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ON NILPOTENT-DUO RINGS

Zhelin Piao*

Abstract. A ring R is called *right* (resp., *left*) *nilpotent-duo* if $N(R)a \subseteq aN(R)$ (resp., $aN(R) \subseteq N(R)a$) for every $a \in R$, where $N(R)$ is the set of all nilpotents in R. In this article we provide other proofs of known results and other computations for known examples in relation with right nilpotent-duo property. Furthermore we show that the left nilpotent-duo property does not go up to a kind of matrix ring.

Throughout this article every ring is associative with identity unless otherwise specified. Let R be a ring. $N(R)$ and $U(R)$ denote the set of all nilpotent elements and the group of all units in R, respectively. We use $J(R)$, $N_*(R)$, and $N^*(R)$ to denote the Jacobson radical, lower nilradical (i.e., prime radical), and upper nilradical (i.e., the sum of all nil ideals) of R, respectively. A nilpotent element is also called a *nilpotent* for simplicity. It is well-known that $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). $R[x]$ (resp., $R[[x]]$) denotes the polynomial ring (resp., power series ring) with an indeterminate x over R.

1. Other proofs for results

In this section, we provide other kinds of proofs of [4, Theorem 1.2(5)] and [4, Proposition 1.6]. These are helpful to see the structure of right nilpotent-duo matrix rings.

Following Feller [3], a ring (possibly without identity) is called *right* (resp., *left*) duo if every right (resp., left) ideal is an ideal; a ring is called

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duo if it is both right and left duo. Following [4], a ring R (possibly without identity) shall be called *right nilpotent-duo* if $N(R)a \subseteq aN(R)$ for every $a \in R$. Left nilpotent-duo rings are defined similarly. A ring shall be called *nilpotent-duo* if it is both left and right nilpotent-duo. Note that a ring R is nilpotent-duo if and only if $N(R)a = aN(R)$ for each $a \in R$. A ring R is usually called *reduced* if $N(R) = 0$. A ring is usually called *abelian* if every idempotent is central. Reduced rings are clearly abelian. It is easily checked that both commutative rings and reduced rings are nilpotent-duo. One-sided nilpotent-duo rings are Abelian by [4, Lemma 1.8(1)].

In the following we give another proof to [4, Theorem 1.2(5)] for the need to use the structure of matrices in such cases.

THEOREM 1.1. [4, Theorem 1.2(5)] *Let R be a reduced ring. Then R is right (resp., left) duo if and only if $D_2(R)$ is right (resp., left) nilpotent-duo.*

Proof. Let $E = D_2(R)$. First note that $N(E) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ because R is reduced. Suppose that R is right duo. Consider BA, for $0 \neq A = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \in E$ and $0 \neq B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in N(E)$. Then $BA = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix}$. Since R is right duo, $ba = ab_1$ for some $b_1 \in R$. Let $B_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}$. Then $B_1 \in N(E)$ and

$$BA = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} = AB_1.$$

So E is right nilpotent-duo.

Conversely, let E be right nilpotent-duo, and consider dc for $0 \neq c, d \in R$. If $dc = 0$ then we are done. So we assume that $dc \neq 0$. Let $C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ and $D = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$. Then $D \in N(E)$. Since E is right nilpotent-duo, there exists $D_1 \in N(E)$ such that $DC = CD_1$. Say $D_1 = \begin{pmatrix} 0 & d_1 \\ 0 & 0 \end{pmatrix}$. Note that $DC \neq 0$ implies $CD_1 \neq 0$. From $\begin{pmatrix} 0 & dc \\ 0 & 0 \end{pmatrix} = DC = CD_1 = \begin{pmatrix} 0 & cd_1 \\ 0 & 0 \end{pmatrix}$, we obtain $dc = cd_1$. Thus R is right duo. The proof for the left case can be done similarly. □

The monomorphism σ in [4, Example 1.5(1)] is not surjective. Hong et al. proved that if σ is surjective then the ring $D_2(R)$ is nilpotent-duo ([4, Proposition 1.6]), by using [4, Theorem 1.2(5)] and a well-known fact. In the following we provide another proof which contains the concrete structure.

