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# An Identity Involving Product of Generalized Hypergeometric Series ${ }_{2} F_{2}$ 

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Abstract. A number of identities associated with the product of generalized hypergeometric series have been investigated. In this paper, we aim to establish an identity involving the product of the generalized hypergeometric series ${ }_{2} F_{2}$. We do this using the generalized Kummer-type II transformation due to Rathie and Pogany and another identity due to Bailey. The result presented here, being general, can be reduced to a number of relatively simple identities involving the product of generalized hypergeometric series, some of which are observed to correspond to known ones.

## 1. Introduction and Preliminaries

The generalized hypergeometric function ${ }_{p} F_{q}$ with $p$ numerator and $q$ denominator parameters is defined by (see, e.g., $[2,5,12]$; see also [16, Section 1.5])

$$
{ }_{p} F_{q}\left[\begin{array}{l}
\alpha_{1}, \ldots, \alpha_{p} ;  \tag{1.1}\\
\beta_{1}, \ldots, \beta_{q} ;
\end{array}\right]={ }_{p} F_{q}\left[\alpha_{1}, \ldots, \alpha_{p} ; \beta_{1}, \ldots, \beta_{q} ; z\right]=\sum_{n=0}^{\infty} \frac{\left(\alpha_{1}\right)_{n} \cdots\left(\alpha_{p}\right)_{n}}{\left(\beta_{1}\right)_{n} \cdots\left(\beta_{q}\right)_{n}} \frac{z^{n}}{n!}
$$

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where $(\lambda)_{\nu}$ denotes the Pochhammer symbol defined (for $\lambda, \nu \in \mathbb{C}$ ), in terms of the Gamma function $\Gamma$, by

$$
(\lambda)_{\nu}:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\left\{\begin{array}{lr}
1 & (\nu=0 ; \lambda \in \mathbb{C} \backslash\{0\})  \tag{1.2}\\
\lambda(\lambda+1) \cdots(\lambda+n-1) & (\nu=n \in \mathbb{N} ; \lambda \in \mathbb{C})
\end{array}\right.
$$

The well-known Kummer-type I transformation for the series ${ }_{1} F_{1}$ is (see, e.g., [12])

$$
e^{-x}{ }_{1} F_{1}\left[\begin{array}{c}
a ;  \tag{1.3}\\
b ;
\end{array}\right]={ }_{1} F_{1}\left[\begin{array}{r}
b-a ; \\
b ;
\end{array},-x\right] .
$$

This result (1.3) was obtained by Kummer [7] and [8] who used the theory of differential equations. Bailey [1] derived this result by employing classical Gauss's summation theorem (see, e.g., [12, p. 48]).

Paris [9] generalized (1.3) in the form

$$
e^{-x}{ }_{2} F_{2}\left[\begin{array}{rr}
a, 1+d ; &  \tag{1.4}\\
b, d ; & x
\end{array}\right]={ }_{2} F_{2}\left[\begin{array}{rr}
b-a-1, f+1 ; & \\
b, f ; & -x
\end{array}\right],
$$

where

$$
\begin{equation*}
f:=\frac{d(a-b+1)}{a-d} \tag{1.5}
\end{equation*}
$$

The particular case $d=\frac{1}{2}$ of (1.4) reduces to Exton's result [4].
Kummer $[7,8]$ also used the theory of differential equations to obtain the following result

$$
e^{-\frac{x}{2}}{ }_{1} F_{1}\left[\begin{array}{rr}
a ; & x  \tag{1.6}\\
2 a ; &
\end{array}\right]={ }_{0} F_{1}\left[\begin{array}{rr}
-; & x^{2} \\
a+\frac{1}{2} ; & \frac{16}{16}
\end{array}\right],
$$

which is often referred to as Kummer-type II transformation.
Using Gauss's summation theorem, the result (1.6) has been re-derived by Bailey [1] and Rathie and Choi [14] (see also [3]).

Motivated by the Kummer-type I transformation (1.4), Rathie and Pogany [15] also generalized the Kummer-type II transformation (1.6) in the form
(1.7) $e^{-\frac{x}{2}}{ }_{2} F_{2}\left[\begin{array}{r}a, 1+d ; \\ 2 a+1, d ;\end{array} \quad x\right]={ }_{0} F_{1}\left[\begin{array}{cc}-; & \frac{x^{2}}{16} \\ a+\frac{1}{2} ; & { }^{16}\end{array}\right]+\frac{x(2 a-d)}{2 d(2 a+1)}{ }_{0} F_{1}\left[\begin{array}{cc}-; & x^{2} \\ a+\frac{3}{2} ; & \frac{x^{16}}{}\end{array}\right]$.

