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DESCRIPTION OF FLOWS NEAR CLOSED SETS WITH COMPACT BOUNDARY

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Abstract. In this paper, we analyze the behaviour of flows near closed sets with compact boundary which contains no semi-orbits. Also, we give a condition that a closed invariant set is to be asymptotically stable.

1. Introduction and Preliminaries

The behaviours of continuous flows in the vicinity of closed sets with compact boundary are analyzed with the related concepts of invariance, attraction, stability and unstability.

In this paper, we analyze the behaviours of flows near closed sets with compact boundary which contains no semi-orbits. Also, we give a condition that a closed invariant set is to be asymptotically stable.

A given continuous flows (X, π) on a locally compact metric space (X, d) will be assumed throughout this paper. The symbols O, D denote, respectively, the orbit and prolongational relations. The unilateral versions of these relations carry the appropriate superscripts $+$ or $-$. Also, $O^+(x)$ and $O^-(x)$ are called positive and negative semi-orbit of x of X, respectively. $\omega(x)$ and $\alpha(x)$ denote, respectively, the positive and negative limit set of x of X. A point x of X is positively (weakly) attracted to a set $M \subset X$ if the positive semi-orbit of x is (frequently) ultimately contained in each neighborhood of M. The region of positive (weak) attraction is denoted by $(A^+_W(M))$ $A^+(M)$ and M is called a positive (weak) attractor if $(A^+_W(M))$ $A^+(M)$ contains a neighborhood of M. A given set M is said to be positively stable if every neighborhood of M contains a positively invariant neighborhood of M and is called positively asymptotically stable if it is a positively stable attractor. Frequently, the adjective "positive" is omitted. A set $M \subset X$ is called a

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saddle set if there exists a neighborhood U of M such that every neighborhood V of M contains at least one point $x \in X$ with $O^+(x) \not\subset U$ and $O^-(x) \not\subset U$.

$U \subset X$ is a neighborhood of $M \subset X$ provided U is an open set containing M. $B(x, \varepsilon)$ denotes the ε-ball and $H(M, \varepsilon)$ means $\{y \in X : d(M, y) = \varepsilon\}$. We denote the closure, interior, boundary and complement of a set $M \subset X$ by $\overline{M}$, M°, ∂M and $X \setminus M$, respectively.

Each of the basic properties of dynamical theory used in this paper are presented in detail in references [1, 2].

2. Basic results

It was shown that a closed weak attractor with compact boundary containing no complete orbits is stable if and only if it is positively invariant [4]. The following result is a generalization of this result for arbitrary closed sets with compact boundary which contain no positive semi-orbits.

LEMMA 2.1. *Let $M \subset X$ be a closed set with compact boundary which contains no positive semi-orbits. Then M is positively invariant if and only if M is asymptotically stable.*

Proof. First, suppose that M is positively invariant and is not stable. Then $D^+(M) \neq M$, So we can select a positive number γ, a sequence of points $\{z_i\}$ and a sequence of positive numbers $\{r_i\}$ satisfying that $z_i \to z \in \partial M$, $z_i r_i \in H(M, \gamma)$. Assume that $z_i r_i \to q \in H(M, \gamma)$. Since M is positively invariant, we may assume that $r_i \to \infty$. Since $z \in \partial M$, $O^+(z)$ is not fully contained in ∂M. As M is positively invariant, there exist positive numbers s and δ such that $B(z, \delta)s \subset M^\circ$. Hence, there is an integer N such that, for every $i > N$, $x_i \in B(z, \delta)$ and $r_i > s$. Then, for all $i > N$, we have $x_i r_i = x_i s(r_i - s) \in M$. Thus q must be in M. But this is absurd because q is chosen in $H(M, \gamma)$. Consequently, M is stable.