PROPOSITION 1.2. [4, Proposition 1.6] *Let K be a field and σ be a monomorphism of K. Consider the skew power series ring $R = K[[x;\sigma]]$ by σ with an indeterminate x over K, where every element is of R is of the form $\sum_{i=0}^{\infty} a_i x^i$, only subject to $xa = \sigma(a)x$ for all $a \in K$. Set $E = D_2(R)$. If σ is bijective, then E is a nilpotent-duo ring.*

Proof. We apply the argument in the proof of [1, Theorem 1.4]. Using the computation in the latter part of [4, Example 1.5(1)], one can prove that E is left nilpotent-duo. We also have that $U(E)$ is the set of all $\begin{pmatrix} f(x) & r(x) \\ 0 & f(x) \end{pmatrix} \in E$, where the constant term of $f(x)$ is nonzero and $r(x) \in R$; and

$$N(E) = \left\{ \begin{pmatrix} 0 & s(x) \\ 0 & 0 \end{pmatrix} \in E \mid s(x) \in R \right\}.$$

Suppose that σ is bijective. Let $0 \neq A = \begin{pmatrix} f(x) & f_1(x) \\ 0 & f(x) \end{pmatrix}$ and $0 \neq B = \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}$ in E. We will show that there exists $C \in N(E)$ such that $BA = AC$.

If $f(x) = 0$, then $BA = 0 = AB$. So assume $f(x) = \sum_{i=m}^{\infty} a_i x^i \in R$ with $a_m \neq 0$, where $m \geq 0$. Let $h(x) = \sum_{k=0}^{\infty} c_k x^k$ with $c_k = a_{m+k}$ for all k. Then $f(x) = h(x)x^m$ with $h(x) \in U(R)$. We first show that $g(x)f(x) \in f(x)R$.

Write $h(x)^{-1}g(x)h(x) = \sum_{l=0}^{\infty} d_l x^l$, noting that $\sum_{l=0}^{\infty} d_l x^l \neq 0$ because $g(x) \neq 0$. Then we obtain

$$\begin{aligned} g(x)f(x) = g(x)h(x)x^m &= h(x)h(x)^{-1}g(x)h(x)x^m \\ &= h(x)x^m(\sum_{l=0}^{\infty} \sigma^{-m}(d_l)x^l) = f(x)(\sum_{l=0}^{\infty} \sigma^{-m}(d_l)x^l), \end{aligned}$$

using the surjection of σ.

Let $k(x) = \sum_{l=0}^{\infty} \sigma^{-m}(d_l)x^l$ and $C = \begin{pmatrix} 0 & k(x) \\ 0 & 0 \end{pmatrix}$. Then $C \in N(E)$, and moreover we get

$$BA = \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(x) & f_1(x) \\ 0 & f(x) \end{pmatrix} = \begin{pmatrix} 0 & g(x)f(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x)k(x) \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} f(x) & f_1(x) \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} 0 & k(x) \\ 0 & 0 \end{pmatrix} = AC.$$

Therefore E is right nilpotent-duo. □

2. Computations and examples

In this section, we provide other computations of the examples in [4, Example 1.3] and [4, Example 1.5(3)]. These are helpful to investigate the structures of various kinds of right nilpotent-duo rings and right duo rings.

In the following we give another computation to show the properties of given rings in [4, Example 1.3].

Example 2.1. There exists a non-reduced right duo ring R over which $D_2(R)$ is not right nilpotent-duo. We refer to the construction and argument of Courter in [2, Example 3.4]. Let K be a field and E be the row finite infinite matrix ring over K. Let S be the K-subspace of E generated by the matrices $\{a_0, a_1, \ldots, a_j, \ldots\}$, where

$a_0 = E_{12},\ a_1 = E_{13},\ a_2 = E_{14}+E_{32},\ \ldots,\ a_j = E_{1(j+2)}+E_{(j+1)2}$ for $j \geq 2$.

Next let I be the identity matrix in E and $R = KI \oplus S$, i.e., R is the K-subspace of E with a basis $\{I, a_0, a_1, \ldots, a_j, \ldots\}$. Then R is a right duo ring by [2, Example 3.4]. R is not reduced since $a_0^2 = E_{12}^2 = 0$. Every element in R is expressed by $kI + \sum_{i=0}^{m} c_i a_i$ with $m \geq 1$ and $k, c_0, \ldots, c_m \in K$. Note that $Sa_0 = 0 = a_0 S$, $a_i a_{i+1} = a_0$ for $i \geq 1$, and $a_i S = Ka_0$. So R is a local ring such that $J(R) = N(R) = N^*(R) = N_*(R) = S$, $J(R)^2 = Ka_0$, and $J(R)^3 = 0$.

