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Some Generating Relations of Extended Mittag-Leffler Functions

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Abstract. Motivated by the results on generating functions investigated by H. Exton and many other authors, we derive certain (presumably) new generating functions for generalized Mittag-Leffler-type functions. Specifically, we introduce a new class of generating relations (which are partly bilateral and partly unilateral) involving the generalized Mittag-Leffler function. Also we present some special cases of our main result.

1. Introduction

In 1903, the Swedish mathematician Gosta Mittag-Leffler [7] introduced the function

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.1}$$

where z is a complex variable, Γ is the Gamma function and $\alpha \geq 0$.

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The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces when $\alpha = 1$. For $0 < \alpha < 1$, it interpolates between the pure exponential and the hypergeometric function $1 - z^{-1}$. Its importance has been realized during the last two decades due to its involvement in problems of physics, chemistry, biology, engineering and applied sciences. The Mittag-Leffler function naturally occurs as the solution of fractional-order differential and integral equations.

Wiman [17] introduced a new generalization of $E_\alpha(z)$ as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{1.2}$$

$$(\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0),$$

which is known as the Wiman function. Properties of the Wiman function $E_{\alpha,\beta}(z)$ and the Mittag-leffler function $E_\alpha(z)$ are very similar (see, [17], [1]). These Mittag-Leffler functions (which are given in (1.1) and (1.2), respectively) are entire functions of order $p = 1/\alpha$. These functions play a very important role in the solution of differential equations of fractional order.

Prabhakar [9] introduced a further generalization of $E_{\alpha,\beta}(z)$ as follows:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$, and $(\gamma)_n$ is the widely used Pochhammer symbol defined by

$$(\gamma)_n := \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1 & (n = 0; \gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma+1)\ldots(\gamma+n-1) & (n \in \mathbb{N}; \gamma \in \mathbb{C}). \end{cases} \tag{1.4}$$

For $\gamma = 1$, (1.3) is easily seen to reduce to (1.2).

The function $E^{\gamma}_{\alpha,\beta}(z)$ is the most natural generalization of the exponential function $\exp(z)$, the Mittag-Leffler function $E_\alpha(z)$ and Wiman's function $E_{\alpha,\beta}(z)$. Kilbas and Saigo [12] and Raina [10] have investigated several properties and applications of these functions.

In a sequel to the above-mentioned works, Shukla and Prajapati [15] defined the following generalization of the Mittag-Leffler function:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{1.5}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \text{ and } \delta \in (0,1)U\mathbb{N}),$$

where $(\gamma)_{\delta n} = \dfrac{\Gamma(\gamma+\delta n)}{\Gamma(\gamma)}$, is the generalized Pochhammer symbol.

In 2009 and 2012, Salim [13] and Salim and Faraj [14] introduced further generalizations of the preceding functions which are given as follows:

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n} \tag{1.6}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \text{ and } \Re(\delta) > 0)$$

and

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} \tag{1.7}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0, \text{ and } q \leq \Re(\alpha) + p).$$

Subsequently, Khan and Ahmad [6] defined the following two interesting generalizations of these functions and investigated their associated properties:

$$E_{\alpha,\beta,\delta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_n} \tag{1.8}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \text{ and } \Re(\delta) > 0, \text{ and } q \in (0,1)U\mathbb{N})$$

and

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}} \tag{1.9}$$

where $(\alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}; p, q > 0; q \leq \Re(\alpha) + p, \text{ and } \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\mu), \Re(\nu), \Re(\rho), \Re(\sigma)\} > 0)$.

2. Generating Relation

An interesting result on generating functions was given by H. Exton [3, p.147(3)]. The modified form of his result due to Pathan and Yasmeen [8] is

$$\exp\left(s + t - \frac{xt}{s}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n F_n^m(x), \tag{2.1}$$

where

$$F_n^m(x) = {}_1F_1(-n; m+1; x)/m!n! = L_n^m(x)/(m+n)!, \tag{2.2}$$

$L_n^m(x)$ denotes the classical Laguerre polynomials (see [2], [12]) and

$$m^* = \max(0, -m), (m = 0, 1, 2, \dots). \tag{2.3}$$

Main result. *Motivated by Exton's result, we derive the following generating relation for the generalized Mittag-Leffler function* (1.9) *given as:*

