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Pascal Distribution Series Connected with Certain Subclasses of Univalent Functions

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Abstract. The aim of this article is to make a connection between the Pascal distribution series and some subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution. For these functions, for linear combinations of these functions and their derivatives, for operators defined by convolution products, and for the Alexander-type integral operator, we find simple sufficient conditions such that these mapping belong to a general class of functions defined and studied by Goodman, Rønning, and Bharati et al.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}, \tag{1.1}$$

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and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{U}$.

Furthermore, for $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ z \in \mathbb{U}$$

the *Hadamard (or convolution) product* of the functions f and g is given by (see [5])

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in \mathbb{U}.$$

If f and g are analytic functions in $\mathbb{U}$, we say that f is *subordinate* to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [3] and [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Goodman in [6, 7], Rønning in [15, 16], and Bharati et al. in [2] introduced and studied the following subclasses:

(i) A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{S}_p(\beta)$ of *k-uniformly starlike functions of order* β if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{U}, \tag{1.2}$$

where $0 \le \beta < 1$ and $k \ge 0$.

(ii) A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{UCV}(\beta)$ of *k-uniformly convex functions of order* β if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \beta\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathbb{U}, \tag{1.3}$$

where $0 \le \beta < 1$ and $k \ge 0$.

It follows from (1.2) and (1.3) that for a function $f \in \mathcal{A}$ we have the equivalence

$$f \in k - \mathcal{UCV}(\beta) \Leftrightarrow zf'(z) \in k - \mathcal{S}_p(\beta).$$

For $\beta = 0$ the classes $k-\ \mathcal{UCV}(\beta)$ and $k - \mathcal{S}_p(\beta)$ reduce to the classes $k-\ \mathcal{UCV}$ and $k-\mathcal{S}_p$ respectively, that have been introduced and studied by Kanas and Wisniowska [10, 11].

Definition 1.1.([18]) For $-1 \le A < B \le 1$ and $|\theta| < \frac{\pi}{2}$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{L}(A, B, \theta)$ if it satisfies the subordination condition

$$e^{i\theta} f'(z) \prec \frac{1 + Az}{1 + Bz} \cos\theta + i \sin\theta.$$

Using the definition of the subordination, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{L}(A, B, \theta)$ if and only if there exists an analytic function w, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that

$$e^{i\theta} f'(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \cos\theta + i \sin\theta, \ z \in \mathbb{U},$$

or equivalently

$$\left| \frac{e^{i\theta}\left(f'(z) - 1\right)}{Be^{i\theta} f'(z) - \left[Be^{i\theta} + (A - B)\cos\theta\right]} \right| < 1, \ z \in \mathbb{U}.$$

We note that for suitable choices of A, B and θ we obtain the following subclasses defined and studied by different authors:

(i) $\mathcal{L}\left(2\alpha - 1, 1, \theta\right) =: \mathcal{L}\left(\theta; \alpha\right)$ $\left(0 \le \alpha < 1, \ |\theta| < \frac{\pi}{2}\right)$ (see Kanas and Srivastava [9]);

(ii) $\mathcal{L}\left(\beta(2\alpha - 1), \beta, 0\right) =: \mathcal{R}\left(\beta; \alpha\right)$ $(0 < \beta \le 1, \ 0 \le \alpha < 1)$ (see Juneja and Mogra [8]);

(iii) $\mathcal{L}\left(\beta(2\alpha - 1), \beta, 0\right) =: \mathcal{D}\left(\beta\right)$ $(0 < \beta \le 1)$ (see Caplinger and Causey [4]);

(iv) $\mathcal{L}\left(A + (B - A)\alpha, B, \theta\right) =: \mathcal{L}\left(A, B, \theta; \alpha\right)$ $\left(-1 \le A < B \le 1, \ |\theta| < \frac{\pi}{2}, \ 0 \le \alpha < 1\right)$ (see Aouf [1]).

