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On the Boundedness of Marcinkiewicz Integrals on Variable Exponent Herz-type Hardy Spaces

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Abstract. The aim of this paper is to prove that Marcinkiewicz integral operators are bounded from $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ when the parameters $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions. Also, we prove the boundedness of μ on variable Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

1. Introduction and Preliminaries

Function spaces with variable exponent are being actively studied not only in the field of real analysis but also in partial differential equations and in applied mathematics. The theory of function spaces with variable exponents has rapidly made progress in the last three decades.

For $0 < \beta \leq 1$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathbb{R}^n) := \left\{ f : \|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Given $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega\left(x'\right) \, \mathrm{d}\sigma(x') = 0$$

where $x' = x/|x|$ for any $x \neq 0$ and S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure.

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The Marcinkiewicz integral μ is defined by

$$\mu(f)(x) := \left(\int_0^\infty |F_\Omega f(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}$$

where

$$F_\Omega f(x) := \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\ \mathrm{d}y.$$

It is well known that the operator μ was first defined by Stein [13] and under the conditions above, Stein proved that μ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$. Benedek et al. [2] showed that μ of type (p,p) with $1 < p < \infty$.

Recently, the boundedness of Marcinkiewicz integral operators μ on variable function spaces have attracted great attention (see [14, 15, 18] and their references).

The purpose of this paper is to generalize some results concerning Marcinkiewicz integral operators μ on variable Herz spaces $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and variable Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. We define the set of variable exponents by

$$\mathcal{P}_0(\mathbb{R}^n) := \{p \text{ measurable: } p(\cdot) : \mathbb{R}^n \to [c, \infty[\text{ for some } c > 0\}.$$

The subset of variable exponents with range $[1,\infty)$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, we use the notation

$$p^- = ess \inf_{x\in\mathbb{R}^n} p(x), \quad p^+ = ess \sup_{x\in\mathbb{R}^n} p(x).$$

Definition 1.1. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This space is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\mu > 0 : \varrho_{p(\cdot)}(\frac{1}{\mu} f) \leq 1\right\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

Definition 1.2. We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-*Hölder continuous at the origin* (or has a log *decay at the origin*). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-*Hölder continuous at infinity* (or has a log *decay at infinity*).

As an example of a function locally log-Hölder continuous, see E. Nakai and Y. Sawano [12, Example 1.3].

The set $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ consist of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The set $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$.

It is well known that if $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ then $p' \in \mathcal{P}^{\log}(\mathbb{R}^n)$, where p' denotes the conjugate exponent of p given by $1/p(\cdot) + 1/p'(\cdot) = 1$.

Definition 1.3. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The *mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \le 1 \right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \gamma > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\gamma}(f_v)_v\Big) \le 1 \right\}.$$

Since $q^+ < \infty$, then we can replace by the simpler expression $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum\limits_v \big\| |f_v|^{q(\cdot)} \big\|_{\frac{p(\cdot)}{q(\cdot)}}$.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. Before giving the definition of variable Herz spaces, let us introduce the following notations

$$B_k := B(0, 2^k)\,, \quad R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}\,, \quad k \in \mathbb{Z}.$$

Definition 1.4. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The *inhomogeneous Herz space* $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\| \Big(2^{k\alpha(\cdot)} f\,\chi_k\Big)_{k\ge 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the *homogeneous Herz space* $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} := \left\| \Big(2^{k\alpha(\cdot)} f\,\chi_k\Big)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The variable Herz spaces $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [8]. In [6] the authors obtained a new equivalent norm of these function spaces. We refer the reader to the paper [5, 16] for further results for these function spaces. If $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are constants, then $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ is the classical Herz space $\dot{K}_{q}^{\alpha,p}(\mathbb{R}^n)$.

The following proposition is very important for the proof of the main results in this paper; it is from D. Drihem and F. Seghiri in [6].

Proposition 1.5. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p,q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(\mathbb{R}^n).$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ have a log *decay at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \left(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f \chi_k\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f \chi_k\|_{p(\cdot)}^{q_\infty}\right)^{1/q_\infty}.$$

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. It was shown in [4, Theorem 4.3.8] that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$, see also [3, Theorem 1.2].

