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INFINITE SERIES IDENTITIES WHICH STEM FROM GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

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Abstract. We establish a few class of infinite series identities from modular transformation formulas for certain series which originally come from generalized non-holomorphic Eisenstein series.

1. Introduction

Bruce C. Berndt established a lot of infinite series identities from transformation formulas for Fourier series of generalized Eisenstein series [1, 2]. In special, those series contain many formulas in Ramanujan's lost notebook. In the sequel of his works, the author obtained lots of identities about infinite series [4]. In addition, the author proved transformation formulas for generalized non-holomorphic Eisenstein series [5] and, using them, found more generalized infinite series identities [6]. In this paper, we obtain this kind of series relations of another type.

We introduce some notations and a principal theorem which will be used to obtain main results. Through this article, the branch of the argument for a complex z is defined by $-\pi \leq \arg z < \pi$. Let $\zeta(s)$ be the Riemann zeta function. Let $\Gamma(s)$ denote the Gamma function and let $(x)_n$ denote the rising factorial defined by

$$(x)_n = x(x+1)\cdots(x+n-1) \text{ for } n>0,\ (x)_0 = 1.$$

The confluent hypergeometric function of the second kind $U(\alpha;\beta;z)$ is defined to be

$$U(\alpha;\beta;z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z),$$

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where

$$ {}_1F_1(\alpha;\beta;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}. $$

We shall use the fact that the function $U(\alpha;\beta;z)$ can be analytically continued to all values of α, β and z [7]. Put $e(z)=e^{2\pi iz}$. Let $\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$, the upper half-plane. For any complex τ, $V\tau=\frac{a\tau+b}{c\tau+d}$ always denotes a modular transformation with $c>0$. Let $r=(r_1,r_2)$ and $h=(h_1,h_2)$ denote real vectors. Let R and H be real vectors defined by

$$ R=(R_1,R_2)=(ar_1+cr_2,br_1+dr_2) $$

and

$$ H=(H_1,H_2)=(dh_1-bh_2,-ch_1+ah_2). $$

For $\tau\in\mathbb{H}$ and $s_1,\ s_2\in\mathbb{C}$, define

$$ \begin{aligned} \mathcal{A}(\tau,s_1,s_2;r,h)=\sum_{m+r_1>0}\sum_{n-h_2>0}&\frac{e\left(mh_1+((m+r_1)\tau+r_2)(n-h_2)\right)}{(n-h_2)^{1-s}}\\ &\times U(s_2;s;4\pi(m+r_1)(n-h_2)\mathrm{Im}(\tau)) \end{aligned} $$

and

$$ \begin{aligned} \bar{\mathcal{A}}(\tau,s_1,s_2;r,h)=\sum_{m+r_1>0}\sum_{n+h_2>0}&\frac{e\left(mh_1-((m+r_1)\bar{\tau}+r_2)(n+h_2)\right)}{(n+h_2)^{1-s}}\\ &\times U(s_1;s;4\pi(m+r_1)(n+h_2)\mathrm{Im}(\tau)), \end{aligned} $$

where $s=s_1+s_2$. Let

$$ \mathcal{H}(\tau,s_1,s_2;r,h)=\mathcal{A}(\tau,s_1,s_2;r,h)+e^{\pi is}\mathcal{A}(\tau,s_1,s_2;-r,-h) $$

and

$$ \bar{\mathcal{H}}(\tau,s_1,s_2;r,h)=\bar{\mathcal{A}}(\tau,s_1,s_2;r,h)+e^{\pi is}\bar{\mathcal{A}}(\tau,s_1,s_2;-r,-h). $$

Let

$$ \mathbf{H}(\tau,\bar{\tau},s_1,s_2;r,h)=\frac{1}{\Gamma(s_1)}\mathcal{H}(\tau,s_1,s_2;r,h)+\frac{1}{\Gamma(s_2)}\bar{\mathcal{H}}(\tau,s_1,s_2;r,h). $$

For real x, y and complex t with Re $t>1$, let

$$ \psi(x,y,t)=\sum_{n+y>0}\frac{e(nx)}{(n+y)^t} $$

and let

$$ \begin{aligned} \Psi(x,\alpha,t)&=\psi(x,\alpha,t)+e^{\pi it}\psi(-x,-\alpha,t),\\ \Psi_{-1}(x,\alpha,t)&=\psi(x,\alpha,t-1)+e^{\pi it}\psi(-x,-\alpha,t-1). \end{aligned} $$

Let λ be the characteristic function of the integers. We now state the principal theorem which makes a role to obtain main results.

THEOREM 1.1. ([5]). *Let* $Q = \{\tau \in \mathbb{H} \mid \mathrm{Re}\ \tau > -d/c\}$ *and* $\varrho = c\{R_2\} - d\{R_1\}$. *Then for* $\tau \in Q$ *and* $s_1, s_2 \in \mathbb{C}$ *with* $s = s_1 + s_2$, *not integers* ≤ 1,

$$\begin{aligned}
z^{-s_1}\bar{z}^{-s_2}\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) &= \mathbf{H}(\tau, \bar{\tau}, s_1, s_2; R, H) \\
&+\lambda(R_1)(2\pi i)^{-s}e^{-\pi i(2R_1H_1+s_2)}\Psi(-H_2, -R_2, s) \\
&-\lambda(r_1)(2\pi i)^{-s}e^{-\pi i(2r_1h_1-s_1)}z^{-s_1}\bar{z}^{-s_2}\Psi(h_2, r_2, s) \\
&+\lambda(H_2)(4\pi\mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1, R_1, s) \\
&-\lambda(h_2)(4\pi\mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1, r_1, s) \\
&+\frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H),
\end{aligned}$$

where $z = c\tau + d$ *and*

$$\begin{aligned}
\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) &:= e(-H_1([R_1] - c) - H_2([R_2] + 1 - d)) \\
&\times \sum_{j=1}^{c} e\left(-H_1 j - H_2\left[\frac{jd+\varrho}{c}\right]\right)\int_0^1 v^{s_1-1}(1-v)^{s_2-1} \\
&\times \int_C u^{s-1}\frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)}}\frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e^{-2\pi i H_2}}dudv,
\end{aligned}$$

where C *is a loop beginning at* $+\infty$, *proceeding in the upper half-plane, encircling the origin in the positive direction so that* $u = 0$ *is the only zero of*

$$(e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)})(e^u - e^{-2\pi i H_2})$$

lying inside the loop, and then returning to $+\infty$ *in the lower half plane. Here, we choose the branch of* u^s *with* $0 < \arg u < 2\pi$.