A variable x is said to have the *Pascal distribution* if it takes the values 0, 1, 2, 3, ... with the probabilities $(1-q)^r$, $\frac{qr(1-q)^r}{1!}$, $\frac{q^2 r(r+1)(1-q)^r}{2!}$, $\frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}$, ..., respectively, where q, r are called the parameters, and thus

$$P(X = k) = \binom{k + r - 1}{r - 1} q^k (1 - q)^r, \ k \in \{0, 1, 2, 3, \dots\}.$$

Now we introduced a power series whose coefficients are probabilities of the Pascal distribution, that is

$$\mathrm{Q}_q^r(z) := z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} q^{n-1} (1 - q)^r z^n, \ z \in \mathbb{U} \quad (r \ge 1, \ 0 \le q \le 1). \tag{1.4}$$

Note that, by using ratio test we deduce that the radius of convergence of the above power series is infinity.

We will define the functions

$$\begin{aligned} \mathrm{N}_{q,\lambda}^r(z) &:= (1 - \lambda)\mathrm{Q}_q^r(z) + \lambda z \left(\mathrm{Q}_q^r(z)\right)' \\ &= z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} \left[1 + \lambda(n - 1)\right] q^{n-1} (1 - q)^r z^n, \ z \in \mathbb{U}, \\ &\quad (r \ge 1, \ 0 \le q \le 1, \ \lambda \ge 0) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\mathrm{M}^r_{q,\lambda,\gamma}(z) &:= (1-\lambda+\gamma)\mathrm{Q}^r_q(z)+(\lambda-\gamma)\,z\left(Q^r_q(z)\right)'+\lambda\gamma z^2\left(Q^r_q(z)\right)'' \\ &= z+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left[1+(n-1)(\lambda-\gamma+n\lambda\gamma)\right]q^{n-1}(1-q)^r z^n,\ z\in\mathbb{U}, \\ &\quad (r\geq 1,\ 0\leq q\leq 1,\ \lambda,\gamma\geq 0)\,,\end{aligned} \tag{1.6}$$

and we introduce the linear operator $\mathrm{P}^r_q:\mathcal{A}\to\mathcal{A}$ defined by

$$\mathrm{P}^r_q(f)(z):=\mathrm{Q}^r_q(z)*f(z)=z+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}(1-q)^r a_n z^n,\ z\in\mathbb{U}, \quad (r\geq 1,\ 0\leq q\leq 1)\,. \tag{1.7}$$

In the present paper we established some sufficient conditions for the Pascal distribution series and other related series to be in some subclasses of analytic functions. Also, we studied similar properties for an integral transform related to this series.

2. Preliminaries

Unless otherwise mentioned, we assume that $-1\leq A<B\leq 1$, $|\theta|<\dfrac{\pi}{2}$, $r\geq 1$, $0\leq q\leq 1$, $\lambda,\gamma\geq 0$ and $\lambda\geq\gamma$. To prove our results we need the following lemmas.

Lemma 2.1.([1, Theorem 4 for $p=1$ and $\alpha=0$]) *If the function $f\in\mathcal{A}$ is of the form* (1.1) *and*

$$\sum_{n=2}^{\infty}n(1+|B|)\,|a_n|\leq(B-A)\cos\theta,$$

then $f\in\mathcal{L}(A,B,\theta)$.

Lemma 2.2.([1, Theorem 1 for $p=1$ and $\alpha=0$]) *If the function $f\in\mathcal{L}(A,B,\theta)$ is of the form* (1.1)*, then*

$$|a_n|\leq\frac{(B-A)\cos\theta}{n}$$

for every $n\geq 2$. The estimate is sharp.

Lemma 2.3.[10, Theorem 3.3] *If $f\in\mathcal{A}$, and for some k $(0\leq k<\infty)$ the following inequality holds*

$$\sum_{n=2}^{\infty}n(n-1)\,|a_n|\leq\frac{1}{k+2},$$

then $f\in k-\mathcal{UCV}$. The number $\dfrac{1}{k+2}$ cannot be increased.

