, \text{ if } t = 0, \\ E_{11}^{(3\times 3)}, \text{ if } 1 \le t < g. \end{cases}$$

Now define a map $\ker d \xrightarrow{\varphi} P(1) \oplus P(g) \oplus P(n)$ as follows:

$$\varphi = \left(\varphi_t : K^{(t)} \to K^{(t)}\right) = \begin{cases} 0_{3\times 2}, \text{ if } t = 0, \\ \left(\frac{1 \quad 0}{0_{2\times 2}}\right), \text{ if } 1 \le t < g, \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ if } t = g, \\ \left(\frac{0_{2\times 2}}{0 \quad 1}\right), \text{ if } g < t \le n. \end{cases} \tag{5.2}$$

Then φ is an isomorphism. Hence we may take $P(1) \oplus P(g) \oplus P(n)$ as P_3. Thus, we have the following result.

Lemma 4. *The $I(g)$-projective resolution of $I(g)$ is given by* $0 \to P(1) \oplus P(g) \oplus P(n) \to P(0)^2 \to P(0)^2 \to P(0)^2 \to I(g) \to 0$.

Next, it is clear that if $i \neq 0$, then $P(i)$ is a submodule of $I(g)$. For $0 < i \le g$, $d^{-1}(P(i)) = \left(K^{(t)}, \alpha_t, \beta_t\right)$ is given by

$$K^{(t)} = \begin{cases} 0^2, \text{ if } t = 0, \\ K \oplus 0, \text{ if } 1 \le t < i, \\ K^2, \text{ if } i \le t < g, \\ K^4, \text{ if } t = g, \\ 0 \oplus K, \text{ if } g < t \le n, \end{cases}$$

$$\alpha_t = \begin{cases} 0_{2\times 2}, \text{ if } t = n, \\ E_{22}^{(2\times 2)}, \text{ if } g < t < n, \\ \left(\frac{E_{22}^{(2\times 2)}}{0_{(2\times 2)}}\right), \text{ if } t = g, \end{cases}$$

$$\beta_t = \begin{cases} 0_{2\times 2}, \text{ if } t = 0, \\ E_{11}^{(2\times 2)}, \text{ if } 1 \le t < i, \\ I_2, \text{ if } i \le t < g-1, \\ \left(\frac{0_{2\times 2}}{I_2}\right), \text{ if } t = g-1. \end{cases}$$

Now consider the module $P(1) \oplus P(i) \oplus P(g) \oplus P(n) = \left(K^{(t)}, \alpha_t, \beta_t\right)$ with the following properties:

$$K^{(t)} = \begin{cases} 0^4, \text{ if } t = 0, \\ K \oplus 0^3, \text{ if } 1 \le t < i, \\ K^2 \oplus 0^2, \text{ if } i \le t < g, \\ K^4, \text{ if } t = g, \\ 0 \oplus K, \text{ if } g < t \le n, \end{cases}$$

$$\alpha_t = \begin{cases} 0_{4\times 4}, \text{ if } t = n, \\ \left(\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & E_{22}^{(2\times 2)} \end{array}\right), \text{ if } g \le t < n, \end{cases}$$

$$\beta_t = \begin{cases} 0_{4\times 4}, \text{ if } t = 0, \\ \left(\begin{array}{c|c} E_{11}^{(2\times 2)} & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{2\times 2} \end{array}\right), \text{ if } 1 \le t < i, \\ \left(\begin{array}{c|c} I_2 & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{2\times 2} \end{array}\right), \text{ if } i \le t < g. \end{cases}$$

Now define a map $d^{-1}\left(P(i)\right) \xrightarrow{\theta} P(1) \oplus P(i) \oplus P(g) \oplus P(n)$ as follows

$$\theta = \left(\theta_t : K^{(t)} \to K^{(t)}\right) = \begin{cases} 0_{4\times 2}, \text{ if } t = 0, \\ \left(\frac{cE_{11}^{(2\times 2)}}{0_{2\times 2}}\right), \text{ if } 1 \le t < i, \\ \left(\frac{cI_2}{0_{2\times 2}}\right), \text{ if } i \le t < g, \\ \left(\begin{array}{c|c} 0_{2\times 2} & I_2 \\ \hline I_2 & 0_{2\times 2} \end{array}\right), \text{ if } t = g, \\ \left(\frac{I_2}{E_{22}^{(2\times 2)}}\right), \text{ if } g < t \le n. \end{cases}$$

Then θ is an isomorphism. Hence we may take $P(1) \oplus P(i) \oplus P(g) \oplus P(n)$ as P_1 and P_2.

Now, let $g < i \leq n$. Then $d^{-1}(P(i)) = \left(K^{(t)}, \alpha_t, \beta_t\right)$ is given by

$$K^{(t)} = \begin{cases} 0^2, \text{ if } t = 0, \\ K \oplus 0, \text{ if } 1 \leq t < g, \\ K^4, \text{ if } t = g, \\ K^2, \text{ if } g < t \leq i, \\ 0 \oplus K, \text{ if } i < t \leq n, \end{cases} \qquad \alpha_t = \begin{cases} 0_{2\times 2}, \text{ if } t = n, \\ E_{22}^{(2\times 2)}, \text{ if } i \leq t < n, \\ I_2, \text{ if } g < t < i, \\ \left(\frac{cI_2}{0_{2\times 2}}\right), \text{ if } t = g, \end{cases}$$

$$\beta_t = \begin{cases} 0_{2\times 2}, \text{ if } t = 0, \\ E_{11}^{(2\times 2)}, \text{ if } 1 \leq t < g-1, \\ \left(\frac{c0_{2\times 2}}{E_{11}^{(2\times 2)}}\right), \text{ if } t = g - 1m. \end{cases}$$