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with supp $\varphi \subseteq B_0$, $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$ and $\varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right)$ for any $t > 0$. Let $\mathcal{M}_\varphi(f)$ be the grand maximal function of f defined by

$$\mathcal{M}_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

Here we give the definition of the homogeneous Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$.

Definition 1.6. Let $p,q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The *homogeneous Herz-type Hardy space* $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\varphi(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \|\mathcal{M}_\varphi(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

It can be shown that if $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ satisfy the conditions of Definition 1.6, then the quasi-norm $\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}$ does not depend, up to the equivalence of quasi-norms, on the choice of the function φ and, hence, the space $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined independently of the choice of φ. If $p \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ with $-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n - \frac{n}{p^-}$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ then $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. If $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$ then $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ coincide with $L^{p(\cdot)}(\mathbb{R}^n)$.

One recognizes immediately that if $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are constants, then the spaces $H\dot{K}^{\alpha,q}_p$ are just the usual Herz-type Hardy spaces were recently studied in [10] and [11].

We refer the reader to the recent monograph [4, section 4.5] for further details, historical remarks and more references on variable exponent spaces.

2. Some Technical Lemmas

In this section, we present six lemmas used to prove our main theorems in Section 3. Recall that the expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

Lemma 2.1 plays an important role in the proof of main results; Lemma 2.2 is a Hardy-type inequality which is easy to prove; Lemma 2.3 presents the Hölder inequality in $L^{p(\cdot)}(\mathbb{R}^n)$; Lemma 2.4 presents the $L^{p(\cdot)}$-boundedness of μ ; Lemma 2.5 treats the boundedness of fractional integral on variable Lebesgue space; and the last Lemma presents the boundedness of homogeneous function of degree zero.

Lemma 2.1.([1]) *Let $p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and let $R = B(0,r) \setminus B(0,\frac{r}{2})$. If $|R| \geq 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.
The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$.

Lemma 2.2.([5]) *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leq c\,I,$$

with $c > 0$ only depending on a and q.

Lemma 2.3.([4]) *Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, $fg \in L^1(\mathbb{R}^n)$ and*

$$\|fg\|_1 \leq 2\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)}\,.$$

Lemma 2.4.([9]) *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then there exists a constant C such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$*

$$\|\mu(f)\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Lemma 2.5.([17]) *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$. Then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|\cdot - y|^{n-\sigma}} dy \right\|_{p_2(\cdot)} \leq c \|f\|_{p_1(\cdot)}.$$

Lemma 2.6.([14]) *If $a > 0, 1 \leq s \leq \infty, 0 < d \leq s$ and $-n + (n-1)d/s < \tau < \infty$, then*

$$\left(\int_{|y| \leq a|x|} |y|^{\tau} |\Omega(x-y)|^d \, dy \right)^{\frac{1}{d}} \leq c\,|x|^{(\tau + n/d)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})}.$$

3. Variable Herz Estimate of Marcinkiewicz Integral Operators

In this section, we present two results concerning the Marcinkiewicz integral operator μ. In the first, we show that μ is bounded from $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ for $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions. Next, we present the boundedness of μ on variable Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Motivated by [9] and [14], we generalize the boundedness for Marcinkiewicz integral operators μ to the case of variable Herz spaces (all exponents are variables). One of our main results can be stated as follows.

Theorem 3.1. *Suppose that $0 < \tau \leq 1, p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty, \Omega \in L^s(\mathrm{S}^{n-1}), s > (p')^-$ and $\alpha \in L^\infty(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q have a* log *decay at the origin such that*

$$-\frac{n}{p(0)} - \frac{n}{s} - \tau < \alpha(0) < n - \frac{n}{p(0)} - \frac{n}{s} - \tau \text{ and } -\frac{n}{p_\infty} - \frac{n}{s} - \tau < \alpha_\infty < n - \frac{n}{p_\infty} - \frac{n}{s} - \tau$$

then μ is bounded from $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ (or $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$) to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ (or $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$).

Remark 3.2. We would like to mention if $\alpha(\cdot)$ and $q(\cdot)$ are constants, then the statements corresponding to Theorem 3.1 can be found in Theorem 2.1 of [14].

