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Evolution of the First Eigenvalue of Weighted p-Laplacian along the Yamabe Flow

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Abstract. Let M be an n-dimensional closed Riemannian manifold with metric g, $d\mu = e^{-\phi(x)}d\nu$ be the weighted measure and $\Delta_{p,\phi}$ be the weighted p-Laplacian. In this article we will study the evolution and monotonicity for the first nonzero eigenvalue problem of the weighted p-Laplace operator acting on the space of functions along the Yamabe flow on closed Riemannian manifolds. We find the first variation formula of it along the Yamabe flow. We obtain various monotonic quantities and give an example.

1. Introduction

Let (M, g) be a Riemannian manifold, $d\nu$ be the Riemannian volume measure on (M, g) and $d\mu = e^{-\phi(x)}d\nu$ is the weighted volume measure. Then triple $(M, g, d\mu)$ is a smooth metric measure space. Such spaces have been used more widely in the work of mathematicians, for instance, Perelman used it in [10]. Let M be an n-dimensional closed Riemannian manifold with metric g.

The geometric flows as the Yamabe flow have been a topic of active research interest in both mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated with some curvature. A smooth one-parameter family of Riemannian metrics $g(t)$ on M with scalar curvature $R = R_{g(t)}$ is a called unnormalized Yamabe flow if

$$\frac{d}{dt}g(t) = -R_{g(t)}g(t), \qquad g(0) = g_0. \tag{1.1}$$

Also, $g(t)$ is a solution of normalized Yamabe flow if

$$\frac{d}{dt}g(t) = -(-R_{g(t)} - r_{g(t)})g(t), \qquad g(0) = g_0, \tag{1.2}$$

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where $r_{g(t)}$ denotes the mean value of the scalar of the metric $g(t)$, i.e.

$$r_{g(t)} = \frac{\int_M R_g dv_{g(t)}}{Vol(M)}.$$

Let $W^{1,p}(M)$ be the Sobolev space and $f : M \to \mathbb{R}$, $f \in W^{1,p}(M)$. For $p \in [1, +\infty)$ and any smooth function f on M, we define the weighed p-Laplacian on M by

$$\Delta_{p,\phi} f = e^{\phi} div \left(e^{-\phi} |\nabla f|^{p-2} \nabla f \right) = \Delta_p f - |\nabla f|^{p-2} \nabla\phi . \nabla f, \tag{1.3}$$

where the p-Laplacian $\Delta_p f$ defined as

$$\Delta_p f = div(|\nabla f|^{p-2}\nabla f) = |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(Hessf)(\nabla f, \nabla f). \tag{1.4}$$

The Witten-Laplacian is defined by $\Delta_\phi = \Delta - \nabla\phi.\nabla$, which is a symmetric diffusion operator on $L^2(M, \mu)$ and is self-adjoint. The weighted p-Laplacian is generalization of p-Laplacian and the Witten-Laplace operators, for instance, when ϕ is a constant function, the weighted p-Laplace operator is just the p-Laplace operator and when $p = 2$, the weighted p-Laplace operator is the Witten-Laplace operator.

We say Λ is an eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ at time $t \in [0, T)$ whenever for some $f \in W^{1,p}(M)$,

$$-\Delta_{p,\phi} f = \Lambda |f|^{p-2} f, \tag{1.5}$$

or equivalently

$$-\int_M f \Delta_{p,\phi} f d\mu = \Lambda \int_M |f|^p d\mu, \tag{1.6}$$

where $d\mu = e^{-\phi(x)} d\nu$ and $d\nu$ is the Riemannian volume measure. We have

$$\int_M |\nabla f|^p \, d\mu = \Lambda \int_M |f|^p d\mu, \tag{1.7}$$

where $f(x, t)$ called eigenfunction corresponding to eigenvalue $\Lambda(t)$. The first non-zero eigenvalue $\lambda(t) = \lambda(M, g(t), d\mu)$ is characterised as follows

$$\lambda(t) = \inf_{0 \neq f \in W_0^{1,p}(M)} \left\{ \int_M |\nabla f|^p d\mu : \quad \int_M |f|^p d\mu = 1 \right\}, \tag{1.8}$$

where $W_0^{1,p}(M)$ is the completion of $C_0^\infty(M)$ with respect Sobolev norm

$$||f||_{W^{1,p}} = \left(\int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu \right)^{\frac{1}{p}}. \tag{1.9}$$

