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SEQUENTIAL PROPERTIES OVER q-COMMUTING ARITHMETIC TABLES

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Abstract. Let $C^{(q)}$ be the q-commuting arithmetic table of $(x+y)^n$ with noncommuting variables. We study sequential properties of diagonal sums of $C^{(q)}$ with various $q > 0$. One of our results shows that each diagonal sum plays like Fibonacci number with some minor conditions.

1. Introduction

An arithmetic table(abbr. AT) is a coefficient table of polynomial. One of well known AT is the Pascal table of $(x+y)^i$, in which the commutativity $xy = yx$ has been assumed implicitly. However in the area of quantum theory, theoretical physicists like P. Dirac and W. Pauli [8] have studied noncommuting variables x and y. In particular when $xy = -yx$, the AT of $(x+y)^i$ $(i \geq 0)$ called the Pauli Pascal table (Pauli table, short) was presented ([4], [5]). As a generalization of noncommutativity, x and y are called q-commuting variables if $yx = qxy$ with $q \in \mathbb{Z}^*$ ([7]). We call the arithmetic table of $(x+y)^i$ with q-commuting variables x, y the q-commuting arithmetic table(AT) denoted by $C^{(q)} = [e^{(q)}_{i,j}]$ such that $(x+y)^i = \sum_{j=0}^{i} e^{(q)}_{i,j} x^{i-j} y^j$.

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Table 1. q-commuting table $C^{(q)} = [e^{(q)}_{i,j}]$ for $i, j \geq 0$

$i \backslash j$	0	1	2	3	4	
0	1					
1	1	1				
2	1	$1+q$	1			
3	1	$1+q(1+q)$	$(1+q)+q^2$	1		
4	1	$1+q+q^2(1+q)$	$1+q(1+q)+q^2((1+q)+q^2)$	$(1+q)+q^2+q^3$	1	$\cdots$

So $C^{(1)}$ is the Pascal table, $C^{(-1)}$ is the Pauli table, and every $e^{(q)}_{i,j}$ in $C^{(q)}$ $(q > 1)$ is the q-binomial coefficient $\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{(1-q^i)(1-q^{i-1})\cdots(1-q^{i-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}$ ([2], [3]) which satisfies the recursive formula

$$e^{(q)}_{i,j} = \begin{bmatrix} i \\ j \end{bmatrix}_q = \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} i-1 \\ j \end{bmatrix}_q = e^{(q)}_{i-1,j-1} + q^j e^{(q)}_{i-1,j} \quad (i, j > 1). \qquad (\star)$$

A purpose of work is to study q-commuting AT $C^{(q)}$ for $q > 0$. Analogous to a fact that each sum of diagonal entries of the Pascal table $C^{(1)}$ is a Fibonacci numbers, we study the sequences of diagonal sums of $C^{(q)}$. One of our result (Theorem 2.4) shows that each diagonal sum is obtained by adding two previous terms like Fibonacci numbers, but one of them must be weighted by $1, q, q^2, \cdots$. A feature of the work is to use algebraic recurrence over the table, without using q-binomial coefficient. Another result in the work is to give relations of diagonal sums of $C^{(q)}$ with different qs by means of Pascal table, for instance the identity $(D^{(q)}_4, D^{(q+1)}_4, D^{(q+2)}_4, D^{(q+3)}_4) \circ \mu_3 = 0$ in Theorem 4.4 imply $D^{(q)}_4 = 3D^{(q+1)}_4 - 3D^{(q+2)}_4 + D^{(q+3)}_4$.

2. Sequential properties on $C^{(q)}$ for $q > 0$

We begin to look at q-commuting tables $C^{(q)} = [e^{(q)}_{i,j}]$ with $q = 2, 3, 4$.

$C^{(2)}$

1				
1	1			
1	3	1		
1	7	7	1	
1	15	35	15	1

$C^{(3)}$

1				
1	1			
1	4	1		
1	13	13	1	
1	40	130	40	1

$C^{(4)}$

1				
1	1			
1	5	1		
1	21	21	1	
1	85	357	85	1

We generally assume $i \geq j$, for $e^{(q)}_{i,j} = 0$ if $i < j$. The next theorem shows a recurrence of $e^{(2)}_{i,j}$ that can be compared to $(\star)$.

THEOREM 2.1. *$e^{(2)}_{i,j} = 2^{i-j} e^{(2)}_{i-1,j-1} + e^{(2)}_{i-1,j}$ and $e^{(2)}_{i,j} = \frac{2^{i-j+1}-1}{2^j-1} e^{(2)}_{i,j-1}$.*

Proof. In $j = 1$th column, the differences of first few consecutive entries are $1, 2, 4, 8, \cdots$, and $e^{(2)}_{5,1} - e^{(2)}_{4,1} = 2^4$. Assume $e^{(2)}_{i,1} - e^{(2)}_{i-1,1} = 2^{i-1}$ for some i. Then
$$e^{(2)}_{i+1,1} - e^{(2)}_{i,1} = (e^{(2)}_{i,0} + 2e^{(2)}_{i,1}) - (e^{(2)}_{i-1,0} + 2e^{(2)}_{i-1,1}) = 2(e^{(2)}_{i,1} - e^{(2)}_{i-1,1}) = 2^i$$
by ($\star$) and $e^{(2)}_{i,0} = e^{(2)}_{i-1,0} = 1$.

In $j = 2$th column, notice $155 = 35 + 2^3 \cdot 15$ and $651 = 155 + 2^4 \cdot 31$. So if assume $e^{(2)}_{i,2} - e^{(2)}_{i-1,2} = 2^{i-2}e^{(2)}_{i-1,1}$ for some i then ($\star$) yields

$$\begin{aligned} e^{(2)}_{i+1,2} - e^{(2)}_{i,2} &= (e^{(2)}_{i,1} + 2^2 e^{(2)}_{i,2}) - (e^{(2)}_{i-1,1} + 2^2 e^{(2)}_{i-1,2}) \\ &= (e^{(2)}_{i,1} - e^{(2)}_{i-1,1}) + 2^2(e^{(2)}_{i,2} - e^{(2)}_{i-1,2}) = 2^{i-1} + 2^2(2^{i-2}e^{(2)}_{i-1,1}) \\ &= 2^{i-1}(e^{(2)}_{i-1,0} + 2e^{(2)}_{i-1,1}) = 2^{i-1}e^{(2)}_{i,1}. \end{aligned}$$