It is obvious that the case $d=2 a$ of (1.7) reduces to (1.6).
Using the theory of differential equations, Preece [10] established the following interesting identity involving product of generalized hypergeometric series

$$
{ }_{1} F_{1}\left[\begin{array}{rr}
a ; & x  \tag{1.8}\\
2 a ; & x
\end{array}\right]{ }_{1} F_{1}\left[\begin{array}{rr}
a ; & -x \\
2 a ; &
\end{array}\right]={ }_{1} F_{2}\left[\begin{array}{rr}
a ; & x^{2} \\
a+\frac{1}{2}, 2 a ; & \frac{1}{4}
\end{array}\right] .
$$

Equation (1.8) is a special case of [6, Exercise 7.24(b)]. In fact, Koepf [6, Example 7.3] showed how such identities can be derived by using the Zeilberger algorithm.

Using (1.3), we can express (1.8) in the form

$$
\left\{{ }_{1} F_{1}\left[\begin{array}{rr}
a ; &  \tag{1.9}\\
2 a ; & x
\end{array}\right\}^{2}=e^{x}{ }_{1} F_{2}\left[\begin{array}{rr}
a ; & x^{2} \\
a+\frac{1}{2}, 2 a ; & \frac{4}{4}
\end{array}\right]\right.
$$

Rathie [13] proved Preece's identity (1.9) by a very short method and obtained two results closely related to it.

By employing the classical Watson's theorem on the sum of a ${ }_{3} F_{2}$ with unit argument (see, e.g., [2, p. 16]), Bailey [1] generalized Preece's identity (1.8) in the form

Rathie and Choi [14] derived the Bailey identity (1.10) by using the same technique given in [13] and obtained four results closely related to it.

In fact, a number of identities associated with the product of generalized hypergeometric series have been investigated (see, e.g., [2, Chapter X]; see also [11, Entires 8.4.45-8.4.49]). In this paper, we aim to establish an identity involving the product of generalized hypergeometric series ${ }_{2} F_{2}$ by using the generalized Kummertype II transformation (1.7) and the following identity due to Bailey [1] (see also [6, Exercise 7.24 (a)])

The result presented here, being general, can be reduced to a number of relatively simple identities involving the product of generalized hypergeometric series, some of which are explicitly indicated to correspond to known ones.

## 2. Product Formula for ${ }_{2} F_{2}$

We present an identity involving product of ${ }_{2} F_{2}$, asserted in the following theorem.

Theorem 2.1. The following identity holds.

$$
{ }_{2} F_{2}\left[\begin{array}{rr}
a, 1+d ; &  \tag{2.1}\\
2 a+1, d ; & x
\end{array}\right]{ }_{2} F_{2}\left[\begin{array}{rr}
b, 1+e ; & \\
2 b+1, e ; &
\end{array}\right]
$$

$$
\begin{aligned}
& =e^{x}\left\{{ }_{2} F_{3}\left[\begin{array}{cc}
\frac{1}{2}(a+b), \frac{1}{2}(a+b+1) ; & \frac{x^{2}}{4} \\
a+\frac{1}{2}, b+\frac{1}{2}, a+b ; &
\end{array}\right]\right. \\
& +\frac{x(2 a-d)}{2 d(2 a+1)}{ }_{2} F_{3}\left[\begin{array}{cc}
\frac{1}{2}(a+b+1), \frac{1}{2}(a+b+2) ; & \frac{x^{2}}{4} \\
a+\frac{3}{2}, b+\frac{1}{2}, a+b+1 ; &
\end{array}\right] \\
& +\frac{x(2 b-e)}{2 e(2 b+1)}{ }_{2} F_{3}\left[\begin{array}{cc}
\frac{1}{2}(a+b+1), \frac{1}{2}(a+b+2) ; & \frac{x^{2}}{4} \\
a+\frac{1}{2}, b+\frac{3}{2}, a+b+1 ; &
\end{array}\right] \\
& \left.+\frac{x^{2}(2 a-d)(2 b-e)}{4 d e(2 a+1)(2 b+1)}{ }_{2} F_{3}\left[\begin{array}{cc}
\frac{1}{2}(a+b+2), & \frac{1}{2}(a+b+3) ; \\
a+\frac{3}{2}, b+\frac{3}{2}, a+b+2 ; & \frac{x^{2}}{4}
\end{array}\right]\right\} .
\end{aligned}
$$