The eigenvalue problem for weighted p-Laplacian has been extensively studied in the literature [11, 12].

The problem of evolution and monotonicity of the eigenvalue of geometric operator is a topic problem. Recently many mathematicians investigate properties of Laplace, p-Laplace, Witten-Laplace and etc geometric operators, under various geometric flows. For first time Perelman in [10] showed that the first eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing along the Ricci flow, where R is scalar curvature.

Then Cao [4] and Zeng and et'al [14] extended the geometric operator $-4\Delta+R$ to the operator $-\Delta + cR$ on closed Riemannian manifolds, and studied the monotonicity of eigenvalues of the operator $-\Delta + cR$ along the Ricci flow and the Ricci-Bourguignon flow, respectively.

Author in [1] studied the evolution for the first eigenvalue of p-Laplacian along the Yamabe flow and in [2] shown that the first eigenvalue of Witten-Laplace operator $-\Delta_\phi$ is monotonic along the Ricci-Bourguignon flow with some assumptions.

In [9], I have been studied the evolution for the first eigenvalue of geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow. For the other recent research in this direction, see [5, 6, 13].

Motivated by the above works, in this paper we will investigate the evolution of the nonzero first eigenvalue of the weighted p-Laplace operator whose metric satisfying the Yamabe flow (1.1) and ϕ evolves by $\frac{\partial \phi}{\partial t} = \Delta\phi$.

2. Preliminaries

In this section, we will first introduce a smooth function where at time t_0 is the first nonzero eigenvalue of the weighted p-Laplace operator $\Delta_{p,\phi}$ then we will find the formula for the evolution of the first non-zero eigenvalue of the weighted p-Laplace operator along the evolution equation system (1.1) on a connected, smooth, closed oriented Riemannnian n-manifold. Let M be a closed oriented Riemannian n-manifold and $(M, g(t), \phi(t))$ be a smooth solution of the evolution equation system (1.1) for $t \in [0, T)$.

The first non-zero eigenvalue of weighted p-Laplacian is nonlinear and its corresponding eigenfunction are not known to be C^1-differentiable. For this reason, we apply techniques of Cao [3] and Wu [13] and assume that at time t_0, $f_0 = f(t_0)$ is the eigenfunction for the first nonzero eigenvalue $\lambda(t_0)$ of $\Delta_{p,\phi}$. Then we get

$$\int_M |f(t_0)|^p \, d\mu_{g(t_0)} = 1. \tag{2.1}$$

We consider the following smooth function

$$h(t) := f_0 \left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))} \right]^{\frac{1}{2(p-1)}}, \tag{2.2}$$

along the Yamabe flow $g(t)$. We assume that

$$f(t) = \frac{h(t)}{(\int_M |h(t)|^p d\mu)^{\frac{1}{p}}}, \tag{2.3}$$

where $f(t)$ is smooth function under the Yamabe flow, satisfied in $\int_M |f|^p d\mu = 1$ and at time t_0, f is the eigenfunction for λ of $\Delta_{p,\phi}$. Therefore if $\int_M |f|^p d\mu = 1$ and

$$\lambda(t, f(t)) = -\int_M f\Delta_{p,\phi} f d\mu, \tag{2.4}$$

then $\lambda(t_0, f(t_0)) = \lambda(t_0)$.

