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Spirallike and Robertson Functions of Complex Order with Bounded Boundary Rotations

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Abstract. Using the concept of bounded boundary rotation, we investigate various properties of two new generalized classes of spirallike and Robertson functions of complex order with bounded boundary rotations.

1. Introduction

Let $\mathbb{D}$ be the unit disc $\{z : |z| < 1\}$ and suppose $\mathcal{A}$ is the class of functions analytic in $\mathbb{D}$ satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Then each function f in $\mathcal{A}$ has the Taylor expression

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

because of the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{V}_k$ denote the family of functions f in $\mathcal{A}$ that map the unit disc $\mathbb{D}$ conformally onto an image domain $f(\mathbb{D})$ of bounded boundary rotation at most $k\pi$.

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The concept of functions of bounded boundary rotation was initiated by Loewner [14] in 1917. However, it was Paatero [22, 23] who systematically studied the class $\mathcal{V}_k$. In Pinchuk [25], it is proved that the functions in $\mathcal{V}_k$ are close-to-convex in $\mathbb{D}$ if $2 \leq k \leq 4$. Brannan in [8] showed that $\mathcal{V}_k$ is a subclass of the class $\mathcal{K}(\alpha)$ of close-to-convex functions of order α for $\alpha = \frac{k}{2} - 1$. For references and survey on bounded boundary rotation, one may refer to a recent survey written by Noor [20].

Failure to settle the Bieberbach conjecture for about 69 years led to the introduction and investigation of several subclasses of $\mathcal{S}$, the subfamily of $\mathcal{A}$ that are univalent in the open unit disc $\mathbb{D}$ (see [2]). In 1932, Spacek [21] proved that if f in $\mathcal{A}$ satisfies the condition $Re[(\eta z f'(z))/f(z)] > 0$ for all $z \in \mathbb{D}$ and a fixed complex number η, then f must be in $\mathcal{S}$. Without loss of generality, we may replace η with $e^{i\lambda}$, $|\lambda| < \pi/2$. Motivated by Spacek [21], Libera [13] in 1967 gave a geometric characterization of $\lambda-$spirallike functions f in $\mathcal{A}$ that satisfy the condition

$$Re\left(e^{i\lambda} z \frac{f'(z)}{f(z)}\right) > 0, \quad (z \in \mathbb{D}, |\lambda| < \frac{\pi}{2}). \tag{1.2}$$

Denote the class of all such functions that satisfy (1.2) by $\mathcal{H}^\lambda$. We observe that $\mathcal{H}^0 = \mathcal{S}^*$, the family of all starlike functions in $\mathbb{D}$. In 1969, Robertson [29] introduced and studied the family

$$\mathcal{M}^\lambda = \{f \in \mathcal{A} : zf' \in \mathcal{H}^\lambda, z \in \mathbb{D}\}.$$

A function f in $\mathcal{M}^\lambda$ is called a $\lambda-$Robertson or a convex $\lambda-$spiral function. In 1991, Ahuja and Silverman [3] surveyed various subclasses of $\mathcal{H}^\lambda$ and $\mathcal{M}^\lambda$, their associated properties and open problems.

Motivated by many earlier researchers [5, 6, 8, 11, 15, 16, 24, 27, 28, 29], we introduce the following:

Definition 1.1. Let $\mathcal{P}_k^\lambda(b)$ be the class of functions p defined in $\mathbb{D}$ that satisfy the property $p(0) = 1$ and the condition

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}p(z) - (1-b)\cos\lambda - i\sin\lambda}{b}\right)\right| d\theta \leq k\pi\cos\lambda, \tag{1.3}$$

where $k \geq 2$, λ real with $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}$ and $z = re^{i\theta}$.

When $\lambda = 0$, $k = 2$ and $b = 1$, the class $\mathcal{P}_2^0(1) = \mathcal{P}$ is a well known class of functions with positive real part in $\mathbb{D}$. In fact, for different values of k, λ and b, $\mathcal{P}_k^\lambda(b)$ reduces to important subclasses studied by various researchers. For instance,

(i) $\mathcal{P}_2^0(1-\alpha) = \mathcal{P}(\alpha)$, $(0 \leq \alpha < 1)$, Robertson [27].

(ii) $\mathcal{P}_k^0(1) = \mathcal{P}_k$, Pinchuk [26].

(iv) $\mathcal{P}_k^0(1-\alpha) = \mathcal{P}_k(\alpha)$, $(0 \leq \alpha < 1)$, Padmanabhan [24].

(v) $\mathcal{P}_k^\lambda(1) = Q_k^\lambda$, Moulis [15].