In the following, we use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

Proof of Theorem 3.1. We show that

$$\|\mu(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\,\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

for all $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$. Using Proposition 1.5, we have $\mu(f)$ in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$-norm is equivalent to

$$\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|\mu(f)\chi_k\|_{p(\cdot)}^{q(0)}\right\}^{1/q(0)} + \left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty(q)_\infty} \|\mu(f)\chi_k\|_{p(\cdot)}^{(q)_\infty}\right\}^{1/(q)_\infty},$$

we write

$$f = \sum_{j\in\mathbb{Z}} f\chi_j = \sum_{j\in\mathbb{Z}} f_j,$$

then

$$\begin{aligned} \|\mu(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=-\infty}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\ & + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k-2}^{k+1} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\ & + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k+2}^{\infty} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\ & + \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=-\infty}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\ & + \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k-2}^{k+1} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\ & + \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k+2}^{\infty} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\ = & : H_1 + H_2 + H_3 + H_4 + H_5 + H_6. \end{aligned}$$

Let us estimate H_1 and H_4. We consider

$$\begin{aligned} |\mu f_j(x)| \leq & \left(\int_0^{|x|} \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\, \mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\ & + \left(\int_{|x|}^{\infty} \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\, \mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\ : & = I_1 + I_2. \end{aligned}$$

Observe that in this case $x \in R_k$, $y \in R_j$ and $j \leq k-2$. So we know that $|x-y| \approx |x| \approx 2^k$, and by mean value theorem, we have

$$\left|\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\right| \leq \frac{c\,|y|}{|x-y|^3}. \tag{3.1}$$

By (3.1), the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_j(y)| \left|\int_{|x-y|}^{|x|} \frac{\mathrm{d}t}{t^3}\right|^{\frac{1}{2}} \mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_j(y)| \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} \mathrm{d}y \\ &\lesssim \frac{2^{j/2}}{|x|^{n+1/2}} \int_{\mathbb{R}^n} |\Omega(x-y)| |f_j(y)| \mathrm{d}y \\ &\lesssim 2^{-nk} \|f_j\|_{p(\cdot)} \|\Omega(x-\cdot)\chi_j\|_{p'(\cdot)}. \end{aligned} \tag{3.2}$$

The estimation of I_2 is the same as before since we never use $|x-y| \approx |x|$, then we obtain

$$I_2 \lesssim 2^{-nk} \|f_j\|_{p(\cdot)} \|\Omega(x-\cdot)\chi_j\|_{p'(\cdot)}.$$

We can obtain that each term (I_1 and I_2) is no more than

$$c2^{-nk} \|f_j\|_{p(\cdot)} \|\Omega(x-\cdot)\chi_j\|_{p'(\cdot)},$$

in the other hand, by Hölder inequality and Lemma 2.6, we obtain

$$\begin{aligned} \|\Omega(x-\cdot)\chi_j\|_{p'(\cdot)} &\leq \|\Omega(x-\cdot)\chi_j\|_s \|\chi_j\|_{\theta(\cdot)} \\ &\leq 2^{-j\tau}\left(\int_{R_j} |y|^{s\tau} |\Omega(x-y)|^s dy\right)^{\frac{1}{s}} \|\chi_j\|_{\theta(\cdot)} \\ &\lesssim 2^{-j\tau} 2^{k(\tau+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|\chi_j\|_{\theta(\cdot)} \end{aligned}$$

where $\frac{1}{p'(\cdot)} = \frac{1}{s} + \frac{1}{\theta(\cdot)}$. Since $\theta \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, we have for any j

$$\|\chi_j\|_{\theta(\cdot)} \approx \|\chi_j\|_{p'(\cdot)} |R_j|^{-\frac{1}{s}},$$

which gives

$$\|\mu(f_j)\chi_k\|_{p(\cdot)} \lesssim 2^{-nk} 2^{-(\frac{n}{s}+\tau)(j-k)} \|f_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \|\chi_k\|_{p(\cdot)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})}. \tag{3.3}$$