Now for any $t < j$, assume $e^{(2)}_{i,t} - e^{(2)}_{i-1,t} = 2^{i-t}e^{(2)}_{i-1,t-1}$ for all i. And on jth column, we may assume $e^{(2)}_{i-1,j} - e^{(2)}_{i-2,j} = 2^{i-j-1}e^{(2)}_{i-2,j-1}$ for some i, since $e^{(2)}_{j,j} = 2^0 e^{(2)}_{j-1,j-1} + e^{(2)}_{j-1,j}$. Then the induction hypothesis together with ($\star$) says

$$\begin{aligned} e^{(2)}_{i,j} - e^{(2)}_{i-1,j} &= (e^{(2)}_{i-1,j-1} + 2^j e^{(2)}_{i-1,j}) - (e^{(2)}_{i-2,j-1} + 2^j e^{(2)}_{i-2,j}) \\ &= (e^{(2)}_{i,j-1} - e^{(2)}_{i-2,j-1}) + 2^j(e^{(2)}_{i-1,j} - e^{(2)}_{i-2,j}) \\ &= 2^{i-j}(e^{(2)}_{i-2,j-2} + 2^{j-1}e^{(2)}_{i-2,j-1}) = 2^{i-j}e^{(2)}_{i-1,j-1}. \end{aligned}$$

Furthermore $e^{(2)}_{i,j} = e^{(2)}_{i-1,j-1} + 2^j e^{(2)}_{i-1,j} = 2^{i-j}e^{(2)}_{i-1,j-1} + e^{(2)}_{i-1,j}$ implies $(2^{i-j} - 1)e^{(2)}_{i-1,j-1} = (2^j - 1)e^{(2)}_{i-1,j}$. □

Theorem 2.1 is explained by the diagram

	$j-1$	j
$i-1$	**A**	**B**
i		**C**

with $C = A + 2^j B = 2^{i-j}A + B$. We may refer [1] for the 2-binomial expression of Theorem 2.1.

THEOREM 2.2. *(1)* $e^{(2)}_{i,j}e^{(2)}_{i-j,j+1} = e^{(2)}_{i,j+1}e^{(2)}_{i-j-1,j}$, *so* $\frac{e^{(2)}_{i,j+1}}{e^{(2)}_{i,j}} = \frac{e^{(2)}_{i-j,j+1}}{e^{(2)}_{i-j-1,j}} = \frac{2^{i-j}-1}{2^{j+1}-1}$.

(2) $C^{(q)} = [e^{(q)}_{i,j}]$ *satisfies* $e^{(q)}_{i+1,j+1} = e^{(q)}_{i,j} + q^{j+1}e^{(q)}_{i,j+1} = q^{i-j}e^{(q)}_{i,j} + e^{(q)}_{i,j+1}$, *and the ratio in a row equals* $\frac{e^{(q)}_{i,j+1}}{e^{(q)}_{i,j}} = \frac{e^{(q)}_{i-j,j+1}}{e^{(q)}_{i-j-1,j}} = \frac{e^{(q)}_{i-j,1}}{e^{(q)}_{j+1,1}} = \frac{q^{i-j}-1}{q^{j+1}-1}$ *for any* $q > 0$.

Proof. Observe $e^{(2)}_{i,1}e^{(2)}_{i-1,2} = e^{(2)}_{i,2}e^{(2)}_{i-2,1}$ and $e^{(2)}_{i,2}e^{(2)}_{i-2,3} = e^{(2)}_{i,3}e^{(2)}_{i-3,2}$. And in general, by Theorem 2.1 we have $\frac{e^{(2)}_{i,j+1}}{e^{(2)}_{i,j}} = \frac{2^{i-j}-1}{2^{j+1}-1}$ and

$$\frac{e^{(2)}_{i-j,j+1}}{e^{(2)}_{i-j-1,j}} = \frac{2^{i-2j-1}e^{(2)}_{i-j-1,j} + e^{(2)}_{i-j-1,j+1}}{e^{(2)}_{i-j-1,j}} = 2^{i-2j-1} + \frac{e^{(2)}_{i-j-1,j+1}}{e^{(2)}_{i-j-1,j}}$$
$$= 2^{i-2j-1} + \frac{2^{i-2j-1}-1}{2^{j+1}-1} = \frac{2^{i-j}-1}{2^{j+1}-1}.$$

(2) is true for $q = 1, 2$. Now for any $q \geq 1$, we have

$$\frac{e^{(q)}_{i-j,j+1}}{e^{(q)}_{i-j-1,j}} = \frac{q^{i-2j-1}e^{(q)}_{i-j-1,j} + e^{(q)}_{i-j-1,j+1}}{e^{(q)}_{i-j-1,j}} = q^{i-2j-1} + \frac{e^{(q)}_{i-j-1,j+1}}{e^{(q)}_{i-j-1,j}}$$
$$= q^{i-2j-1} + \frac{q^{i-2j-1}-1}{q^{j+1}-1} = \frac{q^{i-j}-1}{q^{j+1}-1} = \frac{e^{(q)}_{i,j+1}}{e^{(q)}_{i,j}}.$$

Moreover since $e^{(q)}_{k,1} = 1 + q + \cdots + q^{k-1} = \frac{q^k-1}{q-1}$, we have

$$\frac{e^{(q)}_{i,j}e^{(q)}_{i-j,1}}{e^{(q)}_{i,j+1}} = \frac{e^{(q)}_{i,j}}{e^{(q)}_{i,j+1}}\, e^{(q)}_{i-j,1} = \frac{q^{j+1}-1}{q^{i-j}-1}\frac{q^{i-j}-1}{q-1} = \frac{q^{j+1}-1}{q-1} = e^{(q)}_{j+1,1},$$

so the result follows immediately from $\frac{e^{(q)}_{i,j}}{e^{(q)}_{i,j+1}} = \frac{e^{(q)}_{j+1,1}}{e^{(q)}_{i-j,1}} = \frac{q^{j+1}-1}{q^{i-j}-1}$. □

The diagrams $\begin{array}{c|c|c} & 1 & 2 \\ \hline i-2 & \mathbf{D} & \\ \hline i-1 & & \mathbf{B} \\ \hline i & \mathbf{A} & \mathbf{C} \end{array}$, $\begin{array}{c|c|c} & 2 & 3 \\ \hline i-3 & \mathbf{D} & \\ \hline i-2 & & \mathbf{B} \\ \hline i-1 & & \\ \hline i & \mathbf{A} & \mathbf{C} \end{array}$ and $\begin{array}{c|c|c} & j & j+1 \\ \hline i-j-1 & \mathbf{D} & \\ \hline i-j & & \mathbf{B} \\ \hline \cdots & & \\ \hline i & \mathbf{A} & \mathbf{C} \end{array}$ with $AB = CD$ explain Theorem 2.2.

THEOREM 2.3. *Any* $C^{(q)} = [e^{(q)}_{i,j}]$ *satisfies* $e^{(q)}_{i,j} = \sum_{t=0}^{j} q^{(j-t)(i-j)}e^{(q)}_{i-j-1+t,t}$.