3. Variation of $\lambda(t)$

In this section, we first recall evolution of some geometric structure along the Yamabe flow and then we give some useful evolution formulas for $\lambda(t)$ under the Yamabe flow. From [7] we have

Lemma 3.1. *Under the Yamabe flow equation (1.1), we get*

$$\begin{aligned}
\frac{\partial}{\partial t} g^{ij} &= R g^{ij}, \\
\frac{\partial}{\partial t}(d\nu) &= -\frac{n}{2} R d\nu, \\
\frac{\partial}{\partial t}(d\mu) &= -(\phi_t + \frac{n}{2} R) d\mu, \\
\frac{\partial}{\partial t}(\Gamma_{ij}^k) &= -\frac{1}{2}(\nabla_j R \delta_i^k + \nabla_i R \delta_j^k - \nabla^k R g_{ij}), \\
\frac{\partial}{\partial t} R &= (n-1)\Delta R + R^2,
\end{aligned}$$

and along the normalized Yamabe flow we have

$$\begin{aligned}
\frac{\partial}{\partial t} g^{ij} &= (R - r) g^{ij}, \\
\frac{\partial}{\partial t}(d\nu) &= -\frac{n}{2}(R - r) d\nu, \\
\frac{\partial}{\partial t}(d\mu) &= -(\phi_t + \frac{n}{2}(R - r)) d\mu, \\
\frac{\partial}{\partial t} R &= (n-1)\Delta R + R(R - r).
\end{aligned}$$

Lemma 3.2. *Let $(M, g(t))$, $t \in [0, T)$ be a solution to the Yamabe flow (1.1) on a closed oriented Riemannain manifold. Let $f \in C^\infty(M)$ be a smooth function on $(M, g(t))$. Then we have the following evolutions:*

$$\frac{\partial}{\partial t}|\nabla f|^2 = |\nabla f|^2 + 2g^{ij}\nabla_i f \nabla_j f_t, \tag{3.1}$$

$$\frac{\partial}{\partial t}|\nabla f|^{p-2} = (p-2)|\nabla f|^{p-4}\{\frac{R}{2}|\nabla f|^2 + g^{ij}\nabla_i f \nabla_j f_t\}, \tag{3.2}$$

$$(3.3)\qquad \frac{\partial}{\partial t}(\Delta f) = \Delta f_t + R\Delta f + \frac{2-n}{2}\nabla^k R \nabla_k f,$$

$$(3.4)\qquad \begin{aligned}\frac{\partial}{\partial t}(\Delta_p f) &= R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t)\\ &\quad -\frac{n-2}{2}Zg^{ij}\nabla_i R\nabla_j f,\end{aligned}$$

$$(3.5)\qquad \begin{aligned}\frac{\partial}{\partial t}(\Delta_{p,\phi} f) &= g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - \frac{n-2}{2}Zg^{ij}\nabla_i R\nabla_j f\\ &\quad +R\Delta_{p,\phi}f - Z_t\nabla\phi.\nabla f - Z\nabla\phi_t.\nabla f - Z\nabla\phi.\nabla f_t,\end{aligned}$$

where $Z := |\nabla f|^{p-2}$ *and* $f_t = \frac{\partial f}{\partial t}$.

Proof. By derivative respect to variable time t in local coordinates we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i f\nabla_j f) = \frac{\partial g^{ij}}{\partial t}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\\ &= R|\nabla f|^2 + 2g^{ij}\nabla_i f\nabla_j f_t\end{aligned}$$

which is (3.1). For prove (3.2) by using (3.1) we get

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^{p-2} &= \frac{\partial}{\partial t}(|\nabla f|^2)^{\frac{p-2}{2}} = \frac{p-2}{2}(|\nabla f|^2)^{\frac{p-4}{2}}\frac{\partial}{\partial t}(|\nabla f|^2)\\ &= \frac{p-2}{2}|\nabla f|^{p-4}\left\{R|\nabla f|^2 + 2g^{ij}\nabla_i f\nabla_j f_t\right\}\\ &= (p-2)|\nabla f|^{p-4}\left\{\frac{R}{2}|\nabla f|^2 + g^{ij}\nabla_i f\nabla_j f_t\right\}\end{aligned}$$