2. Infinite series identities

In this section, we obtain a few class of infinite series identities. We first need following lemma which gives the value of $U(s_2; s; z)$ for positive integers $s_1,\ s_2$ with $s = s_1 + s_2$.

LEMMA 2.1. *Let s_1, s_2 be positive integers with $s = s_1 + s_2 \geq 4$. Then*

$$U(s_2; s; z) = \frac{z^{1-s}}{(s_2-1)!} \sum_{k=0}^{s_1-1} \binom{s_1-1}{k} (s-k-2)! z^k.$$

Proof. By definition,

$$\begin{aligned} U(s_2; s; z) &= \frac{\Gamma(1-s)}{\Gamma(1-s_1)} \sum_{k=0}^{\infty} \frac{(s_2)_k}{(s)_k} \frac{z^k}{k!} \\ &\quad + \frac{\Gamma(s-1)}{\Gamma(s_2)} z^{1-s} \sum_{k=0}^{\infty} \frac{(1-s_1)_k}{(2-s)_k} \frac{z^k}{k!}. \end{aligned} \tag{2.1}$$

It is easy to see that

$$\frac{\Gamma(1-s)}{\Gamma(1-s_1)} = \frac{\Gamma(1-s)}{\Gamma(1-s+s_2)} = \frac{1}{(1-s)_{s_2}} = (-1)^{s_2} \frac{(s_1-1)!}{(s-1)!}$$

and

$$\begin{aligned} \frac{(s_2)_k}{(s)_k} \frac{z^k}{k!} &= \frac{(s_2+k-1)!}{(s_2-1)!} \frac{(s+k-1)!}{(s-1)!} \frac{z^k}{k!} \\ &= (s-1)! \binom{s_2+k-1}{k} \frac{z^k}{(s+k-1)!}. \end{aligned}$$

Then the part with the first summation in (2.1) is given by

$$\begin{aligned} &\frac{\Gamma(1-s)}{\Gamma(1-s_1)} \sum_{k=0}^{\infty} \frac{(s_2)_k}{(s)_k} \frac{z^k}{k!} \\ &= (-1)^{s_2} (s_1-1)! \sum_{k=0}^{\infty} \binom{s_2+k-1}{k} \frac{z^k}{(s+k-1)!}. \end{aligned} \tag{2.2}$$

Since

$$\frac{\Gamma(2-s)}{\Gamma(1-s_1)} = \frac{\Gamma(2-s)}{\Gamma(2-s+s_2-1)} = \frac{1}{(2-s)_{s_2-1}} = (-1)^{s_2-1} \frac{(s_1-1)!}{(s-2)!}$$

and

$$\frac{\Gamma(1-s_1+k)}{\Gamma(2-s+k)} = \frac{\Gamma(2-s+k+s_2-1)}{\Gamma(2-s+k)} = (2-s+k)_{s_2-1},$$

we have

$$\frac{(1-s_1)_k}{(2-s)_k} = \frac{\Gamma(2-s)}{\Gamma(1-s_1)}\frac{\Gamma(1-s_1+k)}{\Gamma(2-s+k)}$$
$$= (-1)^{s_2-1}\frac{(s_1-1)!}{(s-2)!}(2-s+k)_{s_2-1}.$$

Then the part with the second summation in (2.2) is given by

$$\frac{\Gamma(s-1)}{\Gamma(s_2)}z^{1-s}\sum_{k=0}^{\infty}\frac{(1-s_1)_k}{(2-s)_k}\frac{z^k}{k!}$$
$$= (-1)^{s_2-1}\frac{(s_1-1)!}{(s_2-1)!}\sum_{k=0}^{\infty}(2-s+k)_{s_2-1}\frac{z^{1-s+k}}{k!}. \tag{2.3}$$

If $s_1 \le k \le s-2$, then $(2-s+k)_{s_2-1} = 0$. Thus

$$\sum_{k=0}^{\infty}(2-s+k)_{s_2-1}\frac{z^k}{k!} = \sum_{k=0}^{s_1-1}(2-s+k)_{s_2-1}\frac{z^k}{k!} + \sum_{k=s-1}^{\infty}(2-s+k)_{s_2-1}\frac{z^k}{k!}.$$

A direct calculation shows that

$$(-1)^{s_2-1}\frac{(s_1-1)!}{(s_2-1)!}\sum_{k=0}^{s_1-1}(2-s+k)_{s_2-1}\frac{z^{1-s+k}}{k!}$$
$$= \frac{(s_1-1)!}{(s_2-1)!}\sum_{k=0}^{s_1-1}\frac{(s-k-2)!}{(s_1-k-1)!}\frac{z^{1-s+k}}{k!}$$
$$= \frac{1}{(s_2-1)!}\sum_{k=0}^{s_1-1}\binom{s_1-1}{k}(s-k-2)!z^{1-s+k}. \tag{2.4}$$