Lemma 2.4.([17, Theorem 2.1]) *If the function $f \in \mathcal{A}$ is of the form* (1.1) *and*

$$\sum_{n=2}^{\infty} [n(1+k) - (k+\beta)] \, |a_n| \le 1 - \beta,$$

then $f \in k - \mathcal{S}_p(\beta)$

3. Main Results

Theorem 3.1. *A sufficient condition for the function Q_q^r given by* (1.4) *to be in the class $\mathcal{L}(A, B, \theta)$ is*

$$\frac{qr}{1-q} - (1-q)^r + 1 \le \frac{(B-A)\cos\theta}{1+|B|}. \tag{3.1}$$

Proof. To prove that $\mathrm{Q}_q^r \in \mathcal{L}(A, B, \theta)$, according to Lemma 2.1 it is sufficient to show that

$$\sum_{n=2}^{\infty} \binom{n+r-2}{r-1} n(1+|B|)|q|^{n-1}|1-q|^r \le (B-A)\cos\theta.$$

In the proofs of all our results we will use the relation

$$\sum_{n=0}^{\infty} \binom{n+r-1}{r-1} q^n = \frac{1}{(1-q)^r}, \ 0 \le q \le 1,$$

and the corresponding ones obtained by replacing the value of r with $r-1$, $r+1$, and $r+2$.

Thus, from the definition relation (1.4) combined with the assumption (3.1) we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} \binom{n+r-2}{r-1} n(1+|B|)q^{n-1}(1-q)^r \\
&= (1+|B|)(1-q)^r \left[\sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (n-1)q^{n-1} + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} q^{n-1} \right] \\
&= (1+|B|)(1-q)^r \left[\sum_{n=2}^{\infty} qr \binom{n+r-2}{r} q^{n-2} + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} q^{n-1} \right] \\
&= (1+|B|)(1-q)^r \left[qr \sum_{n=0}^{\infty} \binom{n+r}{r} q^n + \sum_{n=1}^{\infty} \binom{n+r-1}{r-1} q^n \right] \\
&= (1+|B|)(1-q)^r \left[qr \sum_{n=0}^{\infty} \binom{n+r}{r} q^n + \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} q^n - 1 \right] \\
&= (1+|B|) \left[\frac{qr}{1-q} + 1 - (1-q)^r \right] \le (B-A)\cos\theta,
\end{aligned}$$

and from Lemma 2.1 it follows that $\mathrm{Q}_q^r \in \mathcal{L}(A,B,\theta)$. □

Theorem 3.2. *A sufficient condition for the function* $\mathrm{N}_{q,\lambda}^r$ *defined by* (1.5) *to be in the class* $\mathcal{L}(A,B,\theta)$ *is*

$$\frac{(1+2\lambda)qr}{1-q}+1-(1-q)^r+\frac{\lambda q^2 r(r+1)}{(1-q)^2}\leq\frac{(B-A)\cos\theta}{1+|B|}. \tag{3.2}$$

Proof. From Lemma 2.1, to prove that $\mathrm{N}_{q,\lambda}^r \in \mathcal{L}(A,B,\theta)$ it is sufficient to show that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left[n(1+|B|)\right]\left[1+\lambda(n-1)\right]|q|^{n-1}|1-q|^r\leq(B-A)\cos\theta. \tag{3.3}$$