On the other hand, for each pont x in ∂M, we can choose positive numbers δ_x and t_x satisfying that $B(x, \delta_x)t_x \subset M^\circ$. Let $W = \cup\{B(x, \delta_x) : x \in \partial M\}$. Then $W \cup M^\circ$ is an open neighborhood of M. For each point y in W, $y \in B(q, \delta_q)$ for some q in ∂M and thus $O^+(yt_q) \subset M$. So, the positive semi-orbit $O^+(y)$ is ultimately contained in M. This means that the open neighborhood $W \cup M^\circ$ of M is contained in $A^+(M)$ and M is an attractor.

The converse is trivial and this completes the proof. $\square$

PROPOSITION 2.2. *Let $M \subset X$ be a positive invariant closed set with compact boundary which contains no positive semi-orbits. Then, for each neighborhood U of M, there exist a neighborhood $W \subset U$ and a number $T > 0$ satisfying that $O^+(WT) \subset M$.*

Proof. Let U be a neighborhood of M and let x be in the boundary of M. Since the positive semi-orbit of x is not fully contained in ∂M, there are positive numbers δ_x and t_x with $B(x, \delta_x) \subset U$ and $B(x, \delta_x)t_x \subset M^\circ$. Let $\{B(x_i, \delta_{x_i})\}_{i=1}^n$ is a finite open covering of ∂M. Let $T = \max_{1 \le i \le n}\{t_{x_i}\}$ and $W = \cup_{i=1}^n \{B(x_i, \delta_{x_i})\}$. Then $W \cup M$ is an open neighborhood of M. For a point p in W, let $p \in B(x_{i_0}, \delta_{i_0})$ for some $1 \le i_0 \le n$. then

$$O^+(pT) \subset O^+(B(x_{i_0}, \delta_{i_0})T) \subset O^+(B(x_{i_0}, \delta_{i_0})t_{i_0}) \subset M.$$

Hence, we conclude that $O^+(WT) \subset M$ and this completes the proof. □

For a point x in M, the number of times $O(x)$ leaves M is the cardinality of the class of components of $\{t : xt \in O(x) \setminus M\}$. A weak attractor $M \subset X$ is called recursive provided the positive orbit of x is frequently contained in M for each point x of $A_w^+(M)$. In [3], It was shown that if $M \subset X$ be a closed recursive weak attractor with compact boundary and the boundary of M contains no semi-orbits, then M is a saddle set if and only if some orbit leaves M at least twice. Here, we show the following result holds for arbitrary closed sets which are not weak attractors.

LEMMA 2.3. *Let $M \subset X$ be a closed set with compact boundary. If the boundary of M contains no positive (negative) semi-orbits, then M is a saddle set if and only if some orbit leaves M at least twice.*

Proof. Here, assume that the boundary of M contains no positive semi-orbits. Suppose that M is a saddle set such that each orbit leaves M at most once. Then there are a neighborhood U of M and a sequence of points $\{x_n\}$ in $U \setminus M$ converging to a point x in the boundary of M such that $O^+(x_n) \cap \partial U \ne \emptyset$ and $O^-(x_n) \cap \partial U \ne \emptyset$, for each i. Then we can select numbers $t_n > 0$ and $s_n < 0$ satisfying that $x_n t_n$ and $x_n s_n$ is in ∂U. Let $O^+(x)$ be fully contained in M. Then $\{t_n\}$ tends to infinity. If not, there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ with $t_{n_i} \to t_0$. Then $x_{n_i} t_{n_i} \to xt_0 \in \partial U$ and this is absurd. Since $O^+(x)$ is not fully contained in the boundary of M, $O^+(x) \cap M^\circ$ is nonempty. Hence $xr \in M^\circ$ for some positive number r. Then we can choose an open ball $B(x, \varepsilon)$ with $B(x, \varepsilon)r \subset M^\circ$ and an integer n_0 satisfying that

$$x_{n_0} \in B(x, \varepsilon), \text{ and } x_{n_0} r \in M^\circ, \quad (r < t_{n_0}).$$

Therefore the orbit of $x_{n_0}r$ leaves M at least twice and this is absurd. Therefore $O^+(x)$ is not fully contained in M and thus $O^+(x) \cap (X \setminus M)$ is nonempty.