By replacing k by $k+s-1$, we see that

$$(-1)^{s_2-1}\frac{(s_1-1)!}{(s_2-1)!}\sum_{k=s-1}^{\infty}(2-s+k)_{s_2-1}\frac{z^{1-s+k}}{k!}$$
$$= (-1)^{s_2-1}\frac{(s_1-1)!}{(s_2-1)!}\sum_{k=s-1}^{\infty}\frac{\Gamma(1-s_1+k)}{\Gamma(2-s+k)}\frac{z^{1-s+k}}{k!}$$
$$= (-1)^{s_2-1}\frac{(s_1-1)!}{(s_2-1)!}\sum_{k=0}^{\infty}\frac{\Gamma(s_2+k)}{\Gamma(k+1)}\frac{z^k}{(s+k-1)!}$$
$$= (-1)^{s_2-1}(s_1-1)!\sum_{k=0}^{\infty}\binom{s_2+k-1}{k}\frac{z^k}{(s+k-1)!}. \tag{2.5}$$

Putting (2.4) and (2.5) into (2.3), we have

$$\begin{aligned}
&\frac{\Gamma(s-1)}{\Gamma(s_2)} z^{1-s} \sum_{k=0}^{\infty} \frac{(1-s_1)_k}{(2-s)_k} \frac{z^k}{k!} \\
&= \frac{1}{(s_2-1)!} \sum_{k=0}^{s_1-1} \binom{s_1-1}{k} (s-k-2)! z^{1-s+k} \\
&\quad + (-1)^{s_2-1}(s_1-1)! \sum_{k=0}^{\infty} \binom{s_2+k-1}{k} \frac{z^k}{(s+k-1)!}. \qquad (2.6)
\end{aligned}$$

If we use (2.2) and (2.6) in (2.1), the proof is complete. □

Let $E(n,j)$ be the Eulerian numbers defined by the number of permutations of numbers from 1 to n such that exactly j numbers are greater than the previous elements. For any integer n, the polylogarithm $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

where $z \in \mathbb{C}$ and $|z| < 1$. For any positive integer n, it is known that [3]

$$\mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} E(n,j) z^{n-j}.$$

Let $\{x\}$ be the fractional part of real x. For brevity, let

$$\hat{k} := \left[\frac{k-1}{2}\right], \ \tilde{k} := \left\{\frac{k-1}{2}\right\} \text{ and } \{k\}^* := 1 - \{k\}$$

for any real k. For j and m integral and x complex, let

$$f_k(m,j,x) := \begin{cases} (-1)^{\hat{k}+j+1} \sinh((j+\frac{1}{2})mx), & k \text{ even}, \\ (-1)^{\hat{k}+j+1} \cosh(jmx), & k \text{ odd}. \end{cases}$$

Let $B_n(x)$ denote the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi). \qquad (2.7)$$

Let $\zeta(s,x)$ be the Hurwitz zeta function.

THEOREM 2.2. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let c be a positive integer. For any integers $A \geq 0,\ B \geq 0$ and $N \geq 1$ with $A + B = 2N$,*

$$
\begin{aligned}
&(-1)^B\alpha^{-N}\sum_{k=1}^{A}\binom{A}{k}(2N-k)!\left(\frac{2\alpha}{c}\right)^k{\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(2m,j,(\alpha-i\pi)/c)}{\cosh^{k+1}(m(\alpha-i\pi)/c)}\\
&+(-1)^B\alpha^{-N}\sum_{k=1}^{B}\binom{B}{k}(2N-k)!\left(\frac{2\alpha}{c}\right)^k{\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(2m,j,(\alpha+i\pi)/c)}{\cosh^{k+1}(m(\alpha+i\pi)/c)}\\
&-(-1)^B(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\left(\frac{m^{-2N-1}}{e^{2m(\alpha+i\pi)/c}+1}+\frac{m^{-2N-1}}{e^{2m(\alpha-i\pi)/c}+1}\right)\\
&=(-\beta)^{-N}\sum_{k=1}^{A}\binom{A}{k}(2N-k)!\left(\frac{2\beta}{c}\right)^k{\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(m+\{c/2\}^*)^{k-2N-1}f_k(2m+2\{c/2\}^*,j,(\beta+i\pi)/c)}{\cosh^{k+1}((m+\{c/2\}^*)(\beta+i\pi)/c)}\\
&+(-\beta)^{-N}\sum_{k=1}^{B}\binom{B}{k}(2N-k)!\left(\frac{2\beta}{c}\right)^k{\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(m+\{c/2\}^*)^{k-2N-1}f_k(2m+2\{c/2\}^*,j,(\beta-i\pi)/c)}{\cosh^{k+1}((m+\{c/2\}^*)(\beta-i\pi)/c)}\\
&-(2N)!(-\beta)^{-N}\sum_{m=0}^{\infty}\left(\frac{(m+\{c/2\}^*)^{-2N-1}}{e^{(2m+2\{c/2\}^*)(\beta+i\pi)/c}+1}+\frac{(m+\{c/2\}^*)^{-2N-1}}{e^{(2m+2\{c/2\}^*)(\beta-i\pi)/c}+1}\right)\\
&+(2N)!((-\beta)^{-N}\zeta\left(2N+1,\left\{\frac{c}{2}\right\}\right)-(-1)^B\alpha^{-N}\zeta(2N+1))\\
&+2^{2N}c^{-2N-1}(2^{2N+2}-1)A!B!(\lambda\left(\frac{c}{2}\right)(-1)^B\alpha^{-N-1}-(-1)^N\beta^{-N-1})\zeta(2N+2),
\end{aligned}
$$

where the dash $'$ means that if k is odd, then the value of the term $j = 0$ is multiplied by $\frac{1}{2}$.