(vi) $\mathcal{P}_k^\lambda(1-\alpha) = Q_k^\lambda(\alpha)$, $(0 \le \alpha < 1)$, Moulis [16].

Definition 1.2. Let $\mathcal{V}_k^\lambda(b)$ denote the class of functions f in $\mathcal{A}$ which satisfy the condition

$$1 + z\frac{f''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(b),$$

where k, λ and b are given in Definition 1.1. If $f \in \mathcal{V}_k^\lambda(b)$, then f is called $\lambda-$ Robertson function of complex order b with bounded boundary rotation.

We remark that the class $\mathcal{V}_k^\lambda(b)$ generalizes various known and unknown subclasses of $\mathcal{A}$. For example, for different values of k, λ and b, we get the classes listed in the following table:

Subclasses of $\mathcal{A}$	**Name of a function in the class and References**
$\mathcal{V}_2^0(1) = \mathcal{K}$	Convex functions
$\mathcal{V}_2^0(1-\alpha) = \mathcal{K}(\alpha)$	Convex functions of order α, $0 \le \alpha < 1$, [27]
$\mathcal{V}_2^0(b) = \mathcal{K}(b)$	Convex functions of complex order b, [32]
$\mathcal{V}_k^0(1) = \mathcal{V}_k$	Convex functions with bounded boundary rotation, [22, 23]
$\mathcal{V}_k^0(1-\alpha) = \mathcal{V}_k(\alpha)$	Convex functions of order α with bounded boundary rotation, $0 \le \alpha < 1$, [24]
$\mathcal{V}_2^\lambda(1) = \mathcal{M}^\lambda$	λ- Robertson functions, [29]
$\mathcal{V}_2^\lambda(1-\alpha) = \mathcal{M}^\lambda(\alpha)$	λ- Robertson functions of order α, $0 \le \alpha < 1$, [11]
$\mathcal{V}_2^\lambda(b) = \mathcal{M}^\lambda(b)$	λ- Robertson functions of complex order b, [6]
$\mathcal{V}_k^\lambda(1) = \mathcal{V}_k^\lambda$	λ- Robertson functions with bounded boundary rotation, [15]
$\mathcal{V}_k^\lambda(1-\alpha) = \mathcal{V}_k^\lambda(\alpha)$	λ- Robertson functions of order α with bounded boundary rotation, $0 \le \alpha < 1$, [16].

In view of Definitions 1.1 and 1.2, we immediately get the following.
A function $f \in \mathcal{V}_k^\lambda(b)$ if and only if

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}\left(1 + z\frac{f''(z)}{f'(z)}\right) - (1-b)\cos\lambda - i\sin\lambda}{b} \right) \right| d\theta \le k\pi\cos\lambda. \tag{1.4}$$

We next define another subclass of $\mathcal{P}_k^\lambda(b)$.

Definition 1.3. Let $\mathcal{S}_k^\lambda(b)$ denote the class of functions f in $\mathcal{A}$ which satisfy the condition

$$z\frac{f'(z)}{f(z)} \in \mathcal{P}_k^\lambda(b),$$

where $k \ge 2$, λ real with $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}$. If $f \in \mathcal{S}_k^\lambda(b)$, then f is called $\lambda-$ spirallike function of complex order b with bounded boundary rotation.

For different values of k, λ and b, the class $\mathcal{S}_k^\lambda(b)$ gives rise to several known and unknown subclasses of $\mathcal{A}$. For example, we obtain the following known classes:

Subclasses of $\mathcal{A}$	**Name of a function in the class and References**
$\mathcal{S}_2^0(1) = \mathcal{S}^*$	Starlike functions
$\mathcal{S}_2^0(1-\alpha) = \mathcal{S}^*(\alpha)$	Starlike functions of order α, $0 \le \alpha < 1$, [27]
$\mathcal{S}_2^0(b) = \mathcal{S}(b)$	Starlike functions of complex order b, [17]
$\mathcal{S}_k^0(1-\alpha) = \mathcal{S}_k(\alpha)$	Starlike functions of order α with bounded boundary rotation, $0 \le \alpha < 1$, [18, 24]
$\mathcal{S}_2^\lambda(1) = \mathcal{H}^\lambda$	λ- Spirallike functions, [21]
$\mathcal{S}_2^\lambda(1-\alpha) = \mathcal{H}^\lambda(\alpha)$	λ- Spirallike functions of order α, $0 \le \alpha < 1$, [13]
$\mathcal{S}_2^\lambda(b) = \mathcal{S}^\lambda(b)$	λ- Spirallike functions of complex order b, [5]
$\mathcal{S}_k^\lambda(1) = \mathcal{S}_k^\lambda$	λ- Spirallike functions with bounded boundary rotation, [19].