Thus,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left[n\left(1+|B|\right)\right]\left[1+\lambda(n-1)\right]q^{n-1}(1-q)^r\\
&=(1+|B|)(1-q)^r\left[\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(1+2\lambda)(n-1)q^{n-1}\right.\\
&\qquad\left.+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\lambda(n-1)(n-2)q^{n-1}\right]\\
&=(1+|B|)(1-q)^r\left[\sum_{n=2}^{\infty}qr(1+2\lambda)\binom{n+r-2}{r}q^{n-2}\right.\\
&\qquad\left.+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}+\lambda q^2\sum_{n=3}^{\infty}\binom{n+r-2}{r-1}(n-1)(n-2)q^{n-3}\right]\\
&=(1+|B|)(1-q)^r\left[(1+2\lambda)qr\sum_{n=0}^{\infty}\binom{n+r}{r}q^n+\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}q^n-1\right.\\
&\qquad\left.+\lambda q^2 r(r+1)\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}q^n\right]\\
&=(1+|B|)(1-q)^r\left[(1+2\lambda)qr\frac{1}{(1-q)^{r+1}}+\frac{1}{(1-q)^r}-1\right.\\
&\qquad\left.+\lambda q^2 r(r+1)\frac{1}{(1-q)^{r+2}}\right]\\
&=(1+|B|)\left[\frac{(1+2\lambda)qr}{1-q}+1-(1-q)^r+\frac{\lambda q^2 r(r+1)}{(1-q)^2}\right].
\end{aligned}$$

According to (3.2), from the above identity we conclude it follows that the inequality (3.3) holds, and therefore $\mathrm{N}_{q,\lambda}^r \in \mathcal{L}(A,B,\theta)$. □

Theorem 3.3. *A sufficient condition for the function* $\mathrm{M}^r_{q,\lambda,\gamma}$ *given by* (1.6) *to be in the class* $\mathcal{L}(A,B,\theta)$ *is*

$$\frac{(1+2\lambda-2\gamma+4\lambda\gamma)\,qr}{1-q}+1-(1-q)^r+\frac{(\lambda-\gamma+5\lambda\gamma)\,q^2r(r+1)}{(1-q)^2}+\frac{\lambda\gamma q^3r(r+1)(r+2)}{(1-q)^3}\leq\frac{(B-A)\cos\theta}{1+|B|}. \tag{3.4}$$

Proof. To prove that $\mathrm{M}^r_{q,\lambda,\gamma}\in\mathcal{L}(A,B,\theta)$, from Lemma 2.1 it is sufficient to show that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}n(1+|B|)\left[1+(n-1)(\lambda-\gamma+n\lambda\gamma)\right]|q|^{n-1}|1-q|^r\leq(B-A)\cos\theta. \tag{3.5}$$

From (3.4) it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}n(1+|B|)\left[1+(n-1)\left(\lambda-\gamma+n\lambda\gamma\right)\right]q^{n-1}(1-q)^r\\
&=(1+|B|)(1-q)^r\left[\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(1+2\lambda-2\gamma+4\lambda\gamma)(n-1)q^{n-1}\right.\\
&+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(\lambda-\gamma+5\lambda\gamma)(n-1)(n-2)q^{n-1}\\
&\qquad\left.+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\lambda\gamma(n-1)(n-2)(n-3)q^{n-1}\right]\\
&=(1+|B|)(1-q)^r\left[\sum_{n=2}^{\infty}qr(1+2\lambda-2\gamma+4\lambda\gamma)\binom{n+r-2}{r}q^{n-2}\right.\\
&+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}+\sum_{n=3}^{\infty}(\lambda-\gamma+5\lambda\gamma)\,q^2\binom{n+r-2}{r-1}(n-1)(n-2)q^{n-3}\\
&\qquad\left.+\sum_{n=4}^{\infty}\lambda\gamma q^3\binom{n+r-2}{r-1}q^{n-4}(n-1)(n-2)(n-3)\right]\\
&=(1+|B|)(1-q)^r\left[(1+2\lambda-2\gamma+4\lambda\gamma)qr\sum_{n=0}^{\infty}\binom{n+r}{r}q^n\right.\\
&\quad+\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}q^n-1+(\lambda-\gamma+5\lambda\gamma)q^2r(r+1)\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}q^n\\
&\qquad\left.+\lambda\gamma q^3r(r+1)(r+2)\sum_{n=0}^{\infty}\binom{n+r+2}{r+2}q^n\right]
\end{aligned}$$