Estimation of H_1. In this case, since k and j are negative integers, by Lemma 2.1 and since $\Omega \in L^s(\mathrm{S}^{n-1})$, we have

$$\begin{aligned} \|\mu(f_j)\chi_k\|_{p(\cdot)} &\lesssim 2^{(n-\frac{n}{p(0)}-\tau-\frac{n}{s})(j-k)} \|f_j\|_{p(\cdot)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\ &\lesssim 2^{(n-\frac{n}{p(0)}-\tau-\frac{n}{s})(j-k)} \|f_j\|_{p(\cdot)}, \end{aligned}$$

which gives

$$\begin{aligned} H_1 &\lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n-\frac{n}{p(0)}-\tau-\frac{n}{s})q(0)} \left\| f_j \right\|_{p(\cdot)}^{q(0)} \right) \right\}^{1/q(0)} \\ &= c \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n-\alpha(0)-\frac{n}{p(0)}-\tau-\frac{n}{s})q(0)} \left(2^{j\alpha(0)q(0)} \left\| f_j \right\|_{p(\cdot)}^{q(0)} \right) \right) \right\}^{1/q(0)}, \end{aligned}$$

since $\alpha(0) - n + \frac{n}{p(0)} - \frac{n}{s} - \tau < 0$, then by Lemma 2.2, we have

$$H_1 \leq c \left\{ \sum_{j=-\infty}^{-1} 2^{j\alpha(0)q(0)} \left\| f_j \right\|_{p(\cdot)}^{q(0)} \right\}^{1/q(0)} \leq c \left\| f \right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} .$$

Estimation of H_4. We split

$$\sum_{j=-\infty}^{k-2} \left\| \mu(f_j)\chi_k \right\|_{p(\cdot)} = \sum_{j=-\infty}^{-1} \cdots + \sum_{j=0}^{k-2} \cdots$$

then H_4 can be estimated by

$$H_4^1 + H_4^2,$$

where

$$H_4^1 := \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=-\infty}^{-1} \left\| \mu(f_j)\chi_k \right\|_{p(\cdot)} \right)^{q_\infty} \right\}^{1/q_\infty}$$

and

$$H_4^2 := \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=0}^{k-2} \left\| \mu(f_j)\chi_k \right\|_{p(\cdot)} \right)^{q_\infty} \right\}^{1/q_\infty}$$

Let estimate H_4^1. We have in this case $j < 0 \leq k$. By Lemma 2.1, we obtain

$$\left\| \mu(f_j)\chi_k \right\|_{p(\cdot)} \leq c 2^{k(\frac{n}{p_\infty}-n+\frac{n}{s}+\tau)} 2^{j(n-\frac{n}{s}-\tau-\frac{n}{p(0)})} \left\| f_j \right\|_{p(\cdot)}$$

therefore, H_4^1 is bounded by

$$c \sup_{j\leq 0} 2^{\alpha(0)} \left\| f_j \right\|_{p(\cdot)} \left\{ \sum_{k=0}^{\infty} 2^{k(\alpha_\infty+\frac{n}{p_\infty}-n+\frac{n}{s}+\tau)q_\infty} \left(\sum_{j=-\infty}^{-1} 2^{j(n-\frac{n}{s}-\tau-\frac{n}{p(0)})} \right)^{q_\infty} \right\}^{1/q_\infty},$$

by embedding $\ell^{q(0)} \hookrightarrow \ell^\infty$ and since $\alpha_\infty + \frac{n}{p_\infty} - n + \frac{n}{s} + \tau < 0 < n - \frac{n}{s} - \tau - \frac{n}{p(0)}$, we have

$$H_4^1 \leq c \left(\sum_{j=-\infty}^{-1} 2^{\alpha(0)q(0)j} \left\| f_j \right\|_{p(\cdot)}^{q(0)} \right)^{1/q(0)} \leq c \left\| f \right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} .$$

We can estimate H_4^2 by the same argument used in the estimation of H_1 if we replace $\alpha(0)$, $p(0)$ and $q(0)$ by α_∞, p_∞ and q_∞ respectively, then H_4^2 is bounded by

$$\left\{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-2}2^{(j-k)(n-\frac{n}{p_\infty}-\tau-\frac{n}{s}-\alpha_\infty)}\left(2^{j\alpha_\infty}\left\|f_j\right\|_{p(\cdot)}\right)\right)^{q_\infty}\right\}^{1/q_\infty}$$

$$\leq \quad c\left(\sum_{k=0}^{\infty}2^{j\alpha_\infty q_\infty}\left\|f_j\right\|_{p(\cdot)}^{q_\infty}\right)^{1/q_\infty} \leq c\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