Using Definitions 1.1 and 1.3, we immediately obtain the following.
A function $f \in \mathcal{S}_k^\lambda(b)$ if and only if

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda} z \frac{f'(z)}{f(z)} - (1-b)\cos\lambda - i\sin\lambda}{b} \right) \right| d\theta \le k\pi\cos\lambda. \tag{1.5}$$

Using Definitions 1.2 and 1.3, we obtain the following characterization

$$f \in \mathcal{V}_k^\lambda(b) \quad \text{if and only if} \quad zf' \in \mathcal{S}_k^\lambda(b). \tag{1.6}$$

In view of the relations witnessed in 18 subclasses in the above two tables, we conclude that the notion of generalized classes $\mathcal{V}_k^\lambda(b)$ and $\mathcal{S}_k^\lambda(b)$ unify several known subclasses of $\mathcal{A}$.

We remark that functions in $\mathcal{V}_k^0(1)$ have bounded boundary rotation. But, the functions in the class $\mathcal{V}_k^\lambda(1)$ with $\lambda \ne 0$ may not have bounded boundary rotation. For properties and counter examples of the classes $\mathcal{V}_k^0(1)$ and $\mathcal{V}_k^\lambda(1)$, one may refer to Loewner [14] and Paatero [22, 23] .

In this paper, we investigate various properties of generalized classes $\mathcal{V}_k^\lambda(b)$ and $\mathcal{S}_k^\lambda(b)$.

2. Properties of Class $\mathcal{V}_k^\lambda(b)$

The following result will be helpful in proving representation theorems for the classes $\mathcal{P}_k^\lambda(b)$ and $\mathcal{V}_k^\lambda(b)$.

Lemma 2.1.([22]) *A function $f \in \mathcal{P}_k$ if and only if*

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t),$$

where μ is a real-valued function of bounded variation on $[0, 2\pi]$ for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le k \tag{2.1}$$

for $k \ge 2$.

Lemma 2.2. *If* $p \in \mathcal{P}_k^\lambda(b)$*, then*

$$p(z) = e^{-i\lambda}\left(\frac{\cos\lambda}{2}\int_0^{2\pi}\frac{1+(2b-1)ze^{-it}}{1-ze^{-it}}d\mu(t) + i\sin\lambda\right), \tag{2.2}$$

where $k \geq 2$*,* λ *real with* $|\lambda| < \frac{\pi}{2}$*,* $b \in \mathbb{C} - \{0\}$ *and* μ *is real-valued function of bounded variation satisfying the conditions (2.1).*

Proof. Letting

$$\begin{aligned} f(z) &= 1 + \frac{e^{i\lambda}}{b\cos\lambda}\big(p(z)-1\big) \\ &= \frac{e^{i\lambda}p(z) - (1-b)\cos\lambda - i\sin\lambda}{b\cos\lambda}. \end{aligned}$$

Since $p \in \mathcal{P}_k^\lambda(b)$, it follows from Lemma 2.1, we get

$$\frac{e^{i\lambda}p(z) - (1-b)\cos\lambda - i\sin\lambda}{b\cos\lambda} = \frac{1}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t). \tag{2.3}$$

Equivalently, we obtain

$$e^{i\lambda}p(z) = \frac{b\cos\lambda}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t) + (1-b)\cos\lambda + i\sin\lambda.$$

Since $\int_0^{2\pi} d\mu(t) = 2$, the last equation is equivalent to

$$e^{i\lambda}p(z) = \frac{\cos\lambda}{2}\int_0^{2\pi}\frac{1+(2b-1)ze^{-it}}{1-ze^{-it}}d\mu(t) + i\sin\lambda,$$

where μ is a real-valued function of bounded variation on $[0, 2\pi]$ and satisfies the conditions (2.1). This proves (2.2). □

Motivated by several known results (see for instance [7, 16, 24]) and using Lemma 2.2, we first give the following result for the functions in the family $\mathcal{V}_k^\lambda(b)$.