Proof. Theorem 2.2 implies that

$$\begin{aligned} e^{(q)}_{i,j} &= q^{i-j}e^{(q)}_{i-1,j-1} + e^{(q)}_{i-1,j} = q^{i-j}(q^{i-j}e^{(q)}_{i-2,j-2} + e^{(q)}_{i-2,j-1}) + e^{(q)}_{i-1,j} \\ &= q^{2(i-j)}e^{(q)}_{i-2,j-2} + q^{i-j}e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j} \\ &= q^{2(i-j)}(q^{i-j}e^{(q)}_{i-3,j-3} + e^{(q)}_{i-3,j-2}) + q^{i-j}e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j} \\ &= q^{3(i-j)}e^{(q)}_{i-3,j-3} + q^{2(i-j)}e^{(q)}_{i-3,j-2} + q^{i-j}e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j}. \end{aligned}$$

And continuing this procedure, we have

$$\begin{aligned} e^{(q)}_{i,j} &= q^{(j-1)(i-j)} e^{(q)}_{i-(j-1),j-(j-1)} + q^{(j-2)(i-j)} e^{(q)}_{i-(j-1),j-(j-2)} + \cdots \\ &\quad + q^{2(i-j)} e^{(q)}_{i-3,j-2} + q^{i-j} e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j} \\ &= q^{(j-1)(i-j)} (q^{i-j} e^{(q)}_{i-j,0} + e^{(q)}_{i-j,1}) + q^{(j-2)(i-j)} e^{(q)}_{i-(j-1),j-(j-2)} + \cdots \\ &\quad + q^{2(i-j)} e^{(q)}_{i-2,j-2} + q^{i-j} e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j} \\ &= q^{j(i-j)} e^{(q)}_{i-j,0} + q^{(j-1)(i-j)} e^{(q)}_{i-j,1} + \cdots + q^{i-j} e^{(q)}_{i-2,j-1} + e^{(q)}_{i-1,j} \\ &= \sum_{t=0}^{j} q^{(j-t)(i-j)} e^{(q)}_{i-j-1+t,t}. \end{aligned}$$

□

Theorem 2.3 is diagramed as $\begin{array}{c|lll} & 0 & 1\cdots j & \\ \hline i & \mathbf{A} & & \\ i+1 & & \ddots & \\ \cdots & & & \mathbf{B} \\ & & & \mathbf{C} \end{array}$ with $A + \cdots + B = C$, that can be compared to a hockey stick formula in Pascal table [6]. For example, by looking at $C^{(q)}$ with $q = 4$, if $i = 4$, $j = 3$ then

$$\begin{aligned} \sum_{t=0}^{3} q^{5(3-t)} e^{(q)}_{4+t,t} &= 4^{5\cdot 3} e^{(4)}_{4,0} + 4^{5\cdot 2} e^{(4)}_{5,1} + 4^{5\cdot 1} e^{(4)}_{6,2} + 4^{5\cdot 0} e^{(4)}_{7,3} \\ &= 4^{15} + 4^{10}\cdot 341 + 4^{5}\cdot 93093 + 24208613 \\ &= 1550842085 = e^{(4)}_{8,3}. \end{aligned}$$

In $C^{(q)}$, let $d^{(q)}_n$ be the nth diagonal and $D^{(q)}_n$ be the nth diagonal sum. Then

$$D^{(q)}_n = e^{(q)}_{n,0} + e^{(q)}_{n-1,1} + e^{(q)}_{n-2,2} + \cdots = (e^{(q)}_{n,0}, e^{(q)}_{n-1,1}, e^{(q)}_{n-2,2}, \cdots) \circ (1,1,1,\cdots),$$

where $\circ$ means inner product operation, and we may write $D^{(q)}_n = d^{(q)}_n \circ (1,1,\cdots)$. It is well known that diagonal sums $D^{(1)}_n$ of $C^{(1)}$ yield a Fibonacci sequence. Moreover, for example, the 7th diagonal sum in $C^{(4)}$ is $D^{(4)}_7 = 7248 = 1 + 1365 + 5797 + 85$, in which each terms can be written by $1365 = 1 + 4\cdot 341$, $5797 = 85 + 4^2 \cdot 357$ and $85 = 21 + 4^3 \cdot 1$ by $(\star)$. Thus

$$D^{(4)}_7 = (1 + 85 + 21) + (1 + 4\cdot 341 + 4^2\cdot 357 + 4^3 \cdot 1) = D^{(4)}_5 + X,$$

with $X = (1, 341, 357, 1) \circ (1, 4, 4^2, 4^3) = d^{(4)}_6 \circ (1, 4, 4^2, 4^3)$. Similarly

$$D_7^{(4)} = (1 + 341 + 357 + 1) + (4^5 \cdot 1 + 4^3 \cdot 85 + 4 \cdot 21) = D_6^{(4)} + Y,$$

with $Y = (1, 85, 21) \circ (4^5, 4^3, 4) = d_5^{(4)} \circ (4^5, 4^3, 4)$ by Theorem 2.1.

Let us define the weighted diagonal sums in $C^{(q)}$ by

$$\overrightarrow{D}_n^{(q)} = (e_{n,0}^{(q)}, e_{n-1,1}^{(q)}, e_{n-2,2}^{(q)}, \cdots) \circ (1, q, q^2, \cdots) = d_n^{(q)} \circ (1, q, q^2, \cdots)$$
$$\overleftarrow{D}_n^{(q)} = (e_{n,0}^{(q)}, e_{n-1,1}^{(q)}, e_{n-2,2}^{(q)}, \cdots) \circ (q^n, q^{n-2}, q^{n-4}, \cdots) = d_n^{(q)} \circ (q^n, q^{n-2}, \cdots).$$

Then we can say that $D_7^{(4)}$ satisfies a weighted fibonacci type rule that

$$D_7^{(4)} = \overrightarrow{D}_6^{(2)} + D_5^{(2)} = D_6^{(4)} + \overleftarrow{D}_5^{(4)}.$$

THEOREM 2.4. $D_n^{(q)} = \overrightarrow{D}_{n-1}^{(q)} + D_{n-2}^{(q)} = D_{n-1}^{(q)} + \overleftarrow{D}_{n-2}^{(q)}$, *a weighted fibonacci rule.*