$$= (1+|B|)(1-q)^r \Bigg[(1+2\lambda-2\gamma+4\lambda\gamma)\, qr\frac{1}{(1-q)^{r+1}} + \frac{1}{(1-q)^r} - 1$$
$$+ (\lambda-\gamma+5\lambda\gamma)\, q^2 r(r+1)\frac{1}{(1-q)^{r+2}} + \lambda\gamma q^3 r(r+1)(r+2)\frac{1}{(1-q)^{r+3}}\Bigg]$$
$$= (1+|B|)\Bigg[\frac{(1+2\lambda-2\gamma+4\lambda\gamma)\, qr}{1-q} + 1 - (1-q)^r$$
$$+\frac{(\lambda-\gamma+5\lambda\gamma)\, q^2 r(r+1)}{(1-q)^2} + \frac{\lambda\gamma q^3 r(r+1)(r+2)}{(1-q)^3}\Bigg] \le (B-A)\cos\theta,$$

that is (3.5) holds, hence $\mathrm{M}^r_{q,\lambda,\gamma} \in \mathcal{L}(A,B,\theta)$. □

Theorem 3.4.

(i) If the condition

$$\frac{q^2 r(r+1)}{(1-q)^2} + \frac{3qr}{1-q} + 1 - (1-q)^r \le \frac{(B-A)\cos\theta}{1+|B|} \tag{3.6}$$

holds, then the operator P^r_q *defined by* (1.7) *maps the class* $\mathcal{S}^*$ *to the class* $\mathcal{L}(A,B,\theta)$*, that is* $\mathrm{P}^r_q(\mathcal{S}^*) \subset \mathcal{L}(A,B,\theta)$.

(ii) If the condition (3.1) *is satisfied, then the operator* P^r_q *maps the class* $\mathcal{K}$ *to the class* $\mathcal{L}(A,B,\theta)$*, that is* $\mathrm{P}^r_q(\mathcal{K}) \subset \mathcal{L}(A,B,\theta)$.

Proof. According to Lemma 2.1, to prove that $\mathrm{P}^r_q(f) \in \mathcal{L}(A,B,\theta)$ it is sufficient to show that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1} n(1+|B|)|q|^{n-1}|1-q|^r\, |a_n| \le (B-A)\cos\theta. \tag{3.7}$$

(i) If $f \in \mathcal{S}^*$ has the form (1.1), then the well-known inequality $|a_n| \le n$ holds for all $n \ge 2$ (see [12] and [14]), and using (3.6) we obtain that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1} n(1+|B|)q^{n-1}(1-q)^r\, |a_n|$$
$$\le \sum_{n=2}^{\infty}\binom{n+r-2}{r-1} n^2(1+|B|)q^{n-1}(1-q)^r$$
$$= (1+|B|)(1-q)^r\Bigg[\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(n-1)(n-2)q^{n-1}$$
$$+3\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(n-1)q^{n-1} + \sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}\Bigg]$$

$$= (1+|B|)(1-q)^r \left[q^2 \sum_{n=3}^{\infty} \binom{n+r-2}{r-1} (n-1)(n-2) q^{n-3} \right.$$
$$\left. + 3q \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (n-1) q^{n-2} + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} q^{n-1} \right]$$
$$= (1+|B|)(1-q)^r \left[q^2 r(r+1) \sum_{n=0}^{\infty} \binom{n+r+1}{r+1} q^n + 3qr \sum_{n=0}^{\infty} \binom{n+r}{r} q^n \right.$$
$$\left. + \sum_{n=1}^{\infty} \binom{n+r-1}{r-1} q^n \right]$$
$$= (1+|B|)(1-q)^r \left[\frac{q^2 r(r+1)}{(1-q)^{r+2}} + \frac{3qr}{(1-q)^{r+1}} + \frac{1}{(1-q)^r} - 1 \right]$$
$$= (1+|B|) \left[\frac{q^2 r(r+1)}{(1-q)^2} + \frac{3qr}{1-q} + 1 - (1-q)^r \right] \leq (B-A)\cos\theta,$$

that is (3.7) holds, hence $\mathrm{P}_q^r(f) \in \mathcal{L}(A,B,\theta)$.