Theorem 2.3. *A function* $f_\lambda \in \mathcal{V}_k^\lambda(b)$ *if and only if there exists a function* $f \in \mathcal{V}_k$ *such that*

$$f'_\lambda(z) = [f'(z)]^{be^{-i\lambda}\cos\lambda}, \tag{2.4}$$

where $k \geq 2$*,* λ *real with* $|\lambda| < \frac{\pi}{2}$ *and* $b \in \mathbb{C} - \{0\}$*.*

Proof. Since $f_\lambda \in \mathcal{V}_k^\lambda(b)$, there exists $p \in \mathcal{P}_k^\lambda(b)$ such that

$$1 + z\frac{f''_\lambda(z)}{f'_\lambda(z)} = p(z).$$

By using (2.3), we can write

$$\frac{e^{i\lambda}\left(1+z\frac{f_\lambda''(z)}{f_\lambda'(z)}\right)-(1-b)\cos\lambda-i\sin\lambda}{b\cos\lambda}=\frac{1}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t).$$

Hence

$$e^{i\lambda}\left(1+z\frac{f_\lambda''(z)}{f_\lambda'(z)}\right)=\frac{b\cos\lambda}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t)+(1-b)\cos\lambda+i\sin\lambda. \tag{2.5}$$

In view of Lemma 2.1, there exists a real-valued function μ of bounded variation on $[0,2\pi]$ satisfying conditions (2.1) such that

$$1+z\frac{f''(z)}{f'(z)}=\frac{1}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t). \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$e^{i\lambda}\left(1+z\frac{f_\lambda''(z)}{f_\lambda'(z)}\right)=b\cos\lambda\left(1+z\frac{f''(z)}{f'(z)}\right)+(1-b)\cos\lambda+i\sin\lambda.$$

Calculating the above equality, we get

$$\begin{aligned}\frac{f_\lambda''(z)}{f_\lambda'(z)}&=be^{-i\lambda}\cos\lambda\left(\frac{1}{z}+\frac{f''(z)}{f'(z)}\right)+\frac{(1-b)e^{-i\lambda}\cos\lambda}{z}+\frac{e^{-i\lambda}i\sin\lambda-1}{z}\\&=be^{-i\lambda}\cos\lambda\frac{f''(z)}{f'(z)}.\end{aligned}$$

Integrating both sides, we obtain

$$\ln f_\lambda'(z)=be^{-i\lambda}\cos\lambda\ln f'(z).$$

This gives (2.4). □

The following result is a consequence of Theorem 2.3.

Corollary 2.4. *$f_\lambda\in\mathcal{V}_k^\lambda(b)$ if and only if there exists a function μ with bounded variation on $[0,2\pi]$ satisfying conditions (2.1) and*

$$f_\lambda'(z)=\exp\left[-be^{-i\lambda}\cos\lambda\int_0^{2\pi}\log\left(1-ze^{-it}\right)d\mu(t)\right]. \tag{2.7}$$

Proof. Paatero [23] proved that $f\in\mathcal{V}_k$ if and only if there exists a function μ of bounded variation on $[0,2\pi]$ such that

$$f'(z)=\exp\left[-\int_0^{2\pi}\log\left(1-ze^{-it}\right)d\mu(t)\right],$$

with the conditions given in (2.1). In view of Theorem 2.3, we obtain desired result. □

Theorem 2.5. *If* $f_\lambda(z) = z + a_2z^2 + a_3z^3 + ... \in \mathcal{V}_k^\lambda(b)$, *then*

$$|a_2| \leq \frac{k}{2}|b|\cos\lambda. \tag{2.8}$$

This bound is sharp for the functions of the form

$$f'_\lambda(z) = \left[\frac{(1+z)^{\frac{k}{2}-1}}{(1-z)^{\frac{k}{2}+1}}\right]^{be^{-i\lambda}\cos\lambda}.$$

Proof. Since $f_\lambda(z) = z + a_2z^2 + a_3z^3 + ... \in \mathcal{V}_k^\lambda(b)$, and by using Theorem 2.3 there exists a function $f(z) = z + b_2z^2 + b_3z^3 + ... \in \mathcal{V}_k$ such that

$$f'_\lambda(z) = [f'(z)]^{be^{-i\lambda}\cos\lambda}.$$

That is,

$$1 + 2a_2z + 3a_3z^2 + ... = \left[1 + 2b_2z + 3b_3z^2 + ...\right]^{be^{-i\lambda}\cos\lambda}.$$

Comparing the coefficients of z on both sides, we get

$$a_2 = b_2.$$

In [12], Lehto proved that $|b_2| \leq \frac{k}{2}$. Therefore we obtain

$$|a_2| = |b_2be^{-i\lambda}\cos\lambda| \leq \frac{k}{2}|b|\cos\lambda.$$ □

We need the following two lemmas to prove our next theorem.