$$E_{\alpha_1,\beta_1,\nu_1,\sigma_1,\delta_1,p_1}^{\mu_1,\rho_1,\gamma_1,q_1}(s)E_{\alpha_2,\beta_2,\nu_2,\sigma_2,\delta_2,p_2}^{\mu_2,\rho_2,\gamma_2,q_2}(t)E_{\alpha_3,\beta_3,\nu_3,\sigma_3,\delta_3,p_3}^{\mu_3,\rho_3,\gamma_3,q_3}\left(\frac{-xt}{s}\right)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\ \{_{\alpha_i,\beta_i,\nu_i,\sigma_i,\delta_i,p_i}^{\mu_i,\rho_i,\gamma_i,q_i}F_n^m(x)\}, \tag{2.4}$$

where

$$\{_{\alpha_i,\beta_i,\nu_i,\sigma_i,\delta_i,p_i}^{\mu_i,\rho_i,\gamma_i,q_i}F_n^m(x)\}$$
$$= \sum_{k=0}^{\infty}\frac{(\mu_1)_{\rho_1(m+k)}(\gamma_1)_{q_1(m+k)}(\mu_2)_{\rho_2(n-k)}(\gamma_2)_{q_2(n-k)}(\mu_3)_{\rho_3 k}(\gamma_3)_{q_3 k}}{(\beta_1)_{\alpha_1(m+k)}(\nu_1)_{\sigma_1(m+k)}(\delta_1)_{p_1(m+k)}(\beta_2)_{\alpha_2(n-k)}(\nu_2)_{\sigma_2(n-k)}(\delta_2)_{p_2(n-k)}}$$

$$\times\frac{(-x)^k}{(\beta_3)_{\alpha_3 k}(\nu_3)_{\sigma_3 k}(\delta_3)_{p_3 k}}. \tag{2.5}$$

Proof. On expanding the function

$$M(x,s,t) = E_{\alpha_1,\beta_1,\nu_1,\sigma_1,\delta_1,p_1}^{\mu_1,\rho_1,\gamma_1,q_1}(s)E_{\alpha_2,\beta_2,\nu_2,\sigma_2,\delta_2,p_2}^{\mu_2,\rho_2,\gamma_2,q_2}(t)E_{\alpha_3,\beta_3,\nu_3,\sigma_3,\delta_3,p_3}^{\mu_3,\rho_3,\gamma_3,q_3}\left(\frac{-xt}{s}\right),$$

in series form, we obtain

$$M(x,s,t) = \sum_{k=0}^{\infty}\frac{(\mu_3)_{\rho_3 k}(\gamma_3)_{q_3 k}(-x)^k}{\Gamma(\alpha_3 k+\beta_3)(\nu_3)_{\sigma_3 k}(\delta_3)_{p_3 k}}\sum_{i=0}^{\infty}\frac{(\mu_1)_{\rho_1 i}(\gamma_1)_{q_1 i}s^{(i-k)}}{\Gamma(\alpha_1 i+\beta_1)(\nu_1)_{\sigma_1 i}(\delta_1)_{p_1 i}}$$

$$\times\sum_{j=0}^{\infty}\frac{(\mu_2)_{\rho_2 j}(\gamma_2)_{q_2 j}t^{(j+k)}}{\Gamma(\alpha_2 j+\beta_2)(\nu_2)_{\sigma_2 j}(\delta_2)_{p_2 j}}. \tag{2.6}$$

Replacement of $i-k$ and $j+k$ by m and n respectively, followed by rearrangement justified by the absolute convergence of the above series, then leads to

$$M(x,s,t) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}$$

$$\times \sum_{k=0}^{\infty} \frac{(\mu_1)_{\rho_1(m+k)}(\gamma_1)_{q_1(m+k)}(\mu_2)_{\rho_2(n-k)}(\gamma_2)_{q_2(n-k)}(\mu_3)_{\rho_3 k}(\gamma_3)_{q_3 k}}{(\beta_1)_{\alpha_1(m+k)}(\nu_1)_{\sigma_1(m+k)}(\delta_1)_{p_1(m+k)}(\beta_2)_{\alpha_2(n-k)}(\nu_2)_{\sigma_2(n-k)}(\delta_2)_{p_2(n-k)}}$$