(ii) If $f \in \mathcal{K}$ is of the form (1.1), then the well-known inequality $|a_n| \leq 1$ holds for all $n \geq 2$ (see [12]), and according to (3.1) we obtain that

$$\sum_{n=2}^{\infty} \binom{n+r-2}{r-1} n(1+|B|) q^{n-1} (1-q)^r |a_n|$$
$$\leq \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} n(1+|B|) q^{n-1} (1-q)^r$$
$$= (1+|B|)(1-q)^r \left[\sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (n-1) q^{n-1} + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} q^{n-1} \right]$$
$$= (1+|B|)(1-q)^r \left[\sum_{n=2}^{\infty} q \binom{n+r-2}{r-1} (n-1) q^{n-2} + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} q^{n-1} \right]$$
$$= (1+|B|)(1-q)^r \left[qr \sum_{n=0}^{\infty} \binom{n+r}{r} q^n + \sum_{n=1}^{\infty} \binom{n+r-1}{r-1} q^n \right]$$
$$= (1+|B|)(1-q)^r \left[qr \sum_{n=0}^{\infty} \binom{n+r}{r} q^n + \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} q^n - 1 \right]$$
$$= (1+|B|) \left[\frac{qr}{1-q} + 1 - (1-q)^r \right] \leq (B-A)\cos\theta,$$

hence (3.7) holds, and therefore $\mathrm{P}_q^r(f) \in \mathcal{L}(A,B,\theta)$. □

Theorem 3.5. *If the condition*

$$(B-A)qr\cos\theta \le \frac{1-q}{k+2} \tag{3.8}$$

holds, then the operator P_q^r *maps the class* $\mathcal{L}(A,B,\theta)$ *to the class* $k-\mathcal{UCV}$*, that is* $\mathrm{P}_q^r\left(\mathcal{L}(A,B,\theta)\right) \subset k-\mathcal{UCV}$*.*

Proof. If $f \in \mathcal{L}(A,B,\theta)$ has the form (1.1), since P_q^r is given by (1.7), using Lemma 2.3 we need to prove that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}n(n-1)|q|^{n-1}|1-q|^r\,|a_n| \le \frac{1}{k+2}. \tag{3.9}$$

From Lemma 2.2 and the assumption (3.8) we obtain that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}n(n-1)q^{n-1}(1-q)^r\,|a_n| \\
&\le \sum_{n=2}^{\infty}\binom{n+r-2}{r-1}(n-1)(B-A)q^{n-1}(1-q)^r\cos\theta \\
&= (B-A)(1-q)^r\cos\theta\sum_{n=2}^{\infty}(n-1)\binom{n+r-2}{r-1}q^{n-1} \\
&= (B-A)(1-q)^r qr\cos\theta\sum_{n=0}^{\infty}\binom{n+r}{r}q^n = \frac{(B-A)qr\cos\theta}{1-q} \le \frac{1}{k+2},
\end{aligned}$$

hence (3.9) holds, and consequently $\mathrm{P}_q^r(f) \in k-\mathcal{UCV}$. □

Theorem 3.6. *If the condition*

$$(B-A)\cos\theta\left[k+1-(1-q)^r-\frac{k(1-q)}{q(r-1)}\left(1-(1-q)^{r-1}\right)\right] \le 1 \tag{3.10}$$

holds, then P_q^r *maps the class* $\mathcal{L}(A,B,\theta)$ *to the class* $k-\mathcal{S}_p := k-\mathcal{S}_p(0)$*, that is* $\mathrm{P}_q^r\left(\mathcal{L}(A,B,\theta)\right) \subset k-\mathcal{S}_p$*.*