Lemma 2.6.([28]) *Let* $f \in \mathcal{V}_k$, $2 \leq k < \infty$ *and* $|a| < 1$. *If*

$$F(z) = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{f'(a)(1-|a|^2)} \tag{2.9}$$

for all $z \in \mathbb{D}$, *then* $f \in \mathcal{V}_k$ *and*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{k|z|}{1-|z|^2}. \tag{2.10}$$

Lemma 2.7. *If* $f_\lambda \in \mathcal{V}_k^\lambda(b)$, *then the function* F_λ *defined by*

$$F'_\lambda(z) = \frac{f'_\lambda\left(\frac{z+a}{1+\overline{a}z}\right)}{f'_\lambda(a)(1+\overline{a}z)^{2be^{-i\lambda}\cos\lambda}}, \quad F_\lambda(0) = 0 \tag{2.11}$$

also belongs to $\mathcal{V}_k^\lambda(b)$.

Proof. Let $f_\lambda \in \mathcal{V}_k^\lambda(b)$. By Theorem 2.3, there exists a function $f \in \mathcal{V}_k$ such that

$$f_\lambda'(z) = [f'(z)]^{be^{-i\lambda}\cos\lambda}. \tag{2.12}$$

Since $f \in \mathcal{V}_k$, it follows from Lemma 2.6 that the function F defined by (2.9) is also in $\mathcal{V}_k$. Again, by using the converse part of Theorem 2.3, there exists a function $F_\lambda \in \mathcal{V}_k^\lambda(b)$ such that

$$F_\lambda'(z) = [F'(z)]^{be^{-i\lambda}\cos\lambda}.$$

But, by (2.9) we have

$$F'(z) = \frac{f'\left(\frac{z+a}{1+\overline{a}z}\right)}{f'(a)(1+\overline{a}z)^2},$$

where $|a| < 1$. Therefore, we get

$$\begin{aligned} F_\lambda'(z) &= \frac{[f'\left(\frac{z+a}{1+\overline{a}z}\right)]^{be^{-i\lambda}\cos\lambda}}{[f'(a)]^{be^{-i\lambda}\cos\lambda}(1+\overline{a}z)^{2be^{-i\lambda}\cos\lambda}} \\ &= \frac{f_\lambda'\left(\frac{z+a}{1+\overline{a}z}\right)}{f_\lambda'(a)(1+\overline{a}z)^{2be^{-i\lambda}\cos\lambda}}, \end{aligned}$$

which proves the lemma. □

Theorem 2.8. *If* $f_\lambda \in \mathcal{V}_k^\lambda(b)$ *and* $k|b|\cos\lambda < 1$, *then* f_λ *is univalent in* $\mathbb{D}$ *and*

$$\left| z\frac{f_\lambda''(z)}{f_\lambda'(z)} - \frac{2b|z|^2 e^{-i\lambda}\cos\lambda}{1-|z|^2} \right| \le \frac{k|b||z|\cos\lambda}{1-|z|^2} \tag{2.13}$$

for all $z \in \mathbb{D}$.

Proof. If $f_\lambda \in \mathcal{V}_k^\lambda(b)$, then $F_\lambda'(z)$ defined by (2.11) is also in $\mathcal{V}_k^\lambda(b)$, by Lemma 2.7. Taking differentiation on both sides of (2.11) and letting $z = 0$, we get

$$F_\lambda''(0) = (1-|a|^2)\frac{f_\lambda''(a)}{f_\lambda'(a)} - 2be^{-i\lambda}\overline{a}\cos\lambda.$$

Therefore

$$a_2 = \frac{F_\lambda''(0)}{2!} = \frac{1}{2}\left\{(1-|a|^2)\frac{f_\lambda''(a)}{f_\lambda'(a)} - 2be^{-i\lambda}\overline{a}\cos\lambda\right\}.$$

Replacing a by z and using Theorem 2.5, we get

$$\left|(1-|z|^2)\frac{f_\lambda''(z)}{f_\lambda'(z)} - 2be^{-i\lambda}\overline{z}\cos\lambda\right| \le k|b|\cos\lambda. \tag{2.14}$$

Therefore

$$\left|(1-|z|^2)\frac{zf_\lambda''(z)}{f_\lambda'(z)} - 2be^{-i\lambda}|z|^2\cos\lambda\right| \le k|b||z|\cos\lambda, \quad |z| < 1. \tag{2.15}$$

Letting $c = 2be^{-i\lambda}\cos\lambda$ and by using Ahlfor's [1] criterion for univalence, it follows that f_λ is univalent in $\mathbb{D}$ if $k|b|\cos\lambda < 1$. Dividing both sides of (2.15) by $1-|z|^2$, we get (2.13). □

Remark 2.9. If f is in $\mathcal{V}_k^0(b)$, then f is univalent in $\mathbb{D}$ whenever $|b| < 1/k$ found by Umarani [31].