which is exactly (3.2). Now Lemma 3.1 and $2\nabla^i R_{ij} = \nabla_j R$ result

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}[g^{ij}(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k})]\\ &= \frac{\partial g^{ij}}{\partial t}(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}) + g^{ij}(\frac{\partial^2 f_t}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f_t}{\partial x^k}) - g^{ij}\frac{\partial}{\partial t}(\Gamma^k_{ij})\frac{\partial f}{\partial x^k}\\ &= R\Delta f + \Delta f_t + \frac{1}{2}g^{ij}(\nabla_j R\delta^k_i + \nabla_i R\delta^k_j - \nabla^k R g_{ij})\nabla_k f\\ &= \Delta f_t + R\Delta f + \frac{2-n}{2}\nabla^k R\nabla_k f.\end{aligned}$$

For prove (3.4), let $Z = |\nabla f|^{p-2}$. We obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_p f) &= \frac{\partial}{\partial t}\big(div(|\nabla f|^{p-2}\nabla f)\big) = \frac{\partial}{\partial t}\big(g^{ij}\nabla_i(Z\nabla_j f)\big)\\ &= \frac{\partial}{\partial t}\big(g^{ij}\nabla_i Z\nabla_j f + g^{ij}Z\nabla_i\nabla_j f\big)\\ &= \frac{\partial g^{ij}}{\partial t}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f + Z\frac{\partial}{\partial t}(\Delta f)\end{aligned}$$

$$
\begin{aligned}
&= Rg^{ij}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f \\
&\quad + Z\{\Delta f_t + R\Delta f + \frac{2-n}{2}\nabla^k R\nabla_k f\} \\
&= R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - \frac{n-2}{2}Zg^{ij}\nabla_i R\nabla_j f.
\end{aligned}
$$

Taking derivative with respect to time t of both sides of equation $\Delta_{p,\phi}f = \Delta_p f - |\nabla f|^{p-2}\nabla\phi.\nabla f$ and (3.4) imply that

$$
\begin{aligned}
\frac{\partial}{\partial t}(\Delta_{p,\phi}f) &= \frac{\partial}{\partial t}(\Delta_p f) - Z\frac{\partial g^{ij}}{\partial t}\nabla_i\phi\nabla_j f - Z_t g^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f \\
&\quad - Zg^{ij}\nabla_i\phi\nabla_j f_t \\
&= R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - \frac{n-2}{2}Zg^{ij}\nabla_i R\nabla_j f \\
&\quad - ZRg^{ij}\nabla_i\phi\nabla_j f - Z_t g^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f - Zg^{ij}\nabla_i\phi\nabla_j f_t,
\end{aligned}
$$

it results in (3.5). □

Proposition 3.3. *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the Yamabe flow (1.1) on the smooth closed Riemannain manifold (M^n, g_0) and $\frac{\partial\phi}{\partial t} = \Delta\phi$. If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(x,t)$ under the flow (1.1), then*

$$
\begin{aligned}
\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} &= \frac{n}{2}\lambda(t_0)\int_M R|f|^p\, d\mu + \frac{p-n}{2}\int_M R|\nabla f|^p d\mu \\
&\quad + \lambda(t_0)\int_M (\Delta\phi)|f|^p\, d\mu - \int_M (\Delta\phi)|\nabla f|^p d\mu.
\end{aligned}
\tag{3.6}
$$

Proof. Let $f(t)$ be a smooth function where $f(t_0)$ is the corresponding eigenfunction to $\lambda(t_0) = \lambda(t_0, f(t_0))$. Function $\lambda(t, f(t))$ is smooth and taking derivative $\lambda(t, f(t)) = -\int_M f\Delta_{p,\phi}f\, d\mu$ with respect to t, we get

$$
\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} = -\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\, d\mu. \tag{3.7}
$$