Proof. Let

$$
\mathcal{L}:=e^{-x}{ }_{2} F_{2}\left[\begin{array}{rr}
a, 1+d ; \\
2 a+1, d ; & x
\end{array}\right]{ }_{2} F_{2}\left[\begin{array}{rr}
b, 1+e ; & \\
2 b+1, e ; & x
\end{array}\right] .
$$

Then

$$
\mathcal{L}:=\left\{e^{-\frac{x}{2}}{ }_{2} F_{2}\left[\begin{array}{rr}
a, 1+d ; &  \tag{2.2}\\
2 a+1, d ; & x
\end{array}\right]\right\}\left\{e^{-\frac{x}{2}}{ }_{2} F_{2}\left[\begin{array}{rr}
b, 1+e ; & \\
2 b+1, e ; &
\end{array}\right]\right\} .
$$

Using (1.7) in each factor of (2.2), we get

$$
\begin{align*}
\mathcal{L}= & \left\{{ }_{0} F_{1}\left[\begin{array}{cc}
-; & x^{2} \\
a+\frac{1}{2} ; & \frac{x(2 a-d)}{16}
\end{array}{ }_{0} F_{1}\left[\begin{array}{cc}
-\frac{x}{2 d(2 a+1)} & \frac{x^{2}}{a+\frac{3}{2} ;}
\end{array}\right]\right\}\right.  \tag{2.3}\\
& \times\left\{{ }_{0} F_{1}\left[\begin{array}{cc}
- & x^{2} \\
b+\frac{1}{2} ; & \frac{x(2 b-e)}{16}
\end{array}{ }_{0} F_{1}\left[\begin{array}{cc}
-\frac{x}{2 e(2 b+1)} & \frac{x^{2}}{16} \\
b+\frac{3}{2} ; &
\end{array}\right]\right\} .\right.
\end{align*}
$$

Expanding the right member of (2.3), we obtain

$$
\begin{aligned}
& \mathcal{L}={ }_{0} F_{1}\left[\begin{array}{rl}
-; & \frac{x^{2}}{16}
\end{array}\right]{ }_{0} F_{1}\left[\begin{array}{rl}
-; & \frac{x^{2}}{}\left[\begin{array}{rl}
16
\end{array}\right] \\
b+1 / 2 ; & 1 / 2 ;
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{x(2 b-e)}{2 e(2 b+1)}{ }_{0} F_{1}\left[\begin{array}{rr}
-; & \frac{x^{2}}{16} \\
a+1 / 2 ; & 16{ }_{0} F_{1}\left[\begin{array}{rl}
-; & x^{2} \\
b+3 / 2 ; & 16
\end{array}\right], ~
\end{array}\right. \\
& +\frac{x^{2}(2 a-d)(2 b-e)}{4 d e(2 a+1)(2 b+1)}{ }_{0} F_{1}\left[\begin{array}{rl}
-; & x^{2} \\
a+3 / 2 ; & \overline{16}
\end{array}\right]{ }_{0} F_{1}\left[\begin{array}{rl}
-; & \frac{x^{2}}{}\left[\begin{array}{rl}
16
\end{array}\right] .
\end{array}\right.
\end{aligned}
$$

Finally, using (1.11) in each term of the right member of the last identity, we are led to the right member of (2.1) with the factor $e^{x}$ deleted. This completes the proof.

## 3. Special Cases

The result (2.1), being general, can be reduced to yield a number of relatively simple identities, several of which are considered here.
(i) Setting $d=2 a$ in (2.1), we get

$$
\begin{aligned}
& e^{-x}{ }_{1} F_{1}\left[\begin{array}{c}
a ; \\
2 a ;
\end{array}\right]{ }_{2} F_{2}\left[\begin{array}{c}
b, 1+e ; \\
2 b+1, e ;
\end{array}\right] \\
& ={ }_{2} F_{3}\left[\begin{array}{c}
\frac{1}{2}(a+b), \frac{1}{2}(a+b+1) ; \\
a+\frac{1}{2}, b+\frac{1}{2}, a+b ;
\end{array}\right] \\
& \quad+\frac{x(2 b-e)}{2 e(2 b+1)}{ }_{2} F_{3}\left[\begin{array}{rr}
\frac{1}{2}(a+b+1), & \frac{1}{2}(a+b+2) ; \\
a+\frac{1}{2}, b+\frac{3}{2}, a+b+1 ; & \frac{x^{2}}{4}
\end{array}\right] .
\end{aligned}
$$