Corollary 2.10. *If $f_\lambda \in \mathcal{V}_k^\lambda(b)$, $k|b|\cos\lambda < 1$ and $(2Reb - |b|^2)\cos^2\lambda \le 1$, then f_λ maps*

$$|z| \le r_1 = \frac{2}{k|b|\cos\lambda + \sqrt{(k|b|\cos\lambda)^2 - 4\big(2Reb\cos^2\lambda - 1\big)}}. \tag{2.16}$$

onto a convex domain. This result is sharp.

Proof. For $|z| = r < 1$, the inequality in (2.13) gives

$$\left|\left(1 + \frac{zf_\lambda''(z)}{f_\lambda'(z)}\right) - \frac{1 - r^2 + 2br^2e^{-i\lambda}\cos\lambda}{1-r^2}\right| \le \frac{k|b|r\cos\lambda}{1-r^2}.$$

Thus, we get

$$Re\left(1 + \frac{zf_\lambda''(z)}{f_\lambda'(z)}\right) \ge \frac{1 - k|b|r\cos\lambda + \big(2Re(be^{-i\lambda})\cos\lambda - 1\big)r^2}{1-r^2}.$$

Right side of this inequality is positive for $|z| < r_1$, where r_1 is the positive root of the equation

$$1 - k|b|r\cos\lambda + \big(2Re(be^{-i\lambda})\cos\lambda - 1\big)r^2 = 0.$$

Discriminat of this quadratic equation is

$$\Delta = (k|b|\cos\lambda)^2 - 4\big(2Reb\cos^2\lambda - 1\big) \ge 4\big(1 - (2Reb - |b|^2)\cos^2\lambda\big) \ge 0,$$

provided $(2Reb - |b|^2)\cos^2\lambda \le 1$. Therefore, we obtain radius of convexity as given in (2.16). Sharp function is

$$f_\lambda'(z) = \left[\frac{(1+z)^{\frac{k}{2}-1}}{(1-z)^{\frac{k}{2}+1}}\right]^{be^{-i\lambda}\cos\lambda}.$$

Letting $b = 1$ in Corollary 2.10, we obtain the following radius of convexity for the class $\mathcal{V}_k^\lambda$ defined in [15]. □

Corollary 2.11. *If $f_\lambda \in \mathcal{V}_k^\lambda$, then f_λ is convex for*

$$|z| \le r_1 = \frac{2}{k\cos\lambda + \sqrt{(k\cos\lambda)^2 - 4\cos 2\lambda}}.$$

Remark 2.12. If we let $p = 1$ in Corollary 2 in Silvia [30], we observe that Silvia's result reduces to the corresponding result given in Corollary 2.11.

3. Properties of Class $\mathcal{S}_k^\lambda(b)$

For our result in this section, we need the following principal tool that was found in 1969 by Brannan [8].

Lemma 3.1 *The function f of the form (1.1), belongs to $\mathcal{V}_k$ if and only if there are two functions δ_1 and δ_2 normalized and starlike in $\mathbb{D}$ such that*

$$f'(z) = \frac{\left(\frac{\delta_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{\delta_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}}.$$

Theorem 3.2. *If a function f_λ of the form (1.1) belongs to $\mathcal{V}_k^\lambda(b)$, then there exist two normalized $\lambda-$spirallike functions T_1, T_2 in $\mathbb{D}$ such that*

$$f_\lambda'(z) = \left\{ \frac{\left(\frac{T_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{T_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}} \right\}^b. \tag{3.1}$$

Proof. In view of Lemma 3.1, $f \in \mathcal{V}_k$ if and only if

$$f'(z) = \frac{\left(\frac{\delta_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{\delta_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}},$$

where δ_1, δ_2 are normalized starlike functions in $\mathbb{D}$. If $f_\lambda \in \mathcal{V}_k^\lambda(b)$, then due to Theorem 2.3 we can write

$$f_\lambda'(z) = \left\{ \frac{\left(\frac{\delta_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{\delta_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}} \right\}^{be^{-i\lambda}\cos\lambda}. \tag{3.2}$$

It is well-known that if δ is a starlike function in $\mathbb{D}$, then $T(z) = z[\delta(z)/z]^{e^{-i\lambda}\cos\lambda}$ is $\lambda-$spirallike function (see [7]). Now using this representation, we get desired result follows from (3.2). □

Remark 3.3. If we let $b = 1 - \alpha$, there exist $\lambda-$spirallike functions T_1, T_2 in $\mathbb{D}$ such that

$$f_\lambda'(z) = \left\{ \frac{\left(\frac{T_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{T_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}} \right\}^{1-\alpha},$$

which was obtained by Moulis [16].