Now, by condition $\int_M |f|^p d\mu = 1$ and taking the time derivative of it, we can write

$$
0 = \frac{\partial}{\partial t}\int_M |f|^{p-2}f^2 d\mu = \int_M (p-1)|f|^{p-2}f f_t d\mu + \int_M |f|^{p-2}f\frac{\partial}{\partial t}(f d\mu), \tag{3.8}
$$

hence

$$
\int_M |f|^{p-2}f\left[(p-1)f_t d\mu + \frac{\partial}{\partial t}(f d\mu)\right] = 0. \tag{3.9}
$$

On the other hand, using (3.5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu &= \int_M \frac{\partial}{\partial t}(\Delta_{p,\phi}f)f\,d\mu + \int_M \Delta_{p,\phi}f\,\frac{\partial}{\partial t}(f\,d\mu)\\ &= \int_M R\Delta_{p,\phi}ff\,d\mu + \int_M g^{ij}\nabla_i(Z_t\nabla_j f)f\,d\mu\\ &\quad+\int_M g^{ij}\nabla_i(Z\nabla_j f_t)f\,d\mu - \frac{n-2}{2}\int_M Z\nabla R.\nabla ff\,d\mu\\ &\quad-\int_M Z_t\nabla\phi.\nabla ff\,d\mu - \int_M Z\nabla\phi_t.\nabla ff\,d\mu\\ &\quad-\int_M Z\nabla\phi.\nabla f_t f\,d\mu - \int_M \lambda|f|^{p-2}f\frac{\partial}{\partial t}(fd\mu).\end{aligned}\tag{3.10}$$

By the application of integration by parts we have

$$\int_M g^{ij}\nabla_i(Z_t\nabla_j f)f\,d\mu = -\int_M Z_t|\nabla f|^2 d\mu + \int_M Z_t\nabla f.\nabla\phi f\,d\mu,\tag{3.11}$$

and

$$\int_M g^{ij}\nabla_i(Z\nabla_j f_t)f\,d\mu = -\int_M Z\nabla f_t.\nabla f\,d\mu + \int_M Z\nabla f_t.\nabla\phi f\,d\mu.\tag{3.12}$$

Now, plugging (3.11) and (3.12) into (3.10), we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu &= -\int_M \lambda R|f|^p\,d\mu - \frac{n-2}{2}\int_M Z\nabla R.\nabla ffd\mu\\ &\quad-\int_M Z_t|\nabla f|^2 d\mu - \int_M Z\nabla f_t.\nabla f\,d\mu - \int_M Z\nabla\phi_t.\nabla f\,f\,d\mu\\ &\quad-\int_M \lambda|f|^{p-2}f\frac{\partial}{\partial t}(fd\mu).\end{aligned}\tag{3.13}$$

On the other hand

$$Z_t = \frac{\partial}{\partial t}(|\nabla f|^{p-2}) = (p-2)|\nabla f|^{p-4}\{\frac{R}{2}|\nabla f|^2 + g^{ij}\nabla_i f\nabla_j f_t\}.\tag{3.14}$$

Therefore replacing this into (3.13), we have

$$\begin{aligned}-\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} &= -\lambda(t_0)\int_M R|f|^p\,d\mu - \frac{p-2}{2}\int_M R|\nabla f|^p d\mu\\ &\quad-\frac{n-2}{2}\int_M Z\nabla R.\nabla ffd\mu - (p-1)\int_M Z\nabla f_t.\nabla f\,d\mu\\ &\quad-\int_M Z\nabla\phi_t.\nabla f\,f\,d\mu - \lambda(t_0)\int_M |f|^{p-2}f\frac{\partial}{\partial t}(fd\mu).\end{aligned}\tag{3.15}$$