We claim that $D_2(R)$ is not right nilpotent-duo. Note

$$N(D_2(R)) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in D_2(R) \mid x \in N(R) \right\}.$$

Consider two matrices

$$A = \begin{pmatrix} a_2 & I + a_2 \\ 0 & a_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_1 & I \\ 0 & a_1 \end{pmatrix}$$

in $D_2(R)$. Then $A, B \in N(D_2(R))$ and $BA = \begin{pmatrix} a_0 & a_0 + a_1 + a_2 \\ 0 & a_0 \end{pmatrix}$. Assume that there exists $C = \begin{pmatrix} \sum_{i=0}^m c_i a_i & kI + \sum_{j=0}^n d_j a_j \\ 0 & \sum_{i=0}^m c_i a_i \end{pmatrix} \in N(D_2(R))$ such that $BA = AC$, where $k, c_i, d_j \in K$. Then

$$AC = \begin{pmatrix} a_2(\sum_{i=0}^m c_i a_i) & a_2(kI + \sum_{j=0}^n d_j a_j) + (I + a_2)(\sum_{i=0}^m c_i a_i) \\ 0 & a_2(\sum_{i=0}^m c_i a_i) \end{pmatrix}$$
$$= \begin{pmatrix} a_0 & a_0 + a_1 + a_2 \\ 0 & a_0 \end{pmatrix}$$

implies

$$a_2 \left(\sum_{i=0}^m c_i a_i \right) = a_0 \text{ and } a_2 \left(kI + \sum_{j=0}^n d_j a_j \right) + (I + a_2) \left(\sum_{i=0}^m c_i a_i \right) =$$

$a_0 + a_1 + a_2$. So we have $c_3 = 1$ from

$$c_3 a_0 = c_3 a_2 a_3 = a_2 (\sum_{i=0}^m c_i a_i) = a_0,$$

and thus

$$a_2(kI + \sum_{j=0}^n d_j a_j) + (I + a_2) \left(\sum_{i=0}^m c_i a_i \right) = k a_2 + d_3 a_0 + \sum_{i=0}^m c_i a_i + a_0.$$

It then follows that

$$a_0 + a_1 + a_2 = (1 + c_0 + d_3) a_0 + c_1 a_1 + (k + c_2) a_2 + a_3 + (\sum_{i=4}^m c_i a_i).$$

But this equality is impossible because the right hand side contains a_3. Thus such C cannot exist, and so $D_2(R)$ is not right nilpotent-duo.

In [4, Example 1.5(3)], Hong et al. stated only the existence of a left nilpotent-duo ring which is not right nilpotent-duo. Here we provide the concrete procedure of constructing such rings to see that structure of such rings more deeply.

Example 2.2. There exists a left nilpotent-duo ring which is not right nilpotent-duo. We use a construction that is analogous to one of Example 2.1. Let K be a field and E be the column finite infinite matrix ring over K. Let S be the K-subspace of E generated by the matrices $\{a_0, a_1, \ldots, a_j, \ldots\}$, where

$a_0 = E_{21},\ a_1 = E_{31},\ a_2 = E_{41} + E_{23},\ \ldots,\ a_j = E_{(j+2)1} + E_{2(j+1)}$ for $j \geq 2$.

Next let I be the identity matrix in E and $R = KI \oplus S$, i.e., R is the K-subspace of E with a basis $\{I, a_0, a_1, \ldots, a_j, \ldots\}$. Then R is a left duo ring by applying the argument in [2, Example 3.4]. R is not reduced since $a_0^2 = E_{21}^2 = 0$. Every element in R is expressed by $kI + \sum_{i=0}^{m} c_i a_i$ with $m \geq 1$ and $k, c_0, \ldots, c_m \in K$. Note that

$$Sa_0 = 0 = a_0 S,\ a_1 S = 0 \text{ and } a_{i+1} a_i = a_0,\ Sa_i = Ka_0 \text{ for all } i \geq 1.$$

So R is a local ring such that $J(R) = N(R) = N^*(R) = N_*(R) = S$, $J(R)^2 = Ka_0$, and $J(R)^3 = 0$. Therefore R is left nilpotent-duo by [4, Theorem 1,.2(2)].

Consider a_1 and a_2 in $N(R)$. Then $a_2 a_1 = a_0$ but $a_0 \notin a_1 S = 0$. So R is not right nilpotent-duo. Furthermore $a_2 a_1 = a_0 \notin a_1 R = Ka_1$, and so R is not right duo.

Hong et al. showed that $D_2(R)$ need not be right duo over a right duo ring R, in [4, Example 1.3]. Here we provide another argument to show that.

Example 2.3. There exists a right duo ring R over which $D_2(R)$ is not right duo. Let R be the right duo ring in Example 2.1, and A, B be the same matrices as in Example 2.1. Assume on the contrary that there exists

$$D = \begin{pmatrix} hI + \sum_{i=0}^{m} c_i a_i & kI + \sum_{j=0}^{n} d_j a_j \\ 0 & hI + \sum_{i=0}^{m} c_i a_i \end{pmatrix} \in D_2(R)$$

such that $BA = AD$, where $h, k, c_i, d_j \in K$. Then we get

$$a_2(hI + \sum_{i=0}^{m} c_i a_i) = ha_2 + c_3 a_0 = a_0,$$

hence $h = 0$ and $c_3 = 1$. This yields $D = C$ as in Example 2.1, and the remainder of the argument follows Example 2.1. Then such D cannot exist, and so $D_2(R)$ is not right duo.