Let us estimate H_2+H_5. By the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of μ, we have

$$\begin{aligned} H_2+H_5 &\lesssim \left(\sum_{k=-\infty}^{-1}\|2^{k\alpha(0)}f\chi_k\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)}+\left(\sum_{k=0}^{\infty}\|2^{k\alpha_\infty}f\chi_k\|_{p(\cdot)}^{q_\infty}\right)^{1/q_\infty} \\ &\lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Let us estimate H_3. It is possible to prove the following estimation (similar to the estimate for I_1 and I_2)

$$|\mu(f_j)| \leq c2^{-nj}\left\|f_j\right\|_{p(\cdot)}\left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)}, \tag{3.4}$$

for the detailed proof of this estimation, see [14, p.259-260]. By Hölder inequality and Lemma 2.6, the right-hand of (3.4) is bounded by

$$c2^{-j\tau}\left(\int_{R_j}|y|^{s\tau}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}\left\|\chi_j\right\|_{\theta(\cdot)} \lesssim 2^{-nj}2^{k(\tau+\frac{n}{s})}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\Omega\right\|_{L^s(\mathrm{S}^{n-1})},$$

which gives

$$\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)} \lesssim 2^{-nj}2^{(k-j)(\tau+\frac{n}{s})}\left\|f_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}.$$

We split

$$\sum_{j=k+2}^{\infty}\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)}=\sum_{j=k+2}^{-1}\cdots+\sum_{j=0}^{\infty}\cdots$$

Then H_3 is bounded by

$$\begin{aligned} &\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\left(\sum_{j=k+2}^{-1}\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\ &+\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\left(\sum_{j=0}^{\infty}\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\ =&: H_3^1+H_3^2 \end{aligned}$$

For H_3^1, since j and k are negative integers, we have

$$H_3^1 \lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=k+2}^{-1} 2^{(\alpha(0)+\frac{n}{p(0)}+\tau+\frac{n}{s})(k-j)} 2^{j\alpha(0)} \|f_j\|_{p(\cdot)} \right)^{q(0)} \right\}^{1/q(0)}$$

Since $\alpha(0) + \frac{n}{p(0)} + \tau + \frac{n}{s} > 0$, by Lemma 2.2, we obtain

$$\begin{aligned} H_3^1 &\lesssim \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|f\chi_k\|_{p(\cdot)}^{q(0)} \right)^{1/q(0)} \\ &\lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} . \end{aligned}$$

For H_3^2, since $k < 0 \le j$, then we have

$$\begin{aligned} \|\mu(f_j)\chi_k\|_{p(\cdot)} &\le c2^{(k-j)(\tau+\frac{n}{s})} \|f_j\|_{p(\cdot)} 2^{-\frac{nj}{p_\infty}} 2^{\frac{nk}{p(0)}} \\ &\le c2^{k(\tau+\frac{n}{s}+\frac{n}{p(0)})} 2^{j\alpha_\infty} \|f_j\|_{p(\cdot)} 2^{-j(\frac{n}{p_\infty}+\alpha_\infty+\tau+\frac{n}{s})}, \end{aligned}$$

by Hölder inequality in ℓ^1 and since $\gamma = \frac{n}{p_\infty} + \alpha_\infty + \tau + \frac{n}{s} > 0$, we have

$$\begin{aligned} H_3^2 &\lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{k(\tau+\frac{n}{s}+\frac{n}{p(0)})} \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty} \|f_j\|_{p(\cdot)} 2^{-j\gamma} \right)^{q(0)} \right\}^{1/q(0)} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} 2^{k\eta q(0)} \right)^{1/q(0)} \left(\sum_{j=0}^{\infty} 2^{-j\gamma q'_\infty} \right)^{1/q'_\infty} \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right)^{1/q_\infty} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right)^{1/q_\infty} \\ &\lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} , \end{aligned}$$

where $\eta = \tau + \frac{n}{s} + \frac{n}{p(0)} > 0$.
Let us estimate H_6. In this case, since k and j are non-negative integers, by (3.3) and Lemma 2.1, we have

$$\|\mu(f_j)\chi_k\|_{p(\cdot)} \le c2^{(-n+\frac{n}{p_\infty}+\tau+\frac{n}{s})(k-j)} \|f_j\|_{p(\cdot)} ,$$

which gives

$$H_6 \lesssim \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n-\gamma)q_\infty} \left(2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right) \right) \right\}^{1/q_\infty} ,$$

since $n-\gamma>0$, by Lemma 2.2, we obtain

$$H_6 \leq \left\{ \sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right\}^{1/q_\infty} \leq c\, \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \qquad \square$$

Remark 3.3. A non-homogeneous counterpart of Theorem 3.1 is available. Since $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(\mathbb{R}^n)$, their proof is an immediate consequence of [14, Theorem 2.1].