Proof. Let $s_1 = A+1$ and $s_2 = B+1$ for nonnegative integers $A,\ B$ and let $s = 2N+2$ for an integer $N \geq 1$ in Theorem 1.1. Put also

$$r = \left(0, \frac{1}{2}\right),\ h = (0,0),\ c\tau - c + 1 = \frac{\pi i}{\alpha},\ V\tau = \frac{\tau - 1}{c\tau - c + 1}$$

in Theorem 1.1. Then

$$R = \left(\frac{c}{2}, \frac{1-c}{2}\right),\ H = (0,0),\ V\tau = \frac{1}{c}\left(1 + \frac{\alpha i}{\pi}\right).$$

Here we show how to compute $\mathcal{A}(\tau, s_1, s_2; R, H)$. The rest series $\mathcal{A}(V\tau, s_1, s_2; r, h)$, $\bar{\mathcal{A}}(V\tau, s_1, s_2; r, h)$, $\bar{\mathcal{A}}(\tau, s_1, s_2; R, H)$ are able to be computed by the similar method. For c even, replacing m by $m+1-\frac{c}{2}$ and for c odd, replacing m by $m - [\frac{c}{2}]$, we see that

$$\begin{aligned}
\mathcal{A}(\tau, s_1, s_2; R, H) &= \sum_{m+c/2>0}\sum_{n>0} \frac{e(((m+c/2)\tau + (1-c)/2)n)}{n^{1-s}} \\
&\qquad \times U\left(s_2; s; 4\pi\left(m + \frac{c}{2}\right) n\mathrm{Im}(\tau)\right) \\
&= \sum_{m=0}\sum_{n=1} \frac{e(((m+\{c/2\}^*)(c-1+i\pi/\alpha)/c + (1-c)/2)n)}{n^{-2N-1}} \\
&\qquad \times U\left(B+1; 2N+2; \frac{4}{c}\pi\left(m + \left\{\frac{c}{2}\right\}^*\right) n\beta\right).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&e\left(\left(\frac{1}{c}\left(m + \left\{\frac{c}{2}\right\}^*\right)\left(c - 1 + \frac{i\pi}{\alpha}\right) + \frac{1}{2}(1-c)\right) n\right) \\
&= e^{\pi i n(2m + 2\{c/2\}^* - 2(m+\{c/2\}^*)/c + 1 - c + (m+\{c/2\}^*)\pi i/(c\alpha))} \\
&= (-1)^n e^{-2n((2m+\{c/2\}^*)(\beta+\pi i)/c)}
\end{aligned}$$

and applying Lemma 2.1,

$$\begin{aligned}
&U\left(B+1; 2N+2; \frac{4}{c}\pi\left(m + \left\{\frac{c}{2}\right\}^*\right) n\beta\right) \\
&= \frac{1}{B!}\sum_{k=0}^{A}\binom{A}{k}(2N-k)!\left(\frac{4(m+\{c/2\}^*)n\beta}{c}\right)^{k-2N-1}.
\end{aligned}$$

We now have that

$$\mathcal{A}(\tau, s_1, s_2; R, H) = \frac{1}{B!}\sum_{k=0}^{A}\binom{A}{k}\frac{(2N-k)!}{(4\beta/c)^{2N-k+1}}\sum_{m=0}^{\infty}\frac{\mathrm{Li}_{-k}\left(-e^{-2(m+\{c/2\}^*)(\beta+i\pi)/c}\right)}{(m+\{c/2\}^*)^{2N-k+1}}.$$

For $k > 0$,

$$\begin{aligned}
&\mathrm{Li}_{-k}\left(-e^{-2(m+\{c/2\}^*)(\beta+i\pi)/c}\right)\\
&= \frac{1}{(1+e^{-2(m+\{c/2\}^*)(\beta+i\pi)/c})^{k+1}} \sum_{j=0}^{k-1} \frac{(-1)^{k-j}E(k,j)}{e^{2(m+\{c/2\}^*)(\beta+i\pi)(k-j)/c}}\\
&= \frac{2^{-k-1}}{\cosh^{k+1}((m+\{c/2\}^*)(\beta+i\pi)/c)} \sum_{j=0}^{k-1} \frac{(-1)^{k-j}E(k,j)}{e^{(m+\{c/2\}^*)(\beta+i\pi)(k-2j-1)/c}}
\end{aligned}$$

and, using $E(k,j) = E(k,k-j-1)$,

$$\begin{aligned}
&\sum_{j=0}^{k-1}(-1)^{k-j}E(k,j)e^{-(m+\{c/2\}^*)(\beta+i\pi)(k-2j-1)/c}\\
&= \begin{cases} 2\sum_{j=0}^{\hat{k}}(-1)^{\hat{k}+j+1}E(k,\hat{k}-j)\sinh(\frac{1}{c}(m+\{\frac{c}{2}\}^*)(2j+1)(\beta+i\pi)), & k \text{ even},\\ 2\sum_{j=0}^{\hat{k}}(-1)^{\hat{k}+j+1}E(k,\hat{k}-j)\cosh(\frac{1}{c}(m+\{\frac{c}{2}\}^*)2j(\beta+i\pi)), & k \text{ odd}. \end{cases}
\end{aligned}$$

For $k = 0$,

$$\begin{aligned}
\mathrm{Li}_0\left(-e^{-2(m+\{c/2\}^*)(\beta+i\pi)/c}\right) &= \sum_{n=1}^{\infty}(-1)^n e^{-2(m+\{c/2\}^*)(\beta+i\pi)n/c}\\
&= \frac{-1}{1+e^{2(m+\{c/2\}^*)(\beta+i\pi)/c}}.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
\mathcal{A}(\tau,s_1,s_2;R,H) = &-\frac{(2N)!}{B!}\frac{c^{2N+1}}{(4\beta)^{2N+1}}\sum_{m=0}^{\infty}\frac{(m+\{c/2\}^*)^{-2N-1}}{1+e^{2(m+\{c/2\}^*)(\beta+i\pi)/c}}\\
&+\frac{2^{-2N-1}}{B!}\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\beta/c)^{2N-k+1}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\\
&\times\sum_{m=0}^{\infty}\frac{f_k(2m+2\{c/2\}^*,j,(\beta+i\pi)/c)}{(m+\{c/2\}^*)^{2N-k+1}\cosh^{k+1}((m+\{c/2\}^*)(\beta+i\pi)/c)}.
\end{aligned}$$