Proof. Due to $(\star)$, we have

$$\begin{aligned} D_n^{(q)} &= (e_{n,0}^{(q)}, e_{n-1,1}^{(q)}, e_{n-2,2}^{(q)}, \cdots) \circ (1, 1, \cdots) \\ &= (e_{n,0}^{(q)}, (e_{n-2,0}^{(q)} + q e_{n-2,1}^{(q)}), (e_{n-3,1}^{(q)} + q^2 e_{n-3,2}^{(q)}), \cdots) \circ (1, 1, \cdots) \\ &= ((e_{n-2,0}^{(q)}, e_{n-3,1}^{(q)}, e_{n-4,2}^{(q)}, \cdots) + (e_{n,0}^{(q)}, q e_{n-2,1}^{(q)}, q^2 e_{n-3,2}^{(q)}, \cdots)) \circ (1, 1, \cdots) \\ &= D_{n-2}^{(q)} + (e_{n-1,0}^{(q)}, e_{n-2,1}^{(q)}, e_{n-3,2}^{(q)}, \cdots) \circ (1, q, q^2, \cdots) = D_{n-2}^{(q)} + \overrightarrow{D}_{n-1}^{(q)}. \end{aligned}$$

On the other hand by Theorem 2.1 we also have

$$\begin{aligned} D_n^{(q)} &= (e_{n,0}^{(q)}, (q^{n-2} e_{n-2,0}^{(q)} + e_{n-2,1}^{(q)}), (q^{n-4} e_{n-3,1}^{(q)} + e_{n-3,2}^{(q)}), \cdots) \circ (1, 1, \cdots) \\ &= ((e_{n,0}^{(q)}, e_{n-2,1}^{(q)}, e_{n-3,2}^{(q)}, \cdots) + (q^{n-2} e_{n-2,0}^{(q)}, q^{n-4} e_{n-3,1}^{(q)}, \cdots)) \circ (1, 1, \cdots) \\ &= D_{n-1}^{(q)} + (e_{n-2,0}^{(q)}, e_{n-3,1}^{(q)}, \cdots) \circ (q^{n-2}, q^{n-4}, \cdots) = D_{n-1}^{(q)} + \overleftarrow{D}_{n-2}^{(q)}. \end{aligned}$$

□

If $q = 1$, Theorem 2.4 corresponds the fibonacci recurrence rule. And Table 2 of $D_n^{(q)}$ shows, for example, the 6th diagonal sum in $C^{(5)}$ is

$$\begin{aligned} D_6^{(5)} &= \overrightarrow{D}_5^{(5)} + D_4^{(5)} = (1, 156, 31) \circ (1, 5, 5^2) + (1 + 31 + 1) \\ &= (1 + 156 + 31) + (1, 31, 1) \circ (5^4, 5^2, 1) = D_5^{(5)} + \overleftarrow{D}_4^{(5)} = 1589. \end{aligned}$$

Table 2. Diagonal sums $D_n^{(q)}$

$n\backslash q$	1	2	3	4	5	6	7
3	3	4	5	6	7	8	9
4	5	9	15	23	33	45	59
5	8	23	54	107	188	303	458
6	13	68	253	700	1589	3148	5653

3. Simplified q-commuting table

From Table 1, notice $e_{4,2}^{(q)} = 1 + q + 2q^2 + q^3 + q^4 = (1,1,2,1,1) \circ (1, q, \cdots, q^4)$, where we denote it by $e_{4,2}^{(q)} = \hat{e}_{4,2} \circ (1, \cdots, q^4)$. We often write $\hat{e}_{4,2}$ by $(1,1,2,1,1)$ or 11211 and say its length is $\mathrm{len}(\hat{e}_{4,2}) = 5$. Considering $\hat{e}_{i,j}$ satisfying $e_{i,j} = \hat{e}_{i,j} \circ (1, q, \cdots)$, we make a table $\hat{C} = [\hat{e}_{i,j}]$ called the simplified table of $C^{(q)}$.

THEOREM 3.1. *$\hat{e}_{i,j} = \hat{e}_{i-1,j-1} + 0_{[j]}\ \hat{e}_{i-1,j}$ where $[0]_k$ means $\underbrace{0\cdots0}_{k}$.*

Proof. Since $e_{i,j} = \hat{e}_{i,j} \circ (1, q, q^2, \cdots)$, the recurrence $(\star)$ shows that for instance, $\hat{e}_{5,3} = 1122211$ is obtained by $\hat{e}_{4,2} = 11211$ and $\hat{e}_{4,3} = 1111$ in a way $\begin{pmatrix} 11211 \\ +0001111 \end{pmatrix}$ and taking sums of each column. Similarly $\begin{pmatrix} \hat{e}_{5,3} \\ +0000\hat{e}_{5,4} \end{pmatrix} = \begin{pmatrix} 1122211 \\ +000011111 \end{pmatrix}$ yields $112232211 = \hat{e}_{6,4}$. Hence in general, it follows immediately that

$$\hat{e}_{i,j} = \begin{pmatrix} \hat{e}_{i-1,j-1} \\ +[0]_j\ \hat{e}_{i-1,j} \end{pmatrix} = \hat{e}_{i-1,j-1} + [0]_j\ \hat{e}_{i-1,j}. \qquad \square$$

Table 3. $\hat{C} = [\hat{e}_{i,j}]$

$i\backslash j$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	11	1			
3	1	111	111	1		
4	1	1111	11211	1111	1	
5	1	11111	1122211	1122211	11111	1

length $(\hat{e}_{i,j})$

$i\backslash j$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	5	4	1	
5	1	5	7	7	5	1

THEOREM 3.2. *Let $\hat{D}_n$ be the nth diagonal sum of $\hat{C}$ and $l(n)$ be its digit length. Then $l(n) = 1+(n-2\lfloor \frac{n+2}{4} \rfloor)\lfloor \frac{n+2}{4} \rfloor$ and $\hat{D}_n \circ (1, q, \cdots, q^{l(n)-1}) = D_n^{(q)}$.*

Proof. Length follows from Table 3 by counting digit number of each $\hat{e}_{i,j}$. Notice $\mathrm{len}(\hat{e}_{j,j}) = 1$, $\mathrm{len}(\hat{e}_{j+1,j}) = 1 + j$, $\mathrm{len}(\hat{e}_{j+2,j}) = 1 + 2j$ and

$\mathrm{len}(\hat{e}_{j+3,j}) = 1+3j$. If we assume $\mathrm{len}(\hat{e}_{j+t,j}) = 1+tj$ for some t, then $\hat{e}_{j+(t+1),j} = \hat{e}_{j+t,j-1} + 0_{[j]}\hat{e}_{j+t,j}$ in Theorem 3.1 yields

$$\mathrm{len}(\hat{e}_{j+(t+1),j}) = \mathrm{len}(0_{[j]}\hat{e}_{j+t,j}) = j + (1+tj) = 1+(t+1)j.$$