The identity (3.1) is found to be equivalent to a very recent result due to Kim et al. [5] who used a different method.
(ii) Taking $d=2 a$ and $e=2 b$ in (2.1), we have

$$
\begin{aligned}
& e^{-x}{ }_{1} F_{1}\left[\begin{array}{rr}
a ; & x \\
2 a ; & x
\end{array}\right]{ }_{1} F_{1}\left[\begin{array}{rr}
b ; & x \\
2 b ; &
\end{array}\right] \\
& \quad={ }_{2} F_{3}\left[\begin{array}{cc}
\frac{1}{2}(a+b), & \frac{1}{2}(a+b+1) ; \\
a+\frac{1}{2}, b+\frac{x^{2}}{2}, a+b ; &
\end{array}\right] .
\end{aligned}
$$

Using (1.3) in (3.2), we obtain Bailey's identity (1.10). So the identity (2.1) may be regarded as a generalization of Bailey's identity (1.10).
(iii) Setting $e=d$ and $b=a$ in (2.1), we find

$$
\begin{aligned}
e^{-x}\left\{{ }_{2} F_{2}\left[\begin{array}{rr}
a, 1+d ; & \\
2 a+1, d & ;
\end{array}\right]\right\}^{2}={ }_{1} F_{2}\left[\begin{array}{rr}
a ; & x^{2} \\
a+\frac{1}{2}, 2 a ; & \frac{1}{4}
\end{array}\right] \\
+\frac{x(2 a-d)}{d(2 a+1)}{ }_{1} F_{2}\left[\begin{array}{rr}
a+1 ; & x^{2} \\
a+\frac{3}{2}, 2 a+1 ; & \frac{x^{2}}{4}
\end{array}\right] \\
+\frac{x^{2}(2 a-d)^{2}}{4 d^{2}(2 a+1)^{2}}{ }_{1}{ }_{1} F_{2}\left[\begin{array}{cc}
a+1 ; & x^{2} \\
a+\frac{3}{2}, 2 a+2 ; & 4
\end{array}\right] .
\end{aligned}
$$

Taking $d=2 a$ in (3.3) yields Preece's identity (1.9). Therefore, the identity (3.3) can be looked upon as a generalization of Preece's identity (1.9).

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## References

[1] W. N. Bailey, Products of generalized hypergeometric series, Proc. London Math. Soc. (2), 28(4)(1928), 242-254.
[2] W. N. Bailey, Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics 32, Stechert-Hafner, New York, 1964.
[3] J. Choi, A. K. Rathie and B. Bhojak, On Preece's identity and other contiguous results, Commun. Korean Math. Soc., 20(1)(2005), 169-178.
[4] H. Exton, On the reducibility of the Kampé de Fériet function, J. Comput. Appl. Math., 83(1997), 119-121.
[5] Y. S. Kim, S. Gaboury and A. K. Rathie, Applications of extended Watson's summation theorem, Turkish J. Math., 42(2018), 418-443.
[6] W. Koepf, Hypergeometric summation. An algorithmic approach to summation and special function identities, Advanced Lectures in Mathematics, Friedr. Vieweg \& Sohn, Braunschweig, 1998.
[7] E. E. Kummer, Über die hypergeometrische Reihe $1+\frac{\alpha \cdot \beta x}{1 \cdot \gamma}+\frac{\alpha(\alpha+1) \beta(\beta+1) x^{2}}{1 \cdot 2 \cdot \gamma(\gamma+1)}+\cdots$, J. Reine Angew Math., 15(1836), 127-172.
[8] E. E. Kummer, Collected Papers, Volume II: Function theory, geometry and miscellaneous, Springer-Verlag, Berlin, 1975.
[9] R. Paris, A Kummer-type transformation for $a_{2} F_{2}$ hypergeometric function, J. Comput. Appl. Math., 173(2005), 379-382.
[10] C. T. Preece, The product of two generalized hypergeometric functions, Proc. London Math. Soc. (2), 22(1924), 370-380.
[11] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. 3: More Special Functions, Translated from the Russian by G. G. Gould. Gordon and Breach Science Publishers, New York, 1990.
[12] E. D. Rainville, Special Functions, Macmillan Co., New York, 1960; Reprinted by Chelsea Publishing Co., Bronx, New York, 1971.
[13] A. K. Rathie, A short proof of Preece's identities and other contiguous results, Rev. Mat. Estatist., 15(1997), 207-210.
[14] A. K. Rathie and J. Choi, A note on generalizations of Preece's identity and other contiguous results, Bull. Korean Math. Soc., 35(1998), 339-344.
[15] A. K. Rathie and J. Pogany, New summation formula for ${ }_{3} F_{2}\left(\frac{1}{2}\right)$ and a Kummer-type II transformation of ${ }_{2} F_{2}(x)$, Math. Commun., 13(1)(2008), 63-66.
[16] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and associated series and integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