By our original assumption that each orbit leaves M at most once, $O^-(x)$ must be fully contained in M. If $O^-(x) \cap M^\circ$ is not empty, by the similar arguments, we can show that, for some $x_k \in \{x_n\}$ and $s < 0$, the orbit of $x_k s$ leaves M at least twice. Therefore, $O^-(x)$ is fully contained in the boundary of M. Since the boundary of M is compact, the negative limit set of x is nonempty and is fully contained in the boundary of M. Then every positive semi-orbit of point which is in the negative limit set of x is fully contained in the boundary of M. This contradiction shows that some orbit leaves M at least twice. In the case that the boundary of M contains no negative semi-orbits, the proof is similar.

Conversely, if some orbit leaves M at least twice, then clealy M is a saddle set. This completes the proof of this theorem. □

From the above lemmas we get the following result.

THEOREM 2.4. *Let $M \subset X$ be a closed set with compact boundary which contains no semi-orbits. Then one of the four conditions given below holds.*

(1) *M is positively asymptotically stable,*
(2) *M is negatively asymptotically stable,*
(3) *M is a saddle set (some orbit leaves M at least twice),*
(4) *There are points x and y in M such that $O^+(x) \not\subset M$ and $O^-(y) \not\subset M$.*

Proof. Clearly, M cannot be an invariant set. So this result follows by the above lemmas. □

M is called isolated from closed positively invariant sets provided there exists a neighborhood U of M satisfying that there is no closed positively invariant subsets of U which is disjoint with M. For $M \subset X$, $\alpha(M)$ denotes $\cup_{x \in M} \alpha(x)$. .

THEOREM 2.5. *Let $M \subset X$ be a closed set with compact boundary and be isolated from closed invariant sets. Then, M is positively stable if and only if M is positively invariant and satisfies $\alpha(X \setminus M) \cap M = \emptyset$.*

Proof. First, suppose that M is positively invariant and satisfies the condition $\alpha(X \setminus M) \cap M = \emptyset$. Let U be a neighborhood of M which isolates M from closed positively invariant sets. To show that M is positively stable, assume, on the contrary, that M is not positively stable.

Then $D^+(M) \neq M$ and so there is a point x in $D^+(y) \setminus M$ for some y in M. Let ε be a positive number satisfying that $\overline{B(M,\varepsilon)} \subset U$ and x is not in $\overline{B(M,\varepsilon)}$. Then there are a sequence of points $\{x_i\}$ and a sequence of numbers $\{t_i\}$ in R^+ such that $x_i \to y \in \partial M$ and $x_i t_i \to x$. For each i, define $r_i = \inf\{t > 0 : x_i t \in H(M,\varepsilon)\}$. Since M is positively invariant, we may assume that $x_i[0,\ r_i]$ is contained in the compact set $\overline{B(M,\varepsilon)} \setminus M^\circ$ and $r_i \to \infty$. Let $x_i r_i$ converge to z in $H(M,\varepsilon)$. Suppose $z(-s) \notin \overline{B(M,\varepsilon)}$ for a positive number s. Then, for sufficiently large k, we get $0 < r_k - s < r_k$ and $(x_k r_k)(-s) \notin \overline{B(M,\varepsilon)}$. This contradicts the fact that

$$(x_k r_k)(-s) = x_k(r_k - s) \in x_k[0,\ r_k] \subset \overline{B(M,\varepsilon)}.$$

This shows that $\overline{O^-(z)} \subset \overline{B(M,\varepsilon)} \setminus M^\circ$. Hence $\alpha(z)$ is nonempty and is contained in $\overline{B(M,\varepsilon)} \setminus M^\circ$. Since M is isolated from closed positively invariant sets, $\alpha(z) \cap M \neq \emptyset$ and we conclude $\alpha(X \setminus M) \cap M$ is not empty. This contradiction shows that M is stable.

Next, let M be stable, then M is positively invariant and $D^-(X \setminus M) \cap M$ is empty [1]. Since $\alpha^-(X \setminus M) \cap M \subset D^-(X \setminus M) \cap M$, $\alpha^-(X \setminus M) \cap M$ is empty . This completes the proof of this result. □

From the above theorem, we can obtain the following well-known result as the corollary.