Thus entries in the nth diagonal $\{\hat{e}_{n,0}, \hat{e}_{n-1,1}, \cdots, \hat{e}_{n-j,j}, \cdots, \hat{e}_{\lfloor\frac{n+1}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}\}$ of $\hat{C}$ have lengths that $\mathrm{len}(\hat{e}_{n,0}) = 1$, $\mathrm{len}(\hat{e}_{n-1,1}) = 1+(n-2) = n-1$, $\mathrm{len}(\hat{e}_{n-2,2}) = 1+(n-4)2 = n-7$, and for any $0 \le j < \lfloor\frac{n}{2}\rfloor$, we have

$$\mathrm{len}(\hat{e}_{n-j,j}) = \mathrm{len}(\hat{e}_{j+(n-2j),j}) = 1+(n-2j)j.$$

Hence the longest length in the nth diagonal appear when $j = \lfloor\frac{n}{4}+\frac{1}{2}\rfloor$, so the length of $\hat{D}_n$ is $l(n) = \mathrm{len}(\hat{e}_{j+(n-2j),j}) = 1+(n-2j)j$ with $j = \lfloor\frac{n+2}{4}\rfloor$.

Consider the nth diagonal sum $\hat{D}_n = \hat{e}_{n,0} + \hat{e}_{n-1,1} + \cdots + \hat{e}_{\lfloor\frac{n+1}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}$ of $\hat{C}$. Indeed, $\hat{D}_3 = \begin{pmatrix} 1 \\ +11 \end{pmatrix} = 21$, $\hat{D}_4 = \begin{pmatrix} 1 \\ 111 \\ +1 \end{pmatrix} = 311$ and $\hat{D}_5 = \begin{pmatrix} 1 \\ 1111 \\ +111 \end{pmatrix} = 3221$, so

$\hat{D}_3 \circ (1,q) = (2,1)\circ(1,q) = 2+q = D_3^{(q)}$

$\hat{D}_4 \circ (1,q,q^2) = (3,1,1)\circ(1,q,q^2) = 3+q+q^2 = D_4^{(q)}$

$\hat{D}_5 \circ (1,q,q^2,q^3) = (3,2,2,1)\circ(1,q,q^2,q^3) = D_5^{(q)}$, etc. Thus we have

$$\hat{D}_n \circ (1,q,\cdots,q^{l(n)-1}) = \begin{pmatrix} \hat{e}_{n,0} \\ \hat{e}_{n-1,1} \\ \cdots \\ +\hat{e}_{\lfloor\frac{n+1}{2}\rfloor,\lfloor\frac{n}{2}\rfloor} \end{pmatrix} \circ (1,\cdots,q^{l(n)-1})$$

$$= \begin{pmatrix} \hat{e}_{n,0}\circ(1) \\ \hat{e}_{n-1,1}\circ(1,\cdots,q^{n-1}) \\ \cdots \\ +\hat{e}_{\lfloor\frac{n+1}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}\circ(1,\cdots,q^{l(n)-1}) \end{pmatrix} = \begin{pmatrix} e_{n,0} \\ e_{n-1,1} \\ \cdots \\ +e_{\lfloor\frac{n+1}{2}\rfloor,\lfloor\frac{n}{2}\rfloor} \end{pmatrix} = D_n^{(q)}. \qquad \square$$

In fact, $\hat{D}_6 = \begin{pmatrix} 1 \\ 11111 \\ 11211 \\ +1 \end{pmatrix} = 42322$ is of length 5 and $\hat{D}_7 = \begin{pmatrix} 1 \\ 111111 \\ 1122211 \\ +1111 \end{pmatrix} = 4344321$ of length 7. And $\hat{D}_n$ with $(1,q,q^2,\cdots)$ yields the table of $D_n^{(q)}$.

n	$\hat{D}_n \circ (1,q,\cdots) = D_n^{(q)}$	n	$\hat{D}_n \circ (1,q,\cdots) = D_n^{(q)}$
3	$(2,1)\circ(1,q)$	7	$(4,3,4,4,3,2,1)\circ(1,\cdots,q^6)$
4	$(3,1,1)\circ(1,q,q^2)$	8	$(5,3,5,5,6,4,4,1,1)\circ(1,\cdots,q^8)$
5	$(3,2,2,1)\circ(1,\cdots,q^3)$	9	$(5,4,6,7,8,7,7,5,3,2,1)\circ(1,\cdots,q^{10})$
6	$(4,2,3,2,2)\circ(1,\cdots,q^4)$	10	$(6,4,7,8,11,10,12,9,9,5,4,2,2)\circ(1,\cdots,q^{12})$

From $\hat{D}_n = \begin{pmatrix} \hat{e}_{n,0} \\ \hat{e}_{n-1,1} \\ \hat{e}_{n-2,2} \\ +\cdots \end{pmatrix}$, let $\hat{D}_n^* = \begin{pmatrix} [0]_n \ \hat{e}_{n,0} \\ [0]_{n-2} \ \hat{e}_{n-1,1} \\ [0]_{n-4} \ \hat{e}_{n-2,2} \\ +\cdots \end{pmatrix}$. Then $\hat{D}_4^* = \begin{pmatrix} 00001 \\ 00111 \\ +1 \end{pmatrix} = 10112 = \hat{D}_6 - \hat{D}_5$, and this yields the next theorem.

THEOREM 3.3. *(1)* $\overleftarrow{D}^{(q)}_{n-2} = D_n^{(q)} - D_{n-1}^{(q)} = (\hat{D}_n - \hat{D}_{n-1}) \circ (1, q, \cdots, q^{l(n)})$.
(2) $\hat{D}_n$ *satisfies a fibonacci type rule that* $\hat{D}_n = \hat{D}_{n-1} + \hat{D}^*_{n-2}$.