$$\times \frac{(-x)^k}{(\beta_3)_{\alpha_3 k}(\nu_3)_{\sigma_3 k}(\delta_3)_{p_3 k}}, \tag{2.7}$$

which is our required result. □

3. Special Cases

(1) On setting $\mu_i = \nu_i$ and $\rho_i = \sigma_i$ (where $i = 1, 2, 3$) in (2.4), we get the following result for the Mittag-Leffler function defined by Salim and Faraj [14]:

$$\begin{aligned} &E^{\mu_1,\rho_1,\gamma_1,q_1}_{\alpha_1,\beta_1,\mu_1,\rho_1,\delta_1,p_1}(s)E^{\mu_2,\rho_2,\gamma_2,q_2}_{\alpha_2,\beta_2,\mu_2,\rho_2,\delta_2,p_2}(t)E^{\mu_3,\rho_3,\gamma_3,q_3}_{\alpha_3,\beta_3,\mu_3,\rho_3,\delta_3,p_3}\left(\frac{-xt}{s}\right) \\ &= E^{\gamma_1,\delta_1,q_1}_{\alpha_1,\beta_1,p_1}(s)E^{\gamma_2,\delta_2,q_2}_{\alpha_2,\beta_2,p_2}(t)E^{\gamma_3,\delta_3,q_3}_{\alpha_3,\beta_3,p_3}\left(\frac{-xt}{s}\right) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \; \{{}^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,p_i}F^m_n(x)\}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &\{{}^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,p_i}F^m_n(x)\} \\ &= \sum_{k=0}^{\infty} \frac{(\gamma_1)_{q_1(m+k)}(\gamma_2)_{q_2(n-k)}(\gamma_3)_{q_3 k}(-x)^k}{(\beta_1)_{\alpha_1(m+k)}(\delta_1)_{p_1(m+k)}(\beta_2)_{\alpha_2(n-k)}(\delta_2)_{p_2(n-k)}(\beta_3)_{\alpha_3 k}(\delta_3)_{p_3 k}}. \end{aligned}$$

(2) On setting $\mu_i = \nu_i$, $\rho_i = \sigma_i$ and $p_i = 1$ (where $i = 1, 2, 3$) in (2.4), we get the following interesting result:

$$\begin{aligned} &E^{\mu_1,\rho_1,\gamma_1,q_1}_{\alpha_1,\beta_1,\mu_1,\rho_1,\delta_1,1}(s)E^{\mu_2,\rho_2,\gamma_2,q_2}_{\alpha_2,\beta_2,\mu_2,\rho_2,\delta_2,1}(t)E^{\mu_3,\rho_3,\gamma_3,q_3}_{\alpha_3,\beta_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \; \{{}^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,1}F^m_n(x)\}, \end{aligned} \tag{3.2}$$

where

$$\{{}^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,1}F^m_n(x)\} = \sum_{k=0}^{\infty} \frac{(\gamma_1)_{q_1(m+k)}(\gamma_2)_{q_2(n-k)}(\gamma_3)_{q_3 k}(-x)^k}{(\beta_1)_{\alpha_1(m+k)}(\delta_1)_{(m+k)}(\beta_2)_{\alpha_2(n-k)}(\delta_2)_{(n-k)}(\beta_3)_{\alpha_3 k}(\delta_3)_k}.$$

(3) On setting $\mu_i = \nu_i$, $\rho_i = \sigma_i$ and $q_i = p_i = 1$ (where $i = 1, 2, 3$) in (2.4), we get the following known result for the Mittag-Leffler function defined by Salim in [13]:

$$E^{\mu_1,\rho_1,\gamma_1,1}_{\alpha_1,\beta_1,\mu_1,\rho_1,\delta_1,1}(s)E^{\mu_2,\rho_2,\gamma_2,1}_{\alpha_2,\beta_2,\mu_2,\rho_2,\delta_2,1}(t)E^{\mu_3,\rho_3,\gamma_3,1}_{\alpha_3,\beta_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right)$$
$$= E^{\gamma_1,\delta_1}_{\alpha_1,\beta_1}(s)E^{\gamma_2,\delta_2}_{\alpha_2,\beta_2}(t)E^{\gamma_3,\delta_3}_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\ \{^{\mu_i,\rho_i,\gamma_i,1}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,1}F^m_n(x)\}, \tag{3.3}$$