Now consider the module $P(1) \oplus P(g) \oplus P(i) \oplus P(n) = \left(K^{(t)}, \alpha_t, \beta_t\right)$ with the following properties

$$K^{(t)} = \begin{cases} 0^4, \text{ if } t = 0, \\ K \oplus 0^3, \text{ if } 1 \leq t < g, \\ K^4, \text{ if } t = g, \\ 0^2 \oplus K^2, \text{ if } g < t \leq i, \\ 0^3 \oplus K, \text{ if } i < t \leq n, \end{cases} \qquad \alpha_t = \begin{cases} 0_{4\times 4}, \text{ if } t = n, \\ \left(\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & E_{22}^{(2\times 2)} \end{array}\right), \text{ if } i \leq t < n, \\ \left(\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & I_2 \end{array}\right), \text{ if } g \leq t < i, \end{cases}$$

$$\beta_t = \begin{cases} 0_{4\times 4}, \text{ if } t = 0, \\ \left(\begin{array}{c|c} E_{11}^{(2\times 2)} & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{2\times 2} \end{array}\right), \text{ if } 1 \leq t < g. \end{cases}$$

Now define a map $d^{-1}(P(i)) \xrightarrow{\theta'} P(1) \oplus P(g) \oplus P(n)$ as follows

$$\theta' = \left(\theta'_t : K^{(t)} \to K^{(t)}\right) = \begin{cases} 0_{4\times 2}, \text{ if } t = 0, \\ \left(\frac{E_{11}^{(2\times 2)}}{0_{2\times 2}}\right), \text{ if } 1 \leq t < g, \\ \left(\begin{array}{c|c} 0_{2\times 2} & I_2 \\ \hline I_2 & 0_{2\times 2} \end{array}\right), \text{ if } t = g, \\ \left(\frac{0_{2\times 2}}{I_2}\right), \text{ if } g < t \leq i, \\ \left(\frac{0_{2\times 2}}{E_{22}^{(2\times 2)}}\right), \text{ if } i < t \leq n. \end{cases}$$

Then θ' is an isomorphism. Hence we may take $P(1) \oplus P(g) \oplus P(i) \oplus P(n)$ as P_1 and P_2. We summarize the results for the case $M = I(g)$ while $U = P(i)$ in the following lemma.

Lemma 5. *For $0 < i \leq n$, the $P(i)$-projective resolution of $I(g)$ is given by* $0 \to \bigoplus_{t=1,g,n} P(t) \to \bigoplus_{t=1,i,g,n} P(t) \to \bigoplus_{t=1,i,g,n} P(t) \to P(0)^2 \to I(g) \to 0.$

For the illustration, we choose the quiver $Q_{3,2}$ of type $\tilde{A}_4$ as the underlying quiver. We give here the geometric representation of the $I(i)$-projective resolution of $I(i)$. The dotted lines in Figure 14 represent the U_i's.

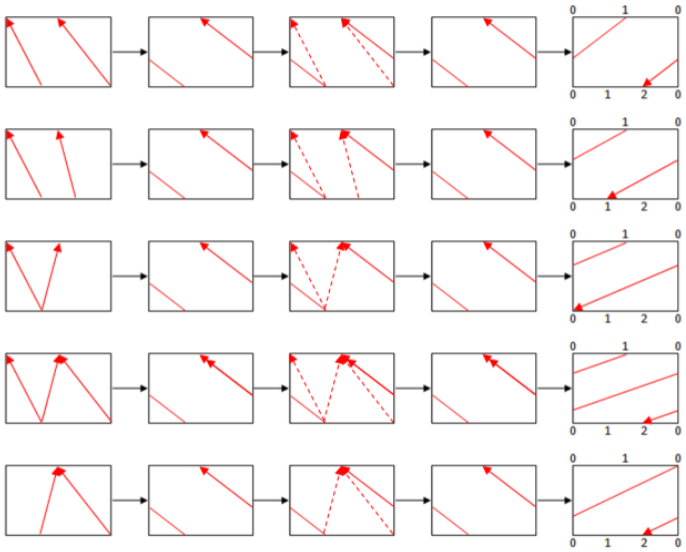

FIGURE 14. Geometric representation for $I(i)$-projective resolution of $kQ_{3,2}$-module $I(i)$. From top to bottom: $i = 0, i = 1, i = 2, i = 3, i = 4$

Acknowledgements. The second author was funded by Sandwich-like Program from Directorate General of Resources for Sciences, Technology, and Higher Education; Ministry of Research and Technology Republic of Indonesia, carried out from October 2017 to January 2018 at the University of Leeds, United Kingdom. He wishes to thank Prof. Robert J. Marsh from the University of Leeds for discussions during the visit. The first and third authors were funded by ASEA UNINET Program 2016. This research was also partially funded by Hibah Desentralisasi DIKTI 2017-2018.

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Karin Baur
Institut für Mathematik und Wissenschaftliches Rechnen Universität Graz
Heinrichstrasse 36 Graz, A-8010, Austria
Email address: karin.baur@uni-graz.at

Yudi Mahatma
Bandung Institute of Technology
Jalan Ganesha 10
Bandung, 40132, Indonesia
Email address: yudi_mahatma@unj.ac.id

Intan Muchtadi-Alamsyah
Bandung Institute of Technology
Jalan Ganesha 10
Bandung, 40132, Indonesia
Email address: ntan@math.itb.ac.id