Corollary 2.6 ([4]). *A closed weak attractor with compact boundary is stable if and only if it is positively invariant and satisfies the condition* $\alpha(X \setminus M) \cap M = \emptyset$.

Proof. The region of weak attraction of M does not contain any closed positively invarinat set which is disjoint with weak attractor M. So, this follows from the above theorem. □

Proposition 2.7. *Let* $M \subset X$ *be an invariant closed set with compact boundary. Suppose that a neighborhood* V *of* M *exists such that every positive semi-orbit of points in* $V \setminus M$ *does mot contained in* $V \setminus M$*. Then* M *is negatively asymptotically stable*

Proof. Let V be a neighborhood of M such the positive semi-orbit of every points in $V \setminus M$ does mot contained in $V \setminus M$. Suppose that M is not negatively stable. Then $D^-(M) \neq M$ and let $p \in D^-(M) \setminus M$. Let ε be a positive number such that $\overline{B(M,\varepsilon)} \subset V$ and $p \notin \overline{B(M,\varepsilon)}$. Since M is invariant, there are a sequence of points $\{x_i\}$ in $B(M,\varepsilon/3)$

and a sequence of negative numbers $\{t_i\}$ such that $x_i \to x \in \partial M$ and $x_i t_i \to p$. For each i, define a number $r_i < 0$ by

$$r_i = \sup\{t < 0 : x_i t \in H(M, \varepsilon)\}.$$

Then, clearly, $x_i(r_i, 0] \subset B(M, \varepsilon)$. If $\{r_i\}$ is bounded, then we can assume that $\{r_i\}$ converges to a number r_0 without loss of generality. This means that $\{x_i r_i\}$ converges to a point $xr_0 \in H(M, \varepsilon)$. But this is absurd since M is invariant. Thus $\{r_i\}$ must be unbounded. Also, for each i, select a number s_i such that $r_i < s_i < 0$ and $x_i s_i \in H(M, \varepsilon/2)$. By the similar arguments, we get $\{s_i\}$ tends to $-\infty$. Assume that $x_i s_i \to q \in H(M, \varepsilon/2)$. Since q is in V, there is a positive number T such that $qT \notin \overline{B(M, \varepsilon)}$. By the continuity of π, we can choose a positive number γ with $B(q, \gamma) \subset B(M, \varepsilon)$ and $B(q, \gamma)T \subset \overline{B(M, \varepsilon)}^c$. Then we can select an integer k satisfying that

$$x_k s_k \in B(q, \gamma), \quad s_k + T < 0 \quad \text{and} \quad x_k(s_k + T) \notin \overline{B(M, \varepsilon)}.$$

Here we have $x_k(s_k + T) \in x_k(r_k, 0)$. But this contradicts the fact that $x_k(r_k, 0] \subset B(M, \varepsilon)$. Hence we conclude that M is negatively stable.

Next, we show that M is a negative attractor. Since M is negatively stable, M has a negatively invariant neighborhood $B(M, \zeta)$ with $\overline{B(M, \zeta)} \subset V$.

Let q be a point in $B(M, \zeta)$ and assume that the negative limit set of q is not fully contained in M. Then $\alpha(q) \subset \overline{O^-(q)} \subset \overline{B(M, \zeta)}$. Let a be a point in $\alpha(q) \setminus (\overline{B(M, \zeta)} \setminus M)$. Then we have

$$O^+(a) \subset \alpha(q) \subset \overline{B(M, \zeta)} \subset V.$$

But this contradicts to our original assumption. Thus we conclude that any negative limit set of points in $B(M, \zeta)$ is contained in $A^+(M)$ and thus M is a negative attractor. This completes the proof of this theorem. □

Corollary 2.8. *Let $M \subset X$ be a invariant closed set with compact boundary. Suppose that a neighborhood V of M exists such that every negative semi-orbit of points in $V \setminus M$ does mot contained in $V \setminus M$. Then M is asymptotically stable*

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