Using (1.6) and Corollary 2.4, we also obtain the following representation theorem for $\mathcal{S}_k^\lambda(b)$.

Corollary 3.4. *A function $f_\lambda \in \mathcal{S}_k^\lambda(b)$ if and only if there exists a function μ with bounded variation on $[0, 2\pi]$ satisfying conditions in (2.1) such that*

$$f_\lambda(z) = z \exp\left[-be^{-i\lambda}\cos\lambda \int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right]. \tag{3.3}$$

4. Integral Operators

In [9], Breaz and Breaz; in [10], Breaz et al. studied the following integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} ... \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt, \tag{4.1}$$

$$G_n(z) = \int_0^z (f_1'(t))^{\gamma_1} ... (f_n'(t))^{\gamma_n} dt, \tag{4.2}$$

for $\gamma_j > 0$, $1 \leq j \leq n$ and $\sum_{j=1}^n \gamma_j \leq n + 1$. They studied starlikeness and convexity of the operators (4.1) and (4.2). In this section, we investigate these integral operators (4.1) and (4.2) for the classes $\mathcal{S}_k^\lambda(b)$ and $\mathcal{V}_k^\lambda(b)$.

Theorem 4.1. *Let $f_j \in \mathcal{S}_k^\lambda(b)$ for $1 \leq j \leq n$ with $k \geq 2$, $b \in \mathbb{C} - \{0\}$. Also, let λ be a real number with $|\lambda| < \frac{\pi}{2}$, $\gamma_j > 0$ $(1 \leq j \leq n)$. Then $F_n \in \mathcal{V}_k^\lambda(\mu)$ for each positive integer n, where $\mu = b\sum_{j=1}^n \gamma_j$.*

Proof. Since $f_j(z) = z + \sum_{n=2}^\infty a_{n,j} z^n$, we have $\frac{f_j(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Therefore by (4.1), we have

$$\frac{F_n''(z)}{F_n'(z)} = \sum_{j=1}^n \gamma_j \left(\frac{f_j'(z)}{f_j(z)} - \frac{1}{z}\right),$$

or equivalently

$$\begin{aligned} & e^{i\lambda}\left(1 + z\frac{F_n''(z)}{F_n'(z)}\right) \\ &\quad = e^{i\lambda} + \sum_{j=1}^n \gamma_j e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - \sum_{j=1}^n \gamma_j e^{i\lambda} \\ &\quad = e^{i\lambda} + (1-b)\cos\lambda \sum_{j=1}^n \gamma_j + \sum_{j=1}^n \gamma_j \left(e^{i\lambda}\frac{zf_j'(z)}{f_j(z)} - (1-b)\cos\lambda\right) - \sum_{j=1}^n \gamma_j e^{i\lambda}. \end{aligned}$$

Taking real part on both sides, we get

$$Re\left(e^{i\lambda}(1 + z\frac{F_n''(z)}{F_n'(z)}) - (1 - b\sum_{j=1}^n \gamma_j)\cos\lambda - i\sin\lambda\right) = \sum_{j=1}^n \gamma_j Re\left(e^{i\lambda}\frac{zf_j'(z)}{f_j(z)} - (1-b)\cos\lambda - i\sin\lambda\right). \tag{4.3}$$

On the other hand, since $f_j \in \mathcal{S}_k^\lambda(b)$ for $1 \le j \le n$ and by using (1.5), we have

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - (1-b)\cos\lambda - i\sin\lambda}{b} \right) \right| d\theta \le k\pi\cos\lambda. \tag{4.4}$$

On letting $\mu = b\sum_{j=1}^n \gamma_j$, (1.4), (4.3) and (4.4) yield

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}\big(1 + z\frac{F_n''(z)}{F_n'(z)}\big) - (1 - b\sum_{j=1}^n \gamma_j)\cos\lambda - i\sin\lambda}{b} \right) \right| d\theta$$
$$\le \sum_{j=1}^n \gamma_j \int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - (1-b)\cos\lambda - i\sin\lambda}{b} \right) \right| d\theta \le k\pi\cos\lambda.$$

Then from above inequality, we obtain

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}\big(1 + z\frac{F_n''(z)}{F_n'(z)}\big) - (1-\mu)\cos\lambda - i\sin\lambda}{\mu} \right) \right| d\theta \le k\pi\cos\lambda.$$

In view of (1.4), it follows that $F_n \in \mathcal{V}_k^\lambda(\mu)$ for each positive integer n. □

We deduce the following known result from Theorem 4.1. We observed in Section 1 that $\mathcal{S}_k^0(1-\alpha) = \mathcal{S}_k(\alpha)$ and $\mathcal{V}_k^0(1-\alpha) = \mathcal{V}_k(\alpha)$ for $0 \le \alpha < 1$ and $k \ge 2$. By letting $\lambda = 0$ and $b = 1-\alpha$, Theorem 4.1 gives the following result obtained in [18].