Hong et al. showed that $D_2(R)$ need not be right nilpotent-duo over a right nilpotent-duo ring R, in [4, Example 1.3]. Here we provide another argument to show that. We also show that $D_2(R)$ need not be left nilpotent-duo over a left nilpotent-duo ring R.

Example 2.4. (1) There exists a right nilpotent-duo ring R over which $D_2(R)$ is not right nilpotent-duo. It suffices to show that the ring R in Example 2.1 is right nilpotent-duo. Let $x \in R$ and $y \in N(R)$. If $x \in U(R)$ then $yx = xx^{-1}yx$. Here $x^{-1}yx \in N(R)$, so we are done.

Assume $x \notin U(R)$. Then we can express them by $x = \sum_{i=0}^{m} e_i a_i$ and $y = \sum_{i=0}^{m} f_i a_i$, where $e_i, f_i \in K$. Then

$$I - y = I + \sum_{i=0}^{m} (-f_i) a_i \in U(R).$$

Next, by help of the argument in [2, Example 3.4], we get $(I-y)x = xv$ for some $v = I + \sum_{i=0}^{l} g_i a_i \in U(R)$, where $g_i \in K$. Then

$$x - yx = (I-y)x = xv = x(I + \sum_{i=0}^{l} g_i a_i) = x + x(\sum_{i=0}^{l} g_i a_i),$$

entailing that $yx = x(\sum_{i=0}^{l}(-g_i)a_i)$ with $\sum_{i=0}^{l}(-g_i)a_i \in N(R)$. Therefore R is right nilpotent-duo.

(2) Let R be the left nilpotent-duo ring in Example 2.2. We claim that $D_2(R)$ is not left nilpotent-duo. Note

$$N(D_2(R)) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in D_2(R) \mid x \in N(R) \right\}.$$

Consider two matrices

$$A = \begin{pmatrix} a_2 & I + a_2 \\ 0 & a_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_1 & I \\ 0 & a_1 \end{pmatrix}$$

in $D_2(R)$. Then $A, B \in N(D_2(R))$ and $AB = \begin{pmatrix} a_0 & a_0 + a_1 + a_2 \\ 0 & a_0 \end{pmatrix}$. Assume that there exists $C = \begin{pmatrix} \sum_{i=0}^{m} c_i a_i & kI + \sum_{j=0}^{n} d_j a_j \\ 0 & \sum_{i=0}^{m} c_i a_i \end{pmatrix} \in N(D_2(R))$ such that $AB = CA$, where $k, c_i, d_j \in K$. Then

$$\begin{aligned} CA &= \begin{pmatrix} (\sum_{i=0}^{m} c_i a_i) a_2 & (\sum_{i=0}^{m} c_i a_i)(I + a_2) + (kI + \sum_{j=0}^{n} d_j a_j) a_2 \\ 0 & (\sum_{i=0}^{m} c_i a_i) a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & a_0 + a_1 + a_2 \\ 0 & a_0 \end{pmatrix} \end{aligned}$$

implies

$$\left(\sum_{i=0}^{m} c_i a_i \right) a_2 = a_0 \text{ and } \left(\sum_{i=0}^{m} c_i a_i \right) (I + a_2) + \left(kI + \sum_{j=0}^{n} d_j a_j \right) a_2 =$$

$a_0 + a_1 + a_2$. So we have $c_3 = 1$ from

$$c_3 a_0 = c_3 a_3 a_2 = (\sum_{i=0}^{m} c_i a_i) a_2 = a_0,$$

and thus

$$\left(\sum_{i=0}^{m} c_i a_i\right)(I + a_2) + \left(kI + \sum_{j=0}^{n} d_j a_j\right) a_2 = \sum_{i=0}^{m} c_i a_i + a_0 + k a_2 + d_3 a_0.$$

It then follows that

$$a_0 + a_1 + a_2 = (1 + c_0 + d_3)a_0 + c_1 a_1 + (k + c_2)a_2 + a_3 + (\sum_{i=4}^{m} c_i a_i).$$

But this equality is impossible because the right hand side contains a_3. Thus such C cannot exist, and so $D_2(R)$ is not left nilpotent-duo.

Applying the method in Example 2.3 to the left case, it is able to show that $D_2(R)$ need not be left duo over the left duo ring R in Example 2.2.

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Department of Mathematics
Yanbian University
Yanji 133002, China
E-mail: zlpiao@ybu.edu.cn