Note that $R_1 = \frac{c}{2},\ R_2 = \frac{1-c}{2},\ H_1 = H_2 = 0$. By direct calculations, we easily deduce that

$$\begin{aligned}\Psi(-H_2, -R_2, s) &= \psi\left(0, \frac{c-1}{2}, 2N+2\right) + e^{2\pi i(N+1)}\psi\left(0, -\frac{c-1}{2}, 2N+2\right)\\ &= \sum_{n+(c-1)/2>0}^{\infty} \frac{1}{(n+(c-1)/2)^{2N+2}} + \sum_{n-(c-1)/2>0}^{\infty} \frac{1}{(n-(c-1)/2)^{2N+2}}\\ &= \begin{cases} 2\zeta(2N+2), & c \text{ odd},\\ (2^{2N+3}-2)\zeta(2N+2), & c \text{ even}.\end{cases}\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\Psi(h_2, r_2, s) = \Psi\left(0, \frac{1}{2}, 2N+2\right) = (2^{2N+3}-2)\zeta(2N+2),\\ &\Psi_{-1}(H_1, R_1, s) = \Psi_{-1}\left(0, \frac{c}{2}, 2N+2\right) = \begin{cases}(2^{2N+3}-2)\zeta(2N+1), & c \text{ odd},\\ 2\zeta(2N+1), & c \text{ even},\end{cases}\\ &\Psi_{-1}(h_1, r_1, s) = \Psi_{-1}(0, 0, 2N+2) = 2\zeta(2N+1).\end{aligned}$$

By the residue theorem and (2.7), we find that

$$\begin{aligned}&\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u - 1}\, dudv\\ &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{N+2-k}((\varrho+jd)/c)}{k!(N+2-k)!}(-z)^{k-1}.\end{aligned} \tag{2.8}$$

Since $s = 2N+2 \geq 4$, the summation in (2.8) is 0. Thus we have

$$\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) = 0.$$

With all the calculations above, we obtain the desired result. □

Corollary 2.3. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any positive integers c and N, we have*

$$\alpha^{-N}\sum_{k=1}^{2N}\frac{(\alpha/c)^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(m,j,(\alpha-i\pi)/c)}{\cosh^{k+1}(m(\alpha-i\pi)/(2c))}$$
$$-\alpha^{-N}\sum_{m=1}^{\infty}\left(\frac{m^{-2N-1}}{e^{m(\alpha+i\pi)/c}+1}+\frac{m^{-2N-1}}{e^{m(\alpha-i\pi)/c}+1}\right)$$
$$=(-\beta)^{-N}\sum_{k=1}^{2N}\frac{(\beta/c)^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(m,j,(\beta+i\pi)/c)}{\cosh^{k+1}(m(\beta+i\pi)/(2c))}$$
$$-(-\beta)^{-N}\sum_{m=1}^{\infty}\left(\frac{m^{-2N-1}}{e^{m(\beta+i\pi)/c}+1}+\frac{m^{-2N-1}}{e^{m(\beta-i\pi)/c}+1}\right)$$
$$+c^{-2N-1}(2^{2N+1}-2^{-1})(\alpha^{-N-1}+(-\beta)^{-N-1})\zeta(2N+2)$$
$$-(\alpha^{-N}-(-\beta)^{-N})\zeta(2N+1).$$

Proof. Let $B = 0$ in Theorem 2.2 and replace c by $2c$. □

Corollary 2.4. *We have*

$$3\pi\sum_{m=1}^{\infty}\frac{\operatorname{sech}^2(m\pi)}{m^2}+6\pi^2\sum_{m=1}^{\infty}\frac{\operatorname{sech}^2(m\pi)\tanh(m\pi)}{m}+6\sum_{m=1}^{\infty}\frac{m^{-3}}{e^{2m\pi}+1}$$
$$=12\pi\sum_{m=0}^{\infty}\frac{\operatorname{csch}^2((2m+1)\pi/2)}{(2m+1)^2}$$
$$+12\pi^2\sum_{m=0}^{\infty}\frac{\operatorname{csch}^2((2m+1)\pi/2)\coth((2m+1)\pi/2)}{2m+1}$$
$$+48\sum_{m=1}^{\infty}\frac{(2m+1)^{-3}}{e^{(2m+1)\pi}-1}+24\zeta(3)-2\pi^3.$$

Proof. Let $N = 1$, $c = 1$ and $\alpha = \beta = \pi$ in Corollary 2.3. It is easy to see that

$$
\begin{aligned}
&f_1(0, m, \pi - i\pi) = -1,\\
&f_2(0, m, \pi - i\pi) = -\sinh\left(\frac{1}{2}m\pi(1-i)\right),\\
&\cosh(m\pi(1-i)) = (-1)^m \cosh(m\pi),\\
&\cosh\left(\frac{1}{2}(2m+1)\pi(1-i)\right) = -i(-1)^m \sinh\left(\frac{1}{2}(2m+1)\pi\right),\\
&\sinh\left(\frac{1}{2}(2m+1)\pi(1-i)\right) = -i(-1)^m \cosh\left(\frac{1}{2}(2m+1)\pi\right).
\end{aligned}
$$