where

$$\{^{\mu_i,\rho_i,\gamma_i,1}_{\alpha_i,\beta_i,\mu_i,\rho_i,\delta_i,1}F^m_n(x)\} = \sum_{k=0}^{\infty}\frac{(\gamma_1)_{(m+k)}(\gamma_2)_{(n-k)}(\gamma_3)_k(-x)^k}{(\beta_1)_{\alpha_1(m+k)}(\delta_1)_{(m+k)}(\beta_2)_{\alpha_2(n-k)}(\delta_2)_{(n-k)}(\beta_3)_{\alpha_3 k}(\delta_3)_k}.$$

(4) For $\mu_i = \nu_i$, $\rho_i = \sigma_i$ and $\delta_i = p_i = 1$ (where $i = 1, 2, 3$), (2.4) reduces to the following known result of Kamarujjama and Khan [5] for the Mittag-Leffler function defined by Shukla and Prajapati [15]:

$$E^{\mu_1,\rho_1,\gamma_1,q_1}_{\alpha_1,\beta_1,\mu_1,\rho_1,1,1}(s)E^{\mu_2,\rho_2,\gamma_2,q_2}_{\alpha_2,\beta_2,\mu_2,\rho_2,1,1}(t)E^{\mu_3,\rho_3,\gamma_3,q_3}_{\alpha_3,\beta_3,\mu_3,\rho_3,1,1}\left(\frac{-xt}{s}\right)$$
$$= E^{\gamma_1,q_1}_{\alpha_1,\beta_1}(s)E^{\gamma_2,q_2}_{\alpha_2,\beta_2}(t)E^{\gamma_3,q_3}_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\ \{^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,1,1}F^m_n(x)\}, \tag{3.4}$$

where

$$\{^{\mu_i,\rho_i,\gamma_i,q_i}_{\alpha_i,\beta_i,\mu_i,\rho_i,1,1}F^m_n(x)\} = \sum_{k=0}^{\infty}\frac{(\gamma_1)_{q_1(m+k)}(\gamma_2)_{q_2(n-k)}(\gamma_3)_{q_3 k}(-x)^k}{(1+m)_k(\beta_1)_{\alpha_1(m+k)}(\beta_2)_{\alpha_2(n-k)}(\beta_3)_{\alpha_3 k}k!}.$$

(5) On setting $\rho_i = q_i = \sigma_i = \alpha_i = pi = 1$, $\mu_3 = \beta_3$ and $\gamma_3 = \delta_3$ (where $i = 1, 2, 3$) in (2.4), we get the following interesting result:

$$E^{\mu_1,1,\gamma_1,1}_{1,\beta_1,\nu_1,1,\delta_1,1}(s)E^{\mu_2,1,\gamma_2,1}_{1,\beta_2,\nu_2,1,\delta_2,1}(t)E^{\mu_3,1,\gamma_3,1}_{1,\mu_3,\nu_3,1,\gamma_3,1}\left(\frac{-xt}{s}\right)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n(\mu_1)_m(\gamma_1)_m(\mu_2)_n(\gamma_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\beta_1)_m(\nu_1)_m(\delta_1)_m(\beta_2)_n(\nu_2)_n(\delta_2)_n}$$
$$\times {}_6F_6\left[\begin{array}{l}\mu_1+m, \gamma_1+m, 1-\beta_2-n, 1-\nu_2-n; 1-\delta_2-n, 1;\\ \beta_1+m, \nu_1+m, 1-\mu_2-n, 1-\gamma_2-n; \delta_1+m, \nu_3;\end{array}\ x\right], \tag{3.5}$$

where ${}_pF_q$ is the *generalized hypergeometric function* defined by (see, [11], [16])

$$ {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}. $$

(6) On setting $\rho_1 = q_1 = \rho_3 = q_3 = \alpha_1 = \sigma_1 = p_1 = \alpha_3 = \sigma_3 = p_3 = 2$, $\rho_2 = q_2 = \alpha_2 = \sigma_2 = p_2 = 1$, $\mu_3 = \beta_3$ and $\gamma_3 = \delta_3$ in (2.4), we get the following result:

$$ E^{\mu_1,2,\gamma_1,2}_{2,\beta_1,\nu_1,1,\delta_1,2}(s) E^{\mu_2,1,\gamma_2,1}_{1,\beta_2,\nu_2,1,\delta_2,1}(t) E^{\mu_3,2,\gamma_3,2}_{2,\mu_3,\nu_3,2,\gamma_3,2}\left(\frac{-xt}{s}\right) $$
$$ = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(s/4)^m t^n (\frac{\mu_1}{2})_m (\frac{\mu_1+1}{2})_m (\frac{\gamma_1}{2})_m (\frac{\gamma_1+1}{2})_m (\mu_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\frac{\beta_1}{2})_m (\frac{\beta_1+1}{2})_m (\frac{\nu_1}{2})_m (\frac{\nu_1+1}{2})_m (\frac{\delta_1}{2})_m (\frac{\delta_1+1}{2})_m (\beta_2)_n (\nu_2)_n (\delta_2)_n} $$
$$ \times {}_8F_9\left[\begin{array}{c} \frac{\mu_1+m}{2}, \frac{\mu_1+1+m}{2}, \frac{\gamma_1+m}{2}, \frac{\gamma_1+1+m}{2}, 1-\beta_2-n, 1-\nu_2-n; 1-\gamma_2-n, 1; \\ \frac{\beta_1+m}{2}, \frac{\beta_1+1+m}{2}, \frac{\nu_1+m}{2}, \frac{\nu_1+1+m}{2}, \frac{\delta_1+m}{2}, \frac{\delta_1+1+m}{2}, 1-\mu_2-n, 1-\gamma_2-n, \nu_3; \end{array} x\right]. \tag{3.6} $$

(7) On setting $q_i = \alpha_i = p_i = 1$, $\mu_i = \nu_i, \rho_i = \sigma_i$ and $\gamma_3 = \beta_3$ (where $i = 1, 2, 3$) in (2.4), we get the following result:

$$ E^{\mu_1,\rho_1,\gamma_1,1}_{1,\beta_1,\mu_1,\rho_1,\delta_1,1}(s) E^{\mu_2,\rho_2,\gamma_2,1}_{1,\beta_2,\mu_2,\rho_2,\delta_2,1}(t) E^{\mu_3,\rho_3,\gamma_3,1}_{1,\gamma_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right) $$
$$ = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n (\gamma_1)_m (\gamma_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\beta_1)_m (\delta_1)_m (\beta_2)_n (\delta_2)_n} $$
$$ \times {}_4F_4\left[\begin{array}{c} \gamma_1+m, 1-\beta_2-n, 1-\delta_2-n, 1; \\ \beta_1+m, 1-\gamma_2-n; \delta_1+m, \delta_3; \end{array} x\right]. \tag{3.7} $$

(8) On setting $q_1 = q_3 = \alpha_1 = p_1 = \alpha_3 = 2$, $p_3 = q_2 = \alpha_2 = p_2 = 1$, $\gamma_3 = \beta_3$, $\mu_i = \nu_i$ and $\rho_i = \sigma_i$ (where $i = 1, 2, 3$) in (2.4), we get the following result:

$$ E^{\mu_1,\rho_1,\gamma_1,2}_{2,\beta_1,\mu_1,\rho_1,\delta_1,2}(s) E^{\mu_2,\rho_2,\gamma_2,1}_{1,\beta_2,\mu_2,\rho_2,\delta_2,1}(t) E^{\mu_3,\rho_3,\gamma_3,2}_{2,\gamma_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right) $$
$$ = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(s/4)^m t^n (\frac{\gamma_1}{2})_m (\frac{\gamma_1+1}{2})_m (\gamma_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\frac{\beta_1}{2})_m (\frac{\beta_1+1}{2})_m (\frac{\delta_1}{2})_m (\frac{\delta_1+1}{2})_m (\beta_2)_n (\delta_2)_n} $$
$$ \times {}_5F_6\left[\begin{array}{c} \frac{\gamma_1+m}{2}, \frac{\gamma_1+1+m}{2}, 1-\beta_2-n, 1-\delta_2-n, 1; \\ \frac{\beta_1+m}{2}, \frac{\beta_1+1+m}{2}, \frac{\delta_1+m}{2}, \frac{\delta_1+1+m}{2}, 1-\gamma_2-n, \delta_3; \end{array} \frac{x}{4}\right]. \tag{3.8} $$

