KYUNGPOOK Math. J. 59(2019), 241-258
https://doi.org/10.5666/KMJ.2019.59.2.241
pISSN 1225-6951 eISSN 0454-8124

Null Controllability of Semilinear Integrodifferential Control Systems in Hilbert Spaces

Ah-ran Park and Jin-Mun Jeong*
Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
e-mail : alanida@naver.com *and* jmjeong@pknu.ac.kr

Abstract. In this paper, we deal with the null controllability of semilinear functional integrodifferential control systems under the Lipschitz continuity of nonlinear terms. Moreover, we establish the regularity and a variation of constant formula for solutions of the given control systems in Hilbert spaces.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. The norm on V (resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$) respectively. Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality, and generate an analytic semigroup $(S(t))_{t\geq 0}$. We study the following the semilinear functional integrodifferential control systems :

$$
(1.1) \qquad \begin{cases} x'(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s))ds + Bu(t), \\ x(0) &= x_0. \end{cases}
$$

Here, a forcing term $k \in L^2(0,T;V^*)$, $x_0 \in H$, and $g : R^+ \times V \to H$ is a nonlinear mapping as detailed in Section 2.. The controller B is a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, where U is some Banach space of control variables.

The existence of solutions for a class of semilinear functional integrodifferential

---

* Corresponding Author.

Received March 8, 2018; revised June 18, 2018; accepted June 26, 2018.
2010 Mathematics Subject Classification: primary 93C20; secondary 45K05.
Key words and phrases: controllability, semilinear control system, regularity for solution, analytic semigroup, integrodifferential control system.
This research was supported by Basic Science Research Program through the National research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2019R1F1A1048077).

control systems has been studied by many authors. For example, one finds parabolic type problems in [3, 17, 18], hyperbolic type problems in [11, 18], and linear cases in [4, 5, 7, 13, 14]. The background of these problems is to serve as an initial value problem for many partial integrodifferential equations which arise in problems connected with heat flow in materials, random dynamical systems, and other physical phenomena. For more details on applications of the theory we refer to the survey of Balachandran and Dauer [2] and the book by Curtain and Zwart [7].

In recent years, as for the controllability of semilinear differential equations, Carrasco and Leiva [6] discussed sufficient conditions for approximate controllability of parabolic equations with delay, Mahmudov [15] in the case that the semilinear equations with nonlocal conditions with condition on the uniform boundedness of the Frechet derivative of nonlinear term, and Sakthivel et al. [19] on impulsive and neutral functional differential equations. As for some considerations on the trajectory set of (1.1) and that of its corresponding linear system (in case $g \equiv 0$) as matters related to (1.1), we refer the reader to Naito [16], Sukavanam and Tomar [22] and references therein.

In [2, 10] the authors dealt with the approximate controllability of a semilinear control system as a particular case of sufficient conditions for the approximate solvability of semilinear equations by assuming that

(1) $S(t)$ is compact operator, and

(2) the linear operator $S^T u := \int_0^T S(t-s)u(s)ds$ has a bounded inverse operator.

The paper [15] replaces the above condition (2) with

(2-1) The Frechet derivative of nonlinear term is uniformly bounded, and

(2-2) the corresponding linear system (1.1) in case $g \equiv 0$ and $x_0 \equiv 0$ is approximately controllable.

In [22] and [23] they studied the control problems of the semilinear equations by assuming conditions (1), (2-2), a Lipschitz continuity of the nonlinear term, and a range condition of the controller B with an inequality constraint.

In this paper we replace the condition (1) by the compactness of the embedding $D(A) \subset V$, and instead of (2.2) and the uniform boundedness f the nonlinear term, we require the following inequality constraint on the range condition of the controller B: for any $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$S^T p = S^T Bu.$$

In Section 2, we will obtain that most parts of the regularity for parabolic linear equations can also be applicable to (1.1) with nonlinear perturbations. The approach used here is similar to that developed in [8, 9, 13, 15] on the general semilnear evolution equations. Moreover, in Section 3, we establish the null controllability of semilinear functional integrodifferential control systems (1.1) under the Lipschitz continuity instead of the uniform boundedness of the Frechet derivative of nonlinear term. It is useful for physical applications of the given equations.

2. Regularity for Solutions

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V.$$

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for the brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \geq c_0 ||u||^2 - c_1 |u|^2,$$

where $c_0 > 0$ and c_1 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u, \ v \in V.$$

Then A is a bounded linear operator from V to V^* by the Lax-Milgram theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Moreover, for each $T > 0$, by using interpolation theory we have

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

From the following inequalities

$$\begin{aligned} c_0 ||u||^2 &\leq \text{Re}\, a(u, u) + c_1 |u|^2 \leq |Au|\, |u| + c_1 |u|^2 \\ &\leq (|Au| + c_1 |u|)|u| \leq \max\{1, c_1\} ||u||_{D(A)} |u|, \end{aligned}$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.2}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^* \tag{2.3}$$

where each space is dense in the next one which is a continuous injection.