Then we have that

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \frac{m^{-2}}{\cosh^2(m\pi(1-i)/2)}\\
&= \sum_{m=1}^{\infty} \frac{(2m)^{-2}}{\cosh^2(m\pi(1-i))} + \sum_{m=0}^{\infty} \frac{(2m+1)^{-2}}{\cosh^2((2m+1)\pi(1-i)/2)}\\
&= \frac{1}{4}\sum_{m=1}^{\infty} \frac{\operatorname{sech}^2(m\pi)}{m^2} - \sum_{m=0}^{\infty} \frac{\operatorname{csch}^2((2m+1)\pi/2)}{(2m+1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \frac{\sinh(m\pi(1-i)/2)}{m\cosh^3(m\pi(1-i)/2)}\\
&= \frac{1}{2}\sum_{m=1}^{\infty} \frac{\sinh(m\pi(1-i))}{m\cosh^3(m\pi(1-i))} + \sum_{m=0}^{\infty} \frac{\sinh((2m+1)\pi(1-i)/2)}{(2m+1)\cosh^3((2m+1)\pi(1-i)/2)}\\
&= \frac{1}{2}\sum_{m=1}^{\infty} \frac{\operatorname{sech}^2(m\pi)\tanh(m\pi)}{m} - \sum_{m=0}^{\infty} \frac{\operatorname{csch}^2((2m+1)\pi/2)\coth((2m+1)\pi/2)}{2m+1}.
\end{aligned}
$$

Finally, apply $\zeta(4) = \frac{\pi^4}{90}$ to complete the proof. □

COROLLARY 2.5. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any positive integers N, we have*

$$\alpha^{-N}\sum_{k=1}^{N}\binom{N}{k}(2N-k)!(2\alpha)^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(2m,j,\alpha)}{\cosh^{k+1}(m\alpha)}$$
$$-(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\frac{m^{-2N-1}}{e^{2m\alpha}+1}$$
$$=2^{2N+1}\beta^{-N}\sum_{k=1}^{N}\binom{N}{k}(2N-k)!\beta^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)$$
$$\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\tilde{k})\beta)}{(2m+1)^{2N-k+1}\sinh^{k+1}((2m+1)\beta/2)}$$
$$+2^{2N+1}(2N)!\beta^{-N}\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)\beta}-1}$$
$$-2^{2N-1}(2^{2N+2}-1)(N!)^2\beta^{-N-1}\zeta(2N+2)$$
$$+2^{-1}(2N)!((2^{2N+1}-1)\beta^{-N}-\alpha^{-N})\zeta(2N+1).$$

Proof. Let $c=1$ and $A=B$ in Theorem 2.2. For k even,

$$\begin{aligned}f_k(2m,j,\alpha\pm i\pi)&=(-1)^{\hat{k}+j+1}\sinh(2m(j+1)(\alpha\pm i\pi))\\&=\frac{1}{2}(-1)^{m+\hat{k}+j+1}\left(e^{(2mj+m)\alpha}-e^{-(2mj+m)\alpha}\right)\\&=(-1)^{m+\hat{k}+j+1}\sinh(2m(j+1)\alpha)\\&=(-1)^m f_k(2m,j,\alpha),\end{aligned}$$

$$\begin{aligned}&f_k(2m+1,j,\beta\pm i\pi)\\&\quad=(-1)^{\hat{k}+j+1}\sinh\left((2m+1)\left(j+\frac{1}{2}\right)(\beta\pm i\pi)\right)\\&\quad=\frac{\pm i}{2}(-1)^{m+\hat{k}+1}\left(e^{(2mj+m+j+1/2)\beta}-e^{-(2mj+m+j+1/2)\beta}\right)\\&\quad=\pm i(-1)^{m+\hat{k}+1}\cosh\left((2m+1)\left(j+\frac{1}{2}\right)\beta\right),\end{aligned}$$

$$\cosh^{k+1}(m(\alpha\pm i\pi))=(-1)^m\cosh^{k+1}(m\alpha),$$

$$\begin{aligned}&\cosh^{k+1}\left(\frac{1}{2}(2m+1)(\beta\pm i\pi)\right)\\&\quad=\frac{(\pm i)^{k+1}}{2^{k+1}}(-1)^m\left(e^{(m+1/2)\beta}+e^{-(m+1/2)\beta}\right)^{k+1}\\&\quad=(\pm i)^{k+1}(-1)^m\sinh^{k+1}\left(\frac{1}{2}(2m+1)\beta\right).\end{aligned}$$

For k odd,

$$\begin{aligned}f_k(2m,j,\alpha\pm i\pi)&=(-1)^{\hat{k}+j+1}\cosh(2mj(\alpha\pm i\pi))\\&=\frac{1}{2}(-1)^{\hat{k}+j+1}\left(e^{2mj\alpha\pm i2mj\pi}+e^{-2mj\alpha\mp i2mj\pi}\right)\\&=(-1)^{\hat{k}+j+1}\cosh(2mj\alpha)\\&=f_k(2m,j,\alpha),\end{aligned}$$

$$\begin{aligned}f_k(2m+1,j,\beta\pm i\pi)&=(-1)^{\hat{k}+j+1}\cosh((2m+1)j(\beta\pm i\pi))\\&=\frac{1}{2}(-1)^{\hat{k}+1}\left(e^{(2m+1)j\beta}+e^{-(2m+1)j\beta}\right)\\&=(-1)^{\hat{k}+1}\cosh((2m+1)j\beta),\end{aligned}$$

$$\cosh^{k+1}(m(\alpha\pm i\pi))=\cosh^{k+1}(m\alpha).$$

□

Corollary 2.6. *We have*

$$\begin{aligned}&\sum_{m=1}^{\infty}\frac{\operatorname{sech}(m\pi)e^{-m\pi}}{m^3}+\pi\sum_{m=1}^{\infty}\frac{\operatorname{sech}^2(m\pi)}{m^2}+7\zeta(3)-\frac{1}{3}\pi^2\\&=-8\sum_{m=0}^{\infty}\frac{\operatorname{csch}((2m+1)\pi/2)e^{-(2m+1)\pi/2}}{(2m+1)^3}-4\pi\sum_{m=0}^{\infty}\frac{\operatorname{csch}^2((2m+1)\pi/2)}{(2m+1)^2}\end{aligned}$$

Proof. Let $N=1$ and $\alpha=\beta=\pi$ in Corollary 2.5. With the similar calculation as in the proof of Corollary 2.4, we complete the proof. □