Corollary 4.2. *Let $f_j \in \mathcal{S}_k(\alpha)$ for $1 \le j \le n$ with $0 \le \alpha < 1$ and $k \ge 2$. Also, let $\gamma_j > 0$ $(1 \le j \le n)$.If*

$$0 \le 1 + (\alpha - 1)\sum_{j=1}^n \gamma_j < 1,$$

then $F_n \in \mathcal{V}_k(\beta)$ with $\beta = 1 + (\alpha-1)\sum_{j=1}^n \gamma_j$.

Remark 4.3. For $n = 1$ and $\gamma_1 = 1$, Theorem 4.1 proves that if $f_1 \in \mathcal{S}_k^\lambda(b)$ with $k \ge 2$, then the integral operator

$$F_1(z) = \int_0^z \frac{f_1(z)}{z} dz \in \mathcal{V}_k^\lambda(b).$$

In particular, for $n = 1$, $\gamma_1 = 1$, $\lambda = 0$, $b = 1$ and $k = 2$, Theorem 4.1 proves that if $f_1 \in \mathcal{S}^*$, then $\int_0^z \frac{f_1(z)}{z} dz \in \mathcal{K}$. This is the famous result found by Alexander [4].

Theorem 4.4. *Let $f_j \in \mathcal{V}_k^\lambda(b)$ for $1 \le j \le n$ with $k \ge 2$, $b \in \mathbb{C} - \{0\}$. Also, let λ be a real number with $|\lambda| < \frac{\pi}{2}$, $\gamma_j > 0$ $(1 \le j \le n)$. Then the integral operator $G_n \in \mathcal{V}_k^\lambda(\mu)$ for each positive integer n, where $\mu = b\sum_{j=1}^n \gamma_j$.*

Proof. In view of (4.2), we have

$$\frac{G_n''(z)}{G_n'(z)} = \gamma_1 \frac{f_1''(z)}{f_1'(z)} + ... + \gamma_n \frac{f_n''(z)}{f_n'(z)},$$

or equivalently

$$e^{i\lambda}\left(1 + z\frac{G_n''(z)}{G_n'(z)}\right) = e^{i\lambda} + \sum_{j=1}^{n} \gamma_j e^{i\lambda}\left(1 + \frac{zf_j''(z)}{f_j'(z)}\right) - \sum_{j=1}^{n} \gamma_j e^{i\lambda}.$$

Subtracting and adding $(1-b)\cos\lambda \sum_{j=1}^{n} \gamma_j$ on the right hand side, taking real part on both sides, and simplifying, we get

$$Re\left(e^{i\lambda}\big(1 + z\frac{G_n''(z)}{G_n'(z)}\big) - (1 - b\sum_{j=1}^{n}\gamma_j)\cos\lambda - i\sin\lambda\right) \tag{4.5}$$
$$= \sum_{j=1}^{n} \gamma_j Re\left(e^{i\lambda}\left(1 + \frac{zf_j''(z)}{f_j'(z)}\right) - (1-b)\cos\lambda - i\sin\lambda\right).$$

Since $f_j \in \mathcal{V}_k^{\lambda}(b)$ for $1 \le j \le n$ and by using (1.4), we obtain

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}\left(1 + \frac{zf_j''(z)}{f_j'(z)}\right) - (1-b)\cos\lambda - i\sin\lambda}{b}\right)\right| d\theta \le k\pi\cos\lambda. \tag{4.6}$$

Therefore letting $\mu = b\sum_{j=1}^{n} \gamma_j$, (1.4), (4.5) and (4.6) give

$$\int_0^{2\pi} \left| Re\left(\frac{e^{i\lambda}\left(1 + z\frac{G_n''(z)}{G_n'(z)}\right) - (1-\mu)\cos\lambda - i\sin\lambda}{\mu}\right)\right| d\theta \le k\pi\cos\lambda.$$

It now follows from (1.4) that $G_n \in \mathcal{V}_k^{\lambda}(\mu)$ for each positive integer n. □

Remark 4.5. For $n = 1$ and $\gamma_1 = 1$, Theorem 4.4 proves that $f_1 \in \mathcal{V}_k^{\lambda}(b)$ with $k \ge 2$, then the integral operator

$$F_1(z) = \int_0^z f_1'(z)dz \in \mathcal{V}_k^{\lambda}(b).$$

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