Also

$$\int_M Z\nabla f_t.\nabla f\, d\mu = -\int_M f_t \Delta_{p,\phi} f\, d\mu = \int_M \lambda |f|^{p-2} f\, f_t\, d\mu. \tag{3.16}$$

Then we arrive at

$$\begin{aligned} -\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} &= -\lambda(t_0)\int_M R|f|^p\, d\mu - \frac{p-2}{2}\int_M R|\nabla f|^p d\mu \\ &\quad -\frac{n-2}{2}\int_M Z\nabla R.\nabla f f d\mu - \int_M Z\nabla\phi_t.\nabla f\, d\mu \\ &\quad -\lambda(t_0)\int_M |f|^{p-2} f\left((p-1) f_t\, d\mu + \frac{\partial}{\partial t}(f d\mu)\right). \end{aligned} \tag{3.17}$$

Hence (3.9) results in

$$\begin{aligned} -\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} &= -\lambda(t_0)\int_M R|f|^p\, d\mu - \frac{p-2}{2}\int_M R|\nabla f|^p d\mu \\ &\quad -\frac{n-2}{2}\int_M Z\nabla R.\nabla f f d\mu - \int_M Z\nabla\phi_t.\nabla f\, d\mu. \end{aligned} \tag{3.18}$$

By integration by parts we get

$$\int_M Z\nabla\phi_t.\nabla f\, f\, d\mu = \int_M \lambda|f|^p(\Delta\phi)\, d\mu - \int_M (\Delta\phi)|\nabla f|^p d\mu \tag{3.19}$$

and

$$\int_M Z\nabla R.\nabla f\, f\, d\mu = \int_M \lambda R|f|^p\, d\mu - \int_M R|\nabla f|^p d\mu. \tag{3.20}$$

Pluggin (3.19) and (3.20) into (3.18) imply that (3.6). □

Corollary 3.4. *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the Yamabe flow (1.1) on the smooth closed Riemannain manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(x,t)$ under the flow (1.1) where ϕ is independent of t, then*

$$\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} = \frac{n}{2}\lambda(t_0)\int_M R|f|^p\, d\mu + \frac{p-n}{2}\int_M R|\nabla f|^p d\mu. \tag{3.21}$$

Theorem 3.5. *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the Yamabe flow (1.1) on the smooth closed Riemannain manifold (M^n, g_0) and $\frac{\partial\phi}{\partial t} = \Delta\phi$. Let $\frac{p-n}{2}R < \Delta\phi$ and $R_{\min}(0) \geq 0$ along the Yamabe flow (1.1). Suppose that $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ then the quantity $\lambda(t)(1-R_{\min}(0)t)^{\frac{p+2}{2}}$ is nondecreasing along the flow (1.1) for $T \leq \frac{1}{R_{\min}(0)}$.*

Proof. According to (3.6) of Proposition 3.3, we have

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} &\geq \frac{n}{2}\lambda(t_0)\int_M R|f|^p\,d\mu+\frac{p-n}{2}\int_M R|\nabla f|^p d\mu\\ &+\frac{p-n}{2}\lambda(t_0)\int_M R|f|^p\,d\mu-\int_M(\Delta\phi)|\nabla f|^p d\mu\\ &\geq \frac{p}{2}\lambda(t_0)\int_M R|f|^p\,d\mu\end{aligned}\tag{3.22}$$

On the other hand, the scalar curvature under the Yamabe flow evolves by

$$\frac{\partial R}{\partial t}=(n-1)\Delta R+R^2\tag{3.23}$$

The solution of the corresponding ODE $y'=y^2$ with initial value $y(0)=R_{\min}(0)$ is

$$\sigma(t)=\frac{R_{\min}(0)}{1-R_{\min}(0)t}$$

on $[0,T')$ where $T'=\min\{T,\frac{1}{R_{\min}(0)}\}$. Using the maximum principle to (3.23), we get $R_{g(t)}\geq\sigma(t)$. Therefore (3.22) becomes