Proof. $\hat{D}_6 - \hat{D}_5 = \begin{pmatrix} 1 \\ 11111 \\ 11211 \\ +1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1111 \\ +111 \end{pmatrix} = \begin{pmatrix} 0 \\ 00001 \\ 00111 \\ +1 \end{pmatrix} = 10112 = \hat{D}_4^*$,
and

$$\begin{aligned} D_6^{(q)} - D_5^{(q)} &= (4,2,3,2,2) \circ (1, q, \cdots, q^4) - (3,2,2,1) \circ (1, q, q^2, q^3) \\ &= (1,0,1,1,2) \circ (1, q, \cdots, q^4) = 1 + 0q + q^2 + q^3 + 2q^4. \end{aligned}$$

So we have $D_6^{(q)} - D_5^{(q)} = (1,0,1,1,2) \circ (1, \cdots, q^4) = \hat{D}_4^* \circ (1, \cdots, q^4)$. With the weighted diagonal sum $\overleftarrow{D}^{(q)}_n$ in Theorem 2.4, we have

$$\begin{aligned} \overleftarrow{D}^{(q)}_{n-2} = D_n^{(q)} - D_{n-1}^{(q)} &= \hat{D}_n \circ (1, \cdots, q^{l(n)-1}) - \hat{D}_{n-1} \circ (1, \cdots, q^{l(n-1)-1}) \\ &= (\hat{D}_n - \hat{D}_{n-1}) \circ (1, q, \cdots, q^{l(n)-1}), \end{aligned}$$

since $l(n-1) \leq l(n)$. But $e_{n,0} - e_{n-1,0} = 0$, $e_{n-1,1} - e_{n-2,1} = q^{n-2} e_{n-2,0}$ and in general $e_{i,j} - e_{i-1,j} = q^{i-j} e_{i-1,j-1}$ $(i, j > 1)$ by Theorem 2.1 implies

$$\hat{D}_n - \hat{D}_{n-1} = \begin{pmatrix} \hat{e}_{n,0} - \hat{e}_{n-1,0} \\ \hat{e}_{n-1,1} - \hat{e}_{n-2,1} \\ \hat{e}_{n-2,2} - \hat{e}_{n-3,2} \\ +\cdots \end{pmatrix} = \begin{pmatrix} 0 \\ [0]_{n-2} \hat{e}_{n-1,0} \\ [0]_{n-4} \hat{e}_{n-3,1} \\ +\cdots \end{pmatrix} = \hat{D}^*_{n-2}.$$

□

4. Interrelationships of $D_n^{(q)}$ with various q

When q is given, some relationships of diagonal sums $D_n^{(q)}$ for all $n \geq 0$ were discussed. When n is given, we turn our attention to study $D_n^{(q)}$ for all $q > 0$. From Table 2 we see $\{D_3^{(q)}\} = \{3, 4, 5, 6, 7, 8, \cdots\}$ and $\{D_4^{(q)}\} = \{5, 9, 15, 23, 33, 45, \cdots\}$, and observe $D_3^{(q)} = q + 2 = 1 + D_3^{(q-1)}$ and $D_4^{(q)} = q(q+1) + 3 = 2q + D_4^{(q-1)} = 3D_4^{(q-1)} - 3D_4^{(q-2)} + D_4^{(q-3)}$. Next theorem provides some interrelationships between nth diagonal sums $D_n^{(q)}$ of $C^{(q)}$ and $D_n^{(q+t)}$ of $C^{(q+t)}$ for $t \geq 0$.

Theorem 4.1. *We have the followings.*

(1) $(D_3^{(q)}, D_3^{(q+1)}) \circ (-1,1) = 1$ *and* $(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ (1,-2,1) = 2$.
(2) $(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}) \circ (-1,3,-3,1) = 6$

Proof. Looking at $\{D_3^{(q)}\}$, $\{D_4^{(q)}\}$ and $\{D_5^{(q)}\} = \{8, 23, 54, 107, 188, 303, \cdots\}$, (1) and (2) can be observed for first few entries. Now $D_3^{(q)} = (2,1) \circ (1,q) = 2+q$ by Theorem 3.2, and it shows

$$(D_3^{(q)}, D_3^{(q+1)}) \circ (-1,1) = (2+q, 2+(q+1)) \circ (-1,1) = 1.$$

And $D_4^{(q)} = (3,1,1) \circ (1,q,q^2) = 3+q+q^2$ implies

$$\begin{aligned}&(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ (1,-2,1)\\ &= (3+q+q^2, 3+(q+1)+(q+1)^2, 3+(q+2)+(q+2)^2)\circ\\ &\quad (1,-2,1) = 2.\end{aligned}$$

Similarly $D_5^{(q)} = (3,2,2,1) \circ (1,q,q^2,q^3) = 3+2q+2q^2+q^3$ yields

$$\begin{aligned}&(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}) \circ (-1,3,-3,1)\\ &= (3+2q+2q^2+q^3, \cdots, 3+2(q+3)+2(q+3)^2+(q+3)^3)\circ\\ &\quad (-1,3,-3,1) = 6\end{aligned}$$

□

Thus some recurrence of $D_n^{(q)}$ are $D_3^{(q+1)} = D_3^{(q)}+1$, $D_4^{(q+2)} = D_4^{(q+1)} - D_4^{(q)}+2$ and $D_5^{(q+3)} = 3D_5^{(q+2)} - 3D_5^{(q+1)} + D_5^{(q)} + 6$. In order to generalize Theorem 4.1, we add a lemma.

Lemma 4.2. *Let* μ_k $(k \geq 0)$ *be the* k*th row of inverse Pascal matrix* P^{-1}. *Then*

(1) *The sum of entries over* μ_k *equals* 0, *i.e.,* $(1, \cdots, 1) \circ \mu_k = 0$.
(2) *For* $a \in \mathbb{Z}$, $(a^i, (a+1)^i, \cdots, (a+k)^i) \circ \mu_k = \begin{cases} 0 & \text{if } 0 \leq i < k \\ k! & \text{if } i = k \end{cases}$.
(3) *Let* $\chi = (b_0, b_1, \cdots, b_k)$ *be any* $(k+1)$ *tuple with* $b_i \in \mathbb{Z}$. *Then*

$$\Big(\chi \circ (1, a, \cdots, a^k), \chi \circ (1, (a+1), \cdots, (a+1)^k),$$
$$\chi \circ (1, (a+2), \cdots, (a+2)^k), \cdots, \chi \circ (1, (a+k), \cdots, (a+k)^k)\Big) \circ \mu_k = b_k k!$$

Proof. Clearly $P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \\ & \cdots & & \end{bmatrix}$ is the arithmetic table of $(x-1)^n$, so the sum of entries on each row μ_k equals 0 by letting $x = 1$, this is (1).

When $i = 0$, (2) corresponds to (1). By simple calculation, for example,

$$\begin{aligned}&(a, (a+1), (a+2), (a+3)) \circ (-1, 3, -3, 1)\\ &\quad = (a^2, (a+1)^2, (a+2)^2, (a+3)^2) \circ (-1, 3, -3, 1) = 0,\end{aligned}$$

while $(a^3, (a+1)^3, (a+2)^3, (a+3)^3) \circ (-1, 3, -3, 1) = 6$.