Proof. If $f \in \mathcal{L}(A,B,\theta)$ has the power expansion series (1.1), and P_q^r is given by (1.7), according to Lemma 2.4 for $\beta = 0$ we need to prove that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left[n(k+1)-k\right]|q|^{n-1}|1-q|^r\,|a_n| \le 1. \tag{3.11}$$

From Lemma 2.2 and the assumption (3.10) we may easily show that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}[n(k+1)-k]\,|q|^{n-1}|1-q|^{r}\,|a_n| \\
&\le \sum_{n=2}^{\infty}\binom{n+r-2}{r-1}[n(k+1)-k]\,q^{n-1}(1-q)^{r}\frac{(B-A)\cos\theta}{n} \\
&= (B-A)(1-q)^{r}\cos\theta\left[\sum_{n=2}^{\infty}(k+1)\binom{n+r-2}{r-1}q^{n-1}-\sum_{n=2}^{\infty}\frac{k}{n}\binom{n+r-2}{r-1}q^{n-1}\right] \\
&= (B-A)(1-q)^{r}\cos\theta\left[(k+1)\left(\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}q^{n}-1\right)\right. \\
&\qquad\left.-\frac{k}{q(r-1)}\left(\sum_{n=0}^{\infty}\binom{n+r-2}{r-2}q^{n}-1-(r-1)q\right)\right] \\
&= (B-A)\cos\theta\left[(k+1)\left[1-(1-q)^{r}\right]\right. \\
&\qquad\left.-\frac{k}{q(r-1)}\left[(1-q)-(1-q)^{r}-q(r-1)(1-q)^{r}\right]\right] \\
&= (B-A)\cos\theta\left[k+1-(1-q)^{r}-\frac{k(1-q)}{q(r-1)}\left(1-(1-q)^{r-1}\right)\right]\le 1,
\end{aligned}$$

that is (3.11) holds, and thus $\mathrm{P}_q^r(f)\in k-\mathcal{S}_p$. □

In the following two theorems we will obtain analogues results in connection with the function I_q^r defined by

$$\mathrm{I}_q^r(z):=\int_0^z\frac{\mathrm{Q}_q^r(t)}{t}\,dt,\ z\in\mathbb{U}.$$

Theorem 3.7. *A sufficient condition for the function* I_q^r *to be in the class* $\mathcal{L}(A,B,\theta)$ *is*

$$(1+|B|)\left[1-(1-q)^{r}\right]\le(B-A)\cos\theta. \tag{3.12}$$

Proof. Since

$$\mathrm{I}_q^r(z)=z+\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}(1-q)^{r}\frac{z^n}{n},\ z\in\mathbb{U}, \tag{3.13}$$

to prove that $\mathrm{I}_q^r\in\mathcal{L}(A,B,\theta)$, according to Lemma 2.1 it is sufficient to show that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\frac{n(1+|B|)}{n}|q|^{n-1}|1-q|^{r}\le(B-A)\cos\theta. \tag{3.14}$$

Using the assumption (3.12), a simple computation shows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\frac{n(1+|B|)}{n}q^{n-1}(1-q)^r\\
&=(1+|B|)(1-q)^r\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}\\
&=(1+|B|)(1-q)^r\sum_{n=1}^{\infty}\binom{n+r-1}{r-1}q^{n}=(1+|B|)(1-q)^r\left[\frac{1}{(1-q)^r}-1\right]\\
&=(1+|B|)\left[1-(1-q)^r\right]\leq(B-A)\cos\theta.
\end{aligned}$$

and from (3.14) if follows that $\mathrm{I}_q^r\in\mathcal{L}(A,B,\theta)$. □

Theorem 3.8. *A sufficient condition for the function* I_q^r *to be in the class* $k-\mathcal{S}_p(\beta)$ *is*

$$k+1-(1-\beta)(1-q)^r-\frac{(k+\beta)(1-q)}{q(r-1)}\left(1-(1-q)^{r-1}\right)\leq 1-\beta. \tag{3.15}$$