Corollary 2.7. *Let* $A \geq 0$, $B \geq 0$ *and* $N \geq 1$ *with* $A + B = 2N$. *If* $A \not\equiv B \pmod 4$, *then*

$$\sum_{k=1}^{A} \binom{A}{k} \frac{(2N-k)!}{(\pi)^{-k}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \Bigg(\sum_{m=1}^{\infty} \frac{(2m)^{k-2N-1} f_k(2m, j, \pi)}{\cosh^{k+1}(m\pi)}$$

$$\sum_{m=1}^{\infty} \frac{(2m+1)^{k-2N-} \cosh((2m+1)(j+\tilde{k})\pi)}{\sinh^{k+1}((2m+1)\pi/2)} \Bigg)$$

$$= -\sum_{k=1}^{B} \binom{B}{k} \frac{(2N-k)!}{(\pi)^{-k}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \Bigg(\sum_{m=1}^{\infty} \frac{(2m)^{k-2N-1} f_k(2m, j, \pi)}{\cosh^{k+1}(m\pi)}$$

$$+ \sum_{m=0}^{\infty} \frac{(2m+1)^{k-2N-1} \cosh((2m+1)(j+\tilde{k})\pi)}{\sinh^{k+1}((2m+1)\pi/2)} \Bigg)$$

$$+2(2N)! \left(\sum_{m=1}^{\infty} \frac{(2m)^{-2N-1}}{e^{2m\pi}+1} - \sum_{m=0}^{\infty} \frac{(2m+1)^{-2N-1}}{e^{(2m+1)\pi}-1} \right)$$

$$-(2N)!\zeta(2N+1) + \frac{1}{2\pi}(2^{2N+2}-1)A!B!\zeta(2N+2).$$

Proof. Let $c = 2$ and $\alpha = \beta = \pi$ in Theorem 2.2. Consider

$$\sum_{m=1}^{\infty} \frac{m^{k-2N-1} f_k(2m, j, (\pi \pm i\pi)/2)}{\cosh^{k+1}(m(\pi \pm i\pi)/2)}$$

$$= \sum_{m=1}^{\infty} \frac{(2m)^{k-2N-1} f_k(4m, j, (\pi \pm i\pi)/2)}{\cosh^{k+1}(m(\pi \pm i\pi))}$$

$$+ \sum_{m=0}^{\infty} \frac{(2m+1)^{k-2N-1} f_k(4m+2, j, (\pi \pm i\pi)/2)}{\cosh^{k+1}((2m+1)(\pi \pm i\pi)/2)}. \tag{2.9}$$

It is easy to see that, for k even,

$$f_k\left(4m, j, \frac{1}{2}(\pi \pm i\pi)\right) = (-1)^{\hat{k}+j+1} \sinh((2mj+m)(\pi \pm i\pi))$$

$$= \frac{1}{2}(-1)^{m+\hat{k}+j+1} \left(e^{(2mj+m)\pi} - e^{-(2mj+m)\pi}\right)$$

$$= (-1)^{m+\hat{k}+j+1} \sinh\left(2m\left(j+\frac{1}{2}\right)\pi\right)$$

$$= (-1)^m f_k(2m, j, \pi) \tag{2.10}$$

and

$$\begin{aligned}
&f_k\left(4m+2, j, \frac{\pi}{2}(1\pm i)\right)\\
&\qquad = (-1)^{\hat{k}+j+1}\sinh\left((4m+2)\left(j+\frac{1}{2}\right)\pi(1\pm i)\right)\\
&\qquad = \frac{\mp i}{2}(-1)^{m+\hat{k}+1}\left(e^{(4mj+2m+2j+1)\pi/2} - e^{-(4mj+2m+2j+1)\pi/2}\right)\\
&\qquad = \mp i(-1)^{m+\hat{k}+1}\cosh\left((4m+2)\left(j+\frac{1}{2}\right)\pi\right). \qquad (2.11)
\end{aligned}$$

Similarly we see that, for k odd,

$$\begin{aligned}
&f_k\left(4m, j, \frac{\pi}{2}(1\pm i)\right) = (-1)^{\hat{k}+j+1}\cosh(2mj\pi) = f_k(2m, j, \pi),\\
&f_k\left(4m+2, j, \frac{\pi}{2}(1\pm i)\right) = (-1)^{\hat{k}+1}\cosh((2m+1)j\pi). \qquad (2.12)
\end{aligned}$$

Putting (2.10), (2.11), (2.12) into (2.9), we obtain that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{m^{k-2N-1}f_k(2m, j, (\pi\pm i\pi)/2)}{\cosh^{k+1}(m(\pi\pm i\pi)/2)}\\
&= \sum_{m=1}^{\infty}\frac{(2m)^{k-2N-1}f_k(2m, j, \pi)}{\cosh^{k+1}(m\pi)}\\
&\qquad + \sum_{m=0}^{\infty}\frac{(2m+1)^{k-2N-1}\cosh((2m+1)(j+\tilde{k})\pi)}{\sinh^{k+1}((2m+1)\pi/2)}.
\end{aligned}$$