And the proof generally follows from the matrix multiplication

$$P^{-1} \begin{bmatrix} 1 & a & a^2 & a^3 & a^4 \\ 1 & a+1 & (a+1)^2 & (a+1)^3 & (a+1)^4 \\ 1 & a+2 & (a+2)^2 & (a+2)^3 & (a+2)^4 \\ 1 & a+3 & (a+3)^2 & (a+3)^3 & (a+3)^4 \\ 1 & a+4 & (a+4)^2 & (a+4)^3 & (a+4)^4 \end{bmatrix} = \begin{bmatrix} 1 & a & a^2 & a^3 & a^4 \\ 0 & 1 & 2a+1 & 3a^2+3a+1 & \cdots \\ 0 & 0 & 2 & 6a+6 & \cdots \\ 0 & 0 & 0 & 6 & 24a+36 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}.$$

Now for (3), when $k = 2$ with $\mu_2 = (1, -2, 1)$, we have

$$\begin{aligned}&\big((b_0, b_1, b_2) \circ (1, a, a^2), (b_0, b_1, b_2) \circ (1, (a+1), (a+1)^2),\\ &\quad (b_0, b_1, b_2) \circ (1, (a+2), (a+2)^2)\big) \circ \mu_2\\ &\quad = (b_0 + b_1 a + b_2 a^2) - 2(b_0 + b_1(a+1) + b_2(a+1)^2)\\ &\qquad + (b_0 + b_1(a+2) + b_2(a+2)^2)\\ &\quad = b_0(1,1,1) \circ \mu_2 + b_1(a, a+1, a+2) \circ \mu_2 + b_2(a^2, (a+1)^2, (a+2)^2) \circ \mu_2\\ &\quad = b_2 2!,\end{aligned}$$

by (2). Similarly, for any $k \geq 0$ we also have

$$\begin{aligned}&\Big(\chi \circ (1, a, \cdots, a^k), \chi \circ (1, (a+1), \cdots, (a+1)^k),\\ &\quad \chi \circ (1, (a+2), \cdots, (a+2)^k), \cdots, \chi \circ (1, (a+k), \cdots, (a+k)^k)\Big) \circ \mu_k\\ &\quad = b_0(1, \cdots, 1) \circ \mu_k + b_1(a, a+1, \cdots, a+k) \circ \mu_k\\ &\qquad + b_2(a^2, (a+1)^2, \cdots, (a+k)^2) \circ \mu_k + \cdots + b_k(a^k, \cdots, (a+k)^k) \circ \mu_k\\ &\quad = b_k k!.\end{aligned}$$

□

THEOREM 4.3. *For any $q \geq 1$, we have $(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ \mu_2 = 2$ and $(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}) \circ \mu_3 = 3!$.*

Proof. By Theorem 3.2 and 3.3, we have

$$D_4^{(q)} = \hat{D}_4 \circ (1,q,q^2) = (\hat{D}_3 + \hat{D}_2^*) \circ (1,q,q^2) = D_3^{(q)} + \hat{D}_2^* \circ (1,q,q^2),$$

where $\hat{D}_2^* = \binom{00\hat{e}_{2,0}}{+\hat{e}_{1,1}} = \binom{001}{+1} = (1,0,1)$. Thus

$$\begin{aligned}
&(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ \mu_2 \\
&\quad = \Big(D_3^{(q)} + \hat{D}_2^* \circ (1,q,q^2), D_3^{(q+1)} + \hat{D}_2^* \circ (1,q+1,(q+1)^2), \\
&\qquad D_3^{(q+2)} + \hat{D}_2^* \circ (1,q+2,(q+2)^2)\Big) \circ \mu_2 \\
&\quad = \Big(D_3^{(q)}, D_3^{(q+1)}, D_3^{(q+2)}\Big) \circ \mu_2 \\
&\qquad + \Big(\hat{D}_2^* \circ (1,q,q^2), \hat{D}_2^* \circ (1,q+1,(q+1)^2), \hat{D}_2^* \circ (1,q+2,(q+2)^2)\Big) \circ \mu_2 \\
&\quad = A + B.
\end{aligned}$$

Here

$$\begin{aligned}
A &= (D_3^{(q)}, D_3^{(q+1)}, D_3^{(q+2)}) \circ \mu_2 = (D_3^{(q)}, D_3^{(q+1)}, D_3^{(q+2)}) \circ (1,-2,1) \\
&= -(D_3^{(q)}, D_3^{(q+1)}) \circ (-1,1) + (D_3^{(q+1)}, D_3^{(q+2)}) \circ (-1,1) = -1+1 = 0
\end{aligned}$$

by Theorem 4.1. And

$$\begin{aligned}
B &= (\hat{D}_2^* \circ (1,q,q^2), \hat{D}_2^* \circ (1,q+1,(q+1)^2), \hat{D}_2^* \circ (1,q+2,(q+2)^2)) \circ \mu_2 \\
&= 1 \cdot 2! = 2
\end{aligned}$$

by Lemma 4.2 with $D_2^* = (1,0,1)$. Similarly, from

$$\begin{aligned}
D_5^{(q)} &= \hat{D}_5 \circ (1,q,q^2,q^3) = (\hat{D}_4 + \hat{D}_3^*) \circ (1,q,q^2,q^3) \\
&= \hat{D}_4 \circ (1,q,q^2) + \hat{D}_3^* \circ (1,q,q^2,q^3) = D_4^{(q)} + \hat{D}_3^* \circ (1,q,q^2,q^3),
\end{aligned}$$

where $\hat{D}_3^* = \binom{000\hat{e}_{3,0}}{+0\hat{e}_{2,1}} = \binom{0001}{+011} = (0,1,1,1)$, we have

$$\begin{aligned}
&(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}) \circ \mu_3 \\
&\quad = \Big(D_4^{(q)} + \hat{D}_3^* \circ (1,q,q^2,q^3), D_4^{(q+1)} + \hat{D}_3^* \circ (1,q+1,(q+1)^2,(q+1)^3), \\
&\qquad\quad D_4^{(q+2)} + \hat{D}_3^* \circ (1,q+2,(q+2)^2,(q+2)^3), \\
&\qquad\quad D_4^{(q+3)} + \hat{D}_3^* \circ (1,q+3,(q+3)^2,(q+3)^3)\Big) \circ \mu_3 \\
&\quad = \Big(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}\Big) \circ \mu_3 \\
&\qquad + \Big(\hat{D}_3^* \circ (1,q,q^2,q^3), \hat{D}_3^* \circ (1,q+1,(q+1)^2,(q+1)^3), \\
&\qquad\qquad \hat{D}_3^* \circ (1,q+2,\cdots,(q+2)^3), \hat{D}_3^* \circ (1,q+3,\cdots,(q+3)^3)\Big) \circ \mu_3 \\
&\quad = A + B.
\end{aligned}$$

Here, again

$$\begin{aligned}
A &= (D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}) \circ (-1,3,-3,1) \\
&= -(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ (1,-2,1) + (D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}) \circ (1,-2,1) \\
&= -2 + 2 = 0
\end{aligned}$$

and due to Lemma 4.2 with $D_3^* = (0,1,1,1)$, we have

$$\begin{aligned}
B = \Big(&\hat{D}_3^* \circ (1,q,q^2,q^3), \hat{D}_3^* \circ (1,q+1,(q+1)^2,(q+1)^3), \\
&\hat{D}_3^* \circ (1,\cdots,(q+2)^3), \hat{D}_3^* \circ (1,\cdots,(q+3)^3)\Big) \circ \mu_3 = 1 \cdot 3! = 3!.
\end{aligned}$$

□

THEOREM 4.4. (1) $(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}) \circ \mu_3 = 0$.
(2) $(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}, D_5^{(q+4)}) \circ \mu_4 = 0$.