$$\frac{d}{dt}\lambda(t,f(t))|_{t=t_0}\geq\frac{p}{2}\lambda(t_0)\sigma(t_0),$$

and this implies that in any sufficiently small neighborhood of t_0 such as I_0, we get

$$\frac{d}{dt}\lambda(t,f(t))\geq\frac{p}{2}\lambda(t,f(t))\sigma(t).$$

Integrating the last inequality with respect to t on $[t_1,t_0]\subset I_0$, we have

$$\ln\frac{\lambda(t_0,f(t_0))}{\lambda(t_1,f(t_1))}>\ln\Big(\frac{1-R_{\min}(0)t_1}{1-R_{\min}(0)t_0}\Big)^{\frac{p}{2}}.$$

Since $\lambda(t_0,f(t_0))=\lambda(t_0)$ and $\lambda(t_1,f(t_1))\geq\lambda(t_1)$ we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)}>\ln\Big(\frac{1-R_{\min}(0)t_1}{1-R_{\min}(0)t_0}\Big)^{\frac{p}{2}},$$

that is, the quantity $\lambda(t)(1-R_{\min}(0)t)^{\frac{p}{2}}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)(1-R_{\min}(0)t)^{\frac{p}{2}}$ is strictly increasing along the Yamabe flow on $[0,T')$. □

Proposition 3.6. *Let $(M,g(t))$, $t\in[0,T)$ be a solution of the normalized Yamabe flow (1.2) on the smooth closed Riemannain manifold (M^n,g_0) and $\frac{\partial\phi}{\partial t}=\Delta\phi$. If*

$\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(x,t)$ under the flow (1.2), then

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} &= -\frac{pr}{2}\lambda(t_0)+\frac{n}{2}\lambda(t_0)\int_M R|f|^p\,d\mu+\frac{p-n}{2}\int_M R|\nabla f|^p d\mu\\ &\quad+\lambda(t_0)\int_M(\Delta\phi)|f|^p\,d\mu-\int_M(\Delta\phi)|\nabla f|^p d\mu. \qquad (3.24)\end{aligned}$$

Proof. In the normalized case, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_{p,\phi}f) &= g^{ij}\nabla_i(Z_t\nabla_j f)+g^{ij}\nabla_i(Z\nabla_j f_t)-\frac{n-2}{2}Zg^{ij}\nabla_i R\nabla_j\\ &\quad+(R-r)\Delta_{p,\phi}f-Z_t\nabla\phi.\nabla f-Z\nabla\phi_t.\nabla f-Z\nabla\phi.\nabla f_t, \qquad (3.25)\end{aligned}$$

and

$$Z_t=\frac{\partial}{\partial t}(|\nabla f|^{p-2})=(p-2)|\nabla f|^{p-4}\{\frac{R-r}{2}|\nabla f|^2+g^{ij}\nabla_i f\nabla_j f_t\}, \qquad (3.26)$$

where $Z=|\nabla f|^{p-2}$. The subsequent process is similar to the proof of Proposition 3.3, direct computation implies that (3.24). □

Corollary 3.7. *Let $(M,g(t))$, $t\in[0,T)$ be a solution of the Yamabe flow (1.1) on the smooth closed Riemannain manifold (M^n,g_0). If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(x,t)$ under the flow (1.1) where ϕ is independent of t, then*

$$\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0}=-\frac{pr}{2}\lambda(t_0)+\frac{n}{2}\lambda(t_0)\int_M R|f|^p\,d\mu+\frac{p-n}{2}\int_M R|\nabla f|^p d\mu. \qquad (3.27)$$

3.1. Variation of $\lambda(t)$ on a Surface

Now, we write Proposition 3.6 and Corollary 3.7 in some remarkable particular cases.