(9) On setting $q_1 = q_3 = \alpha_1 = \alpha_3 = 2$, $p_i = q_2 = \alpha_2 = 1$, $\gamma_3 = \beta_3$, $\mu_i = \nu_i$ and $\rho_i = \sigma_i$ (where $i = 1, 2, 3$) in (2.4), we get the following result:

$$\begin{aligned}
&E^{\mu_1,\rho_1,\gamma_1,2}_{2,\beta_1,\mu_1,\rho_1,\delta_1,1}(s)E^{\mu_2,\rho_2,\gamma_2,1}_{1,\beta_2,\mu_2,\rho_2,\delta_2,1}(t)E^{\mu_3,\rho_3,\gamma_3,2}_{2,\gamma_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(s)^m t^n(\frac{\gamma_1}{2})_m(\frac{\gamma_1+1}{2})_m(\gamma_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\frac{\beta_1}{2})_m(\frac{\beta_1+1}{2})_m(\delta_1)_m(\beta_2)_n(\delta_2)_n}\\
&\times {}_5F_5\left[\begin{array}{c}\frac{\gamma_1+m}{2},\frac{\gamma_1+1+m}{2},1-\beta_2-n,1-\delta_2-n,1;\\ \frac{\beta_1+m}{2},\frac{\beta_1+1+m}{2},1-\delta_2-n,\delta_1+m,\delta_3;\end{array} x\right]. \qquad (3.9)
\end{aligned}$$

(10) On setting $\alpha_1 = 2$, $p_i = q_i = \alpha_2 = \alpha_3 = 1$, $\gamma_3 = \beta_3$, $\mu_i = \nu_i$ and $\rho_i = \sigma_i$ (where $i = 1, 2, 3$) in (2.4), we get the following result:

$$\begin{aligned}
&E^{\mu_1,\rho_1,\gamma_1,1}_{2,\beta_1,\mu_1,\rho_1,\delta_1,1}(s)E^{\mu_2,\rho_2,\gamma_2,1}_{1,\beta_2,\mu_2,\rho_2,\delta_2,1}(t)E^{\mu_3,\rho_3,\gamma_3,1}_{1,\gamma_3,\mu_3,\rho_3,\delta_3,1}\left(\frac{-xt}{s}\right)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(s/4)^m t^n(\gamma_1)_m(\gamma_2)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(\frac{\beta_1}{2})_m(\frac{\beta_1+1}{2})_m(\delta_1)_m(\beta_2)_n(\delta_2)_n}\\
&\times {}_4F_5\left[\begin{array}{c}\gamma_1+m,1-\beta_2-n,1-\delta_2-n,1;\\ \frac{\beta_1+m}{2},\frac{\beta_1+1+m}{2},1-\gamma_2-n,\delta_1+m,\delta_3;\end{array} \frac{x}{4}\right]. \qquad (3.10)
\end{aligned}$$

(11) On setting $\rho_i = q_i = \sigma_i = \alpha_i = p_i = \mu_i = \nu_i = \beta_i = \gamma_i = \delta_i = 1$ (where $i = 1, 2, 3$) in (2.4), we get the following known result due to Exton [3]:

$$\begin{aligned}
E^{1,1,1,1}_{1,1,1,1,1,1}(s)E^{1,1,1,1}_{1,1,1,1,1,1}(t)E^{1,1,1,1}_{1,1,1,1,1,1}\left(\frac{-xt}{s}\right) &= \exp\left(s+t-\frac{xt}{s}\right)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{n!m!}{}_1F_1(-n;m+1;x). \qquad (3.11)
\end{aligned}$$

4. Concluding Remark

In our present investigation, we have studied a number of generating functions for the extended Mittag-Leffler-type functions given in [4, 6, 13, 14, 15]. The main generating function is the further generalization of the result given by Kamarujjama and Khan [5]. The results of this paper, especially (2.4), are of a general nature in the literature on generating functions.

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