Proof. Due to Theorem 4.3, we have

$$\begin{aligned}
&(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}) \circ (-1,3,-3,1) \\
&\quad = -(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+2)}) \circ (1,-2,1) + (D_4^{(q+1)}, D_4^{(q+2)}, D_4^{(q+3)}) \circ (1,-2,1) \\
&\quad = -2 + 2 = 0.
\end{aligned}$$

Similarly

$$(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}, D_5^{(q+4)}) \circ (1, -4, 6, -4, 1)$$
$$= -(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}) \circ (-1, 3, -3, 1)$$
$$+ (D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+3)}, D_5^{(q+4)}) \circ (-1, 3, -3, 1) = -3! + 3! = 0.$$

□

By means of the length $l(n)$ of $\hat{D}_n$, since $l(4) = 3$ and $l(5) = 4$, the identities in Theorem 4.3 and 4.4 can be written by

$(D_4^{(q)}, D_4^{(q+1)}, D_4^{(q+l(4)-1)}) \circ \mu_{l(4)-1} = (l(4) - 1)!$,
$(D_5^{(q)}, D_5^{(q+1)}, D_5^{(q+2)}, D_5^{(q+l(5)-1)}) \circ \mu_{l(5)-1} = (l(5) - 1)!$, and
$(D_4^{(q)}, D_4^{(q+1)}, \cdots, D_4^{(q+l(4))}) \circ \mu_{l(4)} = 0 = (D_5^{(q)}, D_5^{(q+1)}, \cdots, D_5^{(q+l(5))}) \circ \mu_{l(5)}$.

Now it can be generalized as follows.

Theorem 4.5. *Let a be the last digit of $\hat{D}^*_{n-2}$. Then*

(1) $(D_n^{(q)}, D_n^{(q+1)}, \cdots, D_n^{(q+l(n)-1)}) \circ \mu_{l(n)-1} = a\ (l(n) - 1)!$.

(2) $(D_n^{(q)}, D_n^{(q+1)}, \cdots, D_n^{(q+l(n))}) \circ \mu_{l(n)} = 0$.

Proof. When $n = 4, 5$, it is due to Theorem 4.3 and 4.4. Assume the identities are true for n. For convenience write $l(n) = l$ for the length of $\hat{D}_n$. Then

$$\begin{aligned} D_n^{(q)} &= (\hat{D}_{n-1} + \hat{D}^*_{n-2}) \circ (1, \cdots, q^{l-1}) \\ &= \hat{D}_{n-1} \circ (1, \cdots, q^{l-1}) + \hat{D}^*_{n-2} \circ (1, \cdots, q^{l-1}) \\ &= D_{n-1}^{(q)} + \hat{D}^*_{n-2} \circ (1, \cdots, q^{l-1}), \end{aligned}$$

by Theorem 3.2 and 3.3. Hence

$$\begin{aligned} &(D_n^{(q)}, D_n^{(q+1)}, \cdots, D_n^{(q+l-1)}) \circ \mu_{l-1} \\ &= \Big(D_{n-1}^{(q)} + \hat{D}^*_{n-2} \circ (1, q, \cdots, q^{l-1}), \cdots, \\ &\qquad D_{n-1}^{(q+l-1)} + \hat{D}^*_{n-2} \circ (1, q + l - 1, \cdots, (q + l - 1)^{l-1})\Big) \circ \mu_{l-1} \\ &= (D_{n-1}^{(q)}, \cdots, D_{n-1}^{(q+l-1)}) \circ \mu_{l-1} \\ &\quad + \Big(\hat{D}^*_{n-2} \circ (1, q, \cdots, q^{l-1}), \cdots, \hat{D}^*_{n-2} \circ (1, q + l - 1, \cdots, (q + l - 1)^{l-1})\Big) \circ \\ &\qquad \mu_{l-1} \\ &= A + B. \end{aligned}$$

Here

$$A = (D_{n-1}^{(q)}, \cdots, D_{n-1}^{(q+l-1)}) \circ \mu_{l-1}$$
$$= -(D_{n-1}^{(q)}, \cdots, D_{n-1}^{(q+l-2)}) \circ \mu_{l-2} + (D_{n-1}^{(q+1)}, \cdots, D_{n-1}^{(q+l-1)}) \circ \mu_{l-2}$$
$$= -\lambda(l-1)! + \lambda(l-1)! = 0$$

with λ the last digit of D_{n-3}^{*}, by the induction hypothesis. And

B

$$= \left(\hat{D}_{n-2}^{*} \circ (1, q, \cdots, q^{l-1}), \cdots, \hat{D}_{n-2}^{*} \circ (1, q+l-1, \cdots, (q+l-1)^{l-1})\right) \circ \mu_{l-1}$$
$$= a\ (l-1)!$$

by Lemma 4.2, so this proves (1). Furthermore

$$(D_n^{(q)}, D_n^{(q+1)}, \cdots, D_n^{(q+l)})) \circ \mu_{l(n)}$$
$$= -(D_n^{(q)}, D_n^{(q+1)}, \cdots, D_n^{(q+l-1)}) \circ \mu_{l-1} + (D_n^{(q+1)}, \cdots, D_n^{(q+l)}) \circ \mu_{l-1}$$
$$= -a(l-1)! + a(l-1)! = 0.$$

□

For example,

$(D_6^{(q)}, D_6^{(q+1)}, D_6^{(q+2)}, D_6^{(q+3)}, D_6^{(q+4)}) \circ (1, -4, 6, -4, 1) = 2 \cdot 4!$

$(D_7^{(q)}, D_7^{(q+1)}, \cdots, D_7^{(q+6)}) \circ (1, -6, 15, -20, 15, -6, 1) = 720 = 6!$

$(D_8^{(q)}, D_8^{(q+1)}, \cdots, D_8^{(q+8)}) \circ (1, -8, 28, -56, 70, -56, 28, -8, 1) = 40320 = 8!$.

Thus this yields some recurrence rule of $D_n^{(q)}$ immediately.

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