To prove the Theorem 3.6, we need the notation of atomic decomposition.

Definition 3.4. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ and $m \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-atom, if

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\}, r > 0.$

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha(0)/n}, \qquad 0 < r < 1.$

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha_\infty/n}, \qquad r \geq 1.$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leq m.$

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\operatorname{supp} a \subset B(0,r), r \geq 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 3.4, then the corresponding central $(\alpha(\cdot), p(\cdot))$-atom is called a dyadic central $(\alpha(\cdot), p(\cdot))$-atom.

The following theorem presents the atomic decomposition characterization of variable Herz-type Hardy spaces, see [6].

Theorem 3.5. *Let α and q are be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \leq c\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1 - \frac{1}{p^-})$ and $m \geq [\alpha^+ + n(\frac{1}{p^-} - 1)]$, and if holds, then $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \inf \left\{ \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \right\},$$

where the infimum is taken over all the decompositions of f as above.

In the next result we treat the boundedness of Marcinkiewicz integral operators with homogeneous kernel on variable Herz-type Hardy spaces.

Theorem 3.6. *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < 2n$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{1}{2n}$, $\alpha \in L^\infty(\mathbb{R}^n)$, $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n), \Omega \in L^s(\mathrm{S}^{n-1})$ with $s > (p_1')^-$. If α, q_1 and q_2 are* log-*Hölder continuous, both at the origin and at infinity such that*

$$\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-}), q_1(0) \leq q_2(0) \text{ and } (q_1)_\infty \leq (q_2)_\infty .$$

Then μ is bounded from $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)$.

Proof. We must show that

$$\|\mu(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$. Using Theorem 3.5, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and a_i's are $(\alpha(\cdot), p_1(\cdot))$- atom with supp$a_i \subseteq B_i$. Using Proposition 1.5, we have

$$\begin{aligned}
&\|\mu(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)} \\
&\approx \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \|\mu(f)\chi_k\|_{p_2(\cdot)}^{q_2(0)} \right\}^{1/q_2(0)} + \left\{ \sum_{k=0}^{+\infty} 2^{k\alpha_\infty (q_2)_\infty} \|\mu(f)\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty} \right\}^{1/(q_2)_\infty} \\
&\leq \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left(\sum_{i=-\infty}^{k-3} |\lambda_i| \, \|\mu(a_i)\chi_k\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \\
&\quad + \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left(\sum_{i=k-2}^{\infty} |\lambda_i| \, \|\mu(a_i)\chi_k\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \\
&\quad + \left\{ \sum_{k=0}^{+\infty} 2^{k\alpha_\infty (q_2)_\infty} \left(\sum_{i=-\infty}^{k-3} |\lambda_i| \, \|\mu(a_i)\chi_k\|_{p_2(\cdot)} \right)^{(q_2)_\infty} \right\}^{1/(q_2)_\infty} \\
&\quad + \left\{ \sum_{k=0}^{+\infty} 2^{k\alpha_\infty (q_2)_\infty} \left(\sum_{i=k-2}^{+\infty} |\lambda_i| \, \|\mu(a_i)\chi_k\|_{p_2(\cdot)} \right)^{(q_2)_\infty} \right\}^{1/(q_2)_\infty} \\
&=: F_1 + F_2 + F_3 + F_4.
\end{aligned}$$

Let us estimate F_1. We consider

$$\begin{aligned} |\mu(a_i)(x)| &\leq \left(\int_0^{|x|}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_i(y)\ \mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\ &\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_i(y)\ \mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\ : &= Q_1+Q_2. \end{aligned}$$