Corollary 3.1.1. *Let $(M^2,g(t))$, $t\in[0,T)$ be a solution of the normalized Yamabe flow on a closed Riemannian surface (M^2,g_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted p-Laplacian under the flow (1.2), then on g_0, we have*

(1) *If $\frac{\partial\phi}{\partial t}=\Delta\phi$ then*

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} &= -\frac{pr}{2}\lambda(t_0)+\lambda(t_0)\int_M R|f|^p\,d\mu+\frac{p-2}{2}\int_M R|\nabla f|^p d\mu\\ &\quad+\lambda(t_0)\int_M(\Delta\phi)|f|^p\,d\mu-\int_M(\Delta\phi)|\nabla f|^p d\mu. \qquad (2.1)\end{aligned}$$

(2) *If ϕ is independent of t then*

$$\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} = -\frac{pr}{2}\lambda(t_0)+\lambda(t_0)\int_M R|f|^p\,d\mu+\frac{p-2}{2}\int_M R|\nabla f|^p d\mu. \tag{2.2}$$

Remark 3.1.2. Let $(M^2, g(t))$, $t\in[0,T)$ be a solution of the normalized Yamabe flow on a compact Riemannian surface (M^2, g_0) and ϕ be independent of t. Then from [8] for a constant c depending only on g_0, we have

(i) If $r<0$ then $r-ce^{rt}\le R\le r+ce^{rt}$,

(ii) If $r=0$ then $-\frac{c}{1+ct}\le R\le c$,

(iii) If $r>0$ then $-ce^{rt}\le R\le r+ce^{rt}$.

Therefore by using (2.2) we can obtain a lower bound and a upper bound for the first eigenvalue of the weighted p-Laplacian under the flow (1.2).

4. Example

In this section, we find the variational formula of the evolving spectrum of the weighted p-Laplace operator for some of Riemannian manifolds.

Example 4.1. Let (M^n, g_0) be an Einstein manifold i.e. there exists a constant a such that $Ric(g_0)=ag_0$. Assume that we have a solution to the Yamabe flow which is of the form

$$g(t)=u(t)g_0,\quad u(0)=1$$

where $u(t)$ is a positive function. We compute

$$\frac{\partial g}{\partial t}=u'(t)g_0,\;\; Ric(g(t))=Ric(g_0)=ag_0=\frac{a}{u(t)}g(t),\;\; R_{g(t)}=\frac{an}{u(t)},$$

for this to be a solution of the Yamabe flow, we require

$$u'(t)g_0=-R_{g(t)}g(t)=-ang_0$$

this shows that $u(t)=-nat+1$. So $g(t)$ is an Einstein metric. Using equation (3.21), we obtain the following relation

$$\frac{d}{dt}\lambda(t,f(t))|_{t=t_0} \;=\; \frac{n}{2}\lambda(t_0)\frac{an}{-nat_0+1}\int_M|f|^p d\mu+\frac{p-n}{2}\frac{an}{-nat_0+1}\int_M|\nabla f|^p d\mu,$$

or equivalently

$$\frac{d}{dt}\lambda(t,f(t))|_{t=t_0}=\frac{pan\lambda(t_0)}{2(-nat_0+1)}.$$

This implies that in any sufficiently small neighborhood of t_0 such as I_0, we get

$$\frac{d}{dt}\lambda(t,f(t))=\frac{pan\lambda(t,f(t))}{2(-nat+1)}.$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln \frac{\lambda(t_0, f(t_0))}{\lambda(t_1), f(t_1)} = \ln(\frac{-nat_1 + 1}{-nat_0 + 1})^{\frac{p}{2}}.$$

Since $\lambda(t_0, f(t_0)) = \lambda(t_0)$ and $\lambda(t_1, f(t_1)) \geq \lambda(t_1)$ we conclude that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} > \ln(\frac{-nat_1 + 1}{-nat_0 + 1})^{\frac{p}{2}},$$

that is, the quantity $\lambda(t)(-nat + 1)^{\frac{p}{2}}$ is strictly increasing along the flow (1.1) on $[0, T)$.

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