Proof. Since I_q^r has the form (3.13), to prove that $\mathrm{I}_q^r\in k-\mathcal{S}_p(\beta)$, according to Lemma 2.4 it is sufficient to prove that

$$\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\frac{n(k+1)-(k+\beta)}{n}|q|^{n-1}|1-q|^r\leq 1-\beta. \tag{3.16}$$

Thus, from the assumption (3.15) we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\frac{[n(k+1)-(k+\beta)]}{n}q^{n-1}(1-q)^r\\
&=(1-q)^r\left[(k+1)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}q^{n-1}-\sum_{n=2}^{\infty}\frac{k+\beta}{n}\binom{n+r-2}{r-1}q^{n-1}\right]\\
&=(1-q)^r\left[(k+1)\sum_{n=1}^{\infty}\binom{n+r-1}{r-1}q^{n}-\frac{k+\beta}{q(r-1)}\sum_{n=2}^{\infty}\binom{n+r-2}{r-2}q^{n}\right]\\
&=(1-q)^r\Bigg[(k+1)\left(\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}q^{n}-1\right)\\
&\qquad\qquad-\frac{k+\beta}{q(r-1)}\left(\sum_{n=0}^{\infty}\binom{n+r-2}{r-2}q^{n}-1-q(r-1)\right)\Bigg]\\
&=(k+1)\left[1-(1-q)^r\right]-\frac{k+\beta}{q(r-1)}\Big[1-q-\left[1+q(r-1)\right](1-q)^r\Big]\\
&=k+1-(1-\beta)(1-q)^r-\frac{(k+\beta)(1-q)}{q(r-1)}\left(1-(1-q)^{r-1}\right)\leq 1-\beta.
\end{aligned}$$

and from (3.16) we conclude that $\mathrm{I}_q^r \in k - \mathcal{S}_p(\beta)$. □

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References

[1] M. K. Aouf, *On certain subclass of analytic p-valent functions of order alpha*, Rend. Mat. Appl. (7), **8**(1988), 89–104.

[2] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math., **28**(1997), 17–32.

[3] T. Bulboacă, *Differential subordinations and superordinations: Recent results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[4] T. R. Caplinger and W. M. Causey, *A class of univalent functions*, Proc. Amer. Math. Soc., **39**(1973), 357–361.

[5] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[6] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56(1)**(1991), 87–92.

[7] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl., **155**(1991), 364–370.

[8] O. P. Juneja and M. L. Mogra, *A class of univalent functions*, Bull. Sci. Math., **103(4)**(1979), 435–447.

[9] S. Kanas and H. M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transforms Spec. Funct., **9(2)**(2000), 121–132.

[10] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105(1-2)**(1999), 327–336.

[11] S. Kanas and A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures Appl., **45**(2000), 647–657.

[12] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbidungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden*, Leipzig Ber., **69**(1917), 89–106.

[13] S. S. Miller and P. T. Mocanu, *Differential subordination: Theory and applications*, Monographs and Textbooks in Pure and Applied Mathematics **225**, Marcel Dekker Inc., New York and Basel, 2000.

[14] R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Översikt av Finska Vet. Soc. Förh. (A), **63**(1921-1922).

[15] F. Rønning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **45**(1991), 117–122.

[16] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), 189–196.

[17] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci., **55**(2004), 2959–2961

[18] S. L. Shukla and Dashrath, *On a class of univalent functions*, Soochow J. Math., **10**(1984), 117–126.