Observe that in this case $x\in R_k$, $y\in B_i$ and $i\leq k-3$. So we know that $|x-y|\approx|x|\approx 2^k$. By (3.2), we have

$$Q_1\lesssim\int_{B_i}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}}\,|a_i(y)|\ \mathrm{d}y,$$

by the m-order vanishing moments of a_i with $m\geq\left[\alpha^+-n(1-\frac{1}{p_1^-})\right]$, we can subtract the Taylor expansion of $|x-y|^{\frac{1}{2}-n}$ at x, we obtain

$$\begin{aligned} Q_1 &\leq \int_{B_i}\frac{|y|^{m+1}}{|x|^{n+m+\frac{1}{2}}}\,|a_i(y)|\,|\Omega(x-y)|\,dy \\ &\leq c2^{-k(n+m+\frac{1}{2})+i(m+1)}\int_{B_i}|a_i(y)|\,|\Omega(x-y)|\,dy, \end{aligned}$$

The estimation of Q_2 is the same as before since we never use $|x-y|\approx|x|$, we have

$$|\mu(a_i)(x)|\leq c2^{-k(n+m+\frac{1}{2})+i(m+1)}\int_{B_i}|a_i(y)|\,|\Omega(x-y)|\,dy,$$

as the same reason in the proof of (3.3), we get

$$\begin{aligned} \|\mu(a_i)\chi_k\|_{p_2(\cdot)} &\lesssim 2^{-(n-\frac{1}{2})k}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\|\chi_i\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\ &\lesssim 2^{-(n-\frac{1}{2})k}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\|\chi_i\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}, \end{aligned} \tag{3.5}$$

where $\beta=(1+m-\frac{n}{s}-\tau)$.
On the other hand (see [7, p.350] for $\sigma=\frac{1}{2}$), we have

$$\int_{B_k}\frac{\chi_{B_k}(y)}{|x-y|^{n-\frac{1}{2}}}dy\geq\int_{B_k}\frac{dy}{|x-y|^{n-\frac{1}{2}}}\chi_{B_k}(x)\geq c2^{\frac{k}{2}}\chi_{B_k}(x). \tag{3.6}$$

(3.5), (3.6) and Lemma 2.5, gives

$$\begin{aligned}\|\mu(a_i)\chi_k\|_{p_2(\cdot)} &\le 2^{-(n-\frac{1}{2})k}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\\ &\le 2^{-nk}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)}\left\|\int_{B_k}\frac{\chi_{B_k}(y)\,dy}{|\cdot-y|^{n-\frac{1}{2}}}\right\|_{p_2(\cdot)}\\ &\le 2^{-nk}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_{B_k}\|_{p_1(\cdot)},\end{aligned}$$

by Lemma 2.1, we have

$$\begin{aligned}F_1 &= \left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\ &\le c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,2^{(i-k)(\beta-(\alpha+n/p_1)(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)},\end{aligned}$$

since we can choose m large enough such that $\beta-\alpha^{+}+n(1-\frac{1}{p_1^-})>0$, by Lemma 2.2, we obtain

$$F_1\le c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\le c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\le c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}}.$$

Let us estimate F_2. By Lemma 2.4 and applying the size condition of a_i (conditions (ii) and (iii) in Definition 3.4), we have

$$\begin{aligned}F_2 &= \left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{+\infty}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\ &\le c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\ &\le c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{-1}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\ &\quad +c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\end{aligned}$$

$$\leq \quad c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=k-2}^{-1}|\lambda_i|\,2^{(k-i)\alpha(0)}\right)^{q_2(0)}\right\}^{1/q_2(0)}$$
$$+c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,2^{(k-i)\alpha^-+k(\alpha(0)-\alpha^-)+i(\alpha^--\alpha_\infty)}\right)^{q_2(0)}\right\}^{1/q_2(0)}$$

for $k<0\leq i$ and since $\alpha^-\leq\min(\alpha\,(0)\,,\alpha_\infty)$, we have

$$k(\alpha\,(0)-\alpha^-)+i(\alpha^--\alpha_\infty)\leq 0.$$

By Lemma 2.2, we obtain

$$F_2\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

We can estimate F_3 and F_4 by the same arguments used in the estimation of F_1 and F_2 if we replace $\alpha(0)$, $p_2(0)$ and $q_2(0)$ by α_∞, $(p_2)_\infty$ and $(q_2)_\infty$ respectively.

A combination of estimations of F_1, F_2, F_3 and F_4 completes the proof of Theorem 3.6. □

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