□

THEOREM 2.8. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $A \geq 0,\ B \geq 0$ and $N \geq 1$ with $A + B = 2N$,*

$$
\begin{aligned}
&(-1)^B\alpha^{-N}\sum_{k=1}^{A}\binom{A}{k}(2N-k)!(\alpha/c)^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\tilde{k})(\alpha-i\pi)/c)}{(2m+1)^{2N-k+1}\sinh^{k+1}((2m+1)(\alpha-i\pi)/(2c))}\\
&+(-1)^B\alpha^{-N}\sum_{k=1}^{B}\binom{B}{k}(2N-k)!(\alpha/c)^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\tilde{k})(\alpha+i\pi)/c)}{(2m+1)^{2N-k+1}\sinh^{k+1}((2m+1)(\alpha+i\pi)/(2c))}\\
&+(-1)^B(2N)!\alpha^{-N}\left(\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)(\alpha+i\pi)/c}-1}+\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)(\alpha-i\pi)/c}-1}\right)\\
&=(-\beta)^{-N}\sum_{k=1}^{A}\binom{A}{k}(2N-k)!(\beta/c)^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\tilde{k})(\beta+i\pi)/c)}{(2m+1)^{2N-k+1}\sinh^{k+1}((2m+1)(\beta+i\pi)/(2c))}\\
&\quad+(-\beta)^{-N}\sum_{k=1}^{B}\binom{B}{k}(2N-k)!(\beta/c)^k{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\tilde{k})(\beta-i\pi)/c)}{(2m+1)^{2N-k+1}\sinh^{k+1}((2m+1)(\beta-i\pi)/(2c))}\\
&\quad+(2N)!(-\beta)^{-N}\left(\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)(\beta+i\pi)/c}-1}+\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)(\beta-i\pi)/c}-1}\right)\\
&\quad+(2N)!(1-2^{-2N-1})((-1)^N\beta^{-N}-(-1)^B\alpha^{-N})\zeta(2N+1).
\end{aligned}
$$

Proof. Let $s_1,\ s_2,\ h,\ \tau,\ V\tau$ be the same values as those in the proof of Theorem 2.2 and put $r = \left(\frac{1}{2}, 0\right)$ in Theorem 1.1. Then the calculations are similar to those in the proof of Theorem 2.2. □

Corollary 2.9. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. For any positive integer N,*

$$\alpha^{-N}\sum_{k=1}^{2N}\frac{\alpha^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\sum_{m=0}^{\infty}\frac{f_k(2m+1,j,\alpha)}{(2m+1)^{2N-k+1}\cosh^{k+1}((2m+1)\alpha/2)}$$
$$-2\alpha^{-N}\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)\alpha}+1}+\alpha^{-N}(1-2^{-2N-1})\zeta(2N+1)$$
$$=(-\beta)^{-N}\sum_{k=1}^{2N}\frac{\beta^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\sum_{m=0}^{\infty}\frac{f_k(2m+1,j,\beta)}{(2m+1)^{2N-k+1}\cosh^{k+1}((2m+1)\beta/2)}$$
$$-2(-\beta)^{-N}\sum_{m=0}^{\infty}\frac{(2m+1)^{-2N-1}}{e^{(2m+1)\beta}+1}+(-\beta)^{-N}(1-2^{-2N-1})\zeta(2N+1).$$

Proof. Let $B = 0$ and $c = 1$ in Theorem 2.8. Short calculations show that, for k odd,

$$\cosh((2m+1)(j+\tilde{k})(\alpha\pm i\pi)) = \frac{1}{2}(-1)^j\left(e^{(2m+1)j\alpha}+e^{-(2m+1)j\alpha}\right)$$
$$=(-1)^j\cosh((2m+1)j\alpha),$$

for k even,

$$\cosh((2m+1)(j+\tilde{k})(\alpha\pm i\pi))$$
$$=\frac{\pm i}{2}(-1)^{m+j}\left(e^{(2m+1)(j+1/2)\alpha}+e^{-(2m+1)(j+1/2)\alpha}\right)$$
$$=\pm i(-1)^{m+j}\sinh\left((2m+1)(j+\frac{1}{2})\alpha\right)$$

and

$$\sinh\left(\frac{1}{2}(2m+1)(\alpha\pm i\pi)\right)=\frac{\pm i}{2}(-1)^m\left(e^{(2m+1)\alpha/2}+e^{-(2m+1)\alpha/2}\right)$$
$$=\pm i(-1)^m\cosh\left(\frac{1}{2}(2m+1)\alpha\right).$$

Thus we obtain that

$$\frac{\cosh((2m+1)(j+\tilde{k})(\alpha \pm i\pi))}{\sinh^{k+1}((2m+1)(\alpha \pm i\pi)/2)}$$

$$= \begin{cases} \dfrac{(-1)^{\hat{k}+j+1}\cosh((2m+1)j\alpha)}{\cosh^{k+1}((2m+1)\alpha/2)}, & k \text{ odd}, \\ \dfrac{(-1)^{\hat{k}+j+1}\sinh((2m+1)(j+1/2)\pi)}{\cosh^{k+1}((2m+1)\alpha/2)}, & k \text{ even} \end{cases}$$

$$= \frac{f_k(2m+1,j,\alpha)}{\cosh^{k+1}((2m+1)\alpha/2)}.$$

□

COROLLARY 2.10. *For any positive integer N,*

$$\sum_{k=1}^{4N-2} \frac{\pi^k}{k!} {\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j) \sum_{m=0}^{\infty} \frac{f_k(2m+1,j,\pi)}{(2m+1)^{4N-k-1}\cosh^{k+1}((2m+1)\pi/2)}$$

$$= 2\sum_{m=0}^{\infty} \frac{(2m+1)^{1-4N}}{e^{(2m+1)\pi}+1} - (1-2^{1-4N})\zeta(4N-1).$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 2.9 and replace N by $2N-1$. □

COROLLARY 2.11. *We have*

$$4\pi^2 \sum_{m=0}^{\infty} \frac{\operatorname{sech}^2((2m+1)\pi/2)\tanh((2m+1)\pi/2)}{2m+1} + 4\pi \sum_{m=0}^{\infty} \frac{\operatorname{sech}^2((2m+1)\pi/2)}{(2m+1)^2}$$

$$= -8\sum_{m=0}^{\infty} \frac{\operatorname{sech}((2m+1)\pi/2)e^{-(2m+1)\pi}}{(2m+1)^3} + 7\zeta(3).$$

Proof. Let $N=1$ in Corollary 2.10. It is easy to see that

$$f_1(2m+1,0,\pi) = -1, \quad f_2(2m+1,0,\pi) = -\sinh\left(\frac{1}{2}(2m+1)\pi\right).$$

□

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