Lemma 2.1. *With the notations (2.1)-(2.3), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [21]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity, we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of A.

Lemma 2.2. *Let $T > 0$. Then*

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\}.$$

Proof. Put $u(t) = e^{tA}x$ for $x \in H$. Then,

$$u'(t) = Au(t), \quad u(0) = x.$$

As in Theorem 4.1 of Chapter 4 of [14], the solution u belongs to $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$, hence we obtain that

$$\int_0^T ||Ae^{tA}x||_*^2 dt = \int_0^T ||u'(s)||_*^2 ds < \infty.$$

Conversely, suppose that $x \in V^*$ and $\int_0^T ||Ae^{tA}x||_*^2 dt < \infty$. Put $u(t) = e^{tA}x$. Then since A is an isomorphism operator from V to V^* there exists a constant $c > 0$ such that

$$\int_0^T ||u(t)||^2 dt \leq c\int_0^T ||Au(t)||_*^2 dt = c\int_0^T ||Ae^{tA}x||_*^2 dt.$$

From the assumptions and $u^{'}(t) = Ae^{tA}x$ it follows

$$u \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Therefore, $x = u(0) \in H$. □

By Lemma 2.1, from Theorem 3.5.3 of Butzer and Berens [5], we can see that

$$(V, V^*)_{1/2,2} = H.$$

Consider the following linear system

$$(2.4)\qquad \begin{cases} x'(t) &= Ax(t) + h(t), \\ x(0) &= x_0. \end{cases}$$

By virtue of Theorem 3.3 of [4](or Theorem 3.1 of [9]), we have the following result on the corresponding linear equation of (2.4).

Proposition 2.1. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*

(1) *Let $V = (D(A), H)_{1/2,2}$ where $(D(A), H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H(see [[21]; section 1.3.3], or Lemma 2.1). For $x_0 \in V$ and $h \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.4) belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$(2.5)\qquad ||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||h||_{L^2(0,T;H)}),$$

where C_1 is a constant depending on T.

(2) *Let $x_0 \in H$ and $h \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.4) belonging to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$(2.6)\qquad ||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + ||h||_{L^2(0,T;V^*)}),$$

where C_1 is a constant depending on T.

For the sake of simplicity, we assume that solution semigroup $S(t)$ generated by A is uniformly bounded:

$$||S(t)|| \le M \quad t \ge 0.$$

First, we consider the following inequalities.

Lemma 2.3. *Suppose that $h \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)h(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$(2.7)\qquad ||x||_{L^2(0,T;D(A))} \le C_1||h||_{L^2(0,T;H)},$$

$$(2.8)\qquad ||x||_{L^2(0,T;H)} \le C_2T||h||_{L^2(0,T;H)},$$

and

$$(2.9)\qquad ||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||h||_{L^2(0,T;H)}.$$

Proof. The assertion (2.7) is immediately obtained by (2.5). Since

$$\begin{aligned}||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)h(s)ds|^2 dt \le M\int_0^T (\int_0^t |h(s)|ds)^2 dt\\ &\le M\int_0^T t\int_0^t |h(s)|^2 dsdt \le M\frac{T^2}{2}\int_0^T |h(s)|^2 ds\end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||h||_{L^2(0,T;H)}.$$

From (2.3), (2.7), and (2.8) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||h||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

Consider the following initial value problem for the abstract semilinear parabolic equation (1.1). Let U be a Banach space and the controller operator B be a bounded linear operator from U to H.

Let $g : R^+ \times V \to H$ be a nonlinear mapping satisfying the following:

(F1) For any $x \in V$, the mapping $g(\cdot, x)$ is strongly measurable;

(F2) There exist positive constants L_0, L_1 such that

$$|g(t,x) - g(t,\hat{x})| \le L_1||x - \hat{x}||,$$
$$|g(t,0)| \le L_0$$

for all $t \in R^+$, and $x, \hat{x} \in V$.

For $x \in L^2(0,T;V)$, we set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds,$$

where k belongs to $L^2(0,T)$.

Lemma 2.4. *Let $x \in L^2(0,T;V)$ for any $T > 0$. Then $f(\cdot,x) \in L^2(0,T;H)$ and*

$$||f(\cdot,x)||_{L^2(0,T;H)} \le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}L_1\sqrt{T}||x||_{L^2(0,T;V)}. \tag{2.10}$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, then

$$||f(\cdot,x) - f(\cdot,\hat{x})||_{L^2(0,T;H)} \le ||k||_{L^2(0,T)}L_1\sqrt{T}||x - \hat{x}||_{L^2(0,T;V)}. \tag{2.11}$$

Proof. From (F1), (F2), and using the Hölder inequality, it is easily seen that

$$\begin{aligned}
&||f(\cdot,x)||_{L^2(0,T;H)} \le ||f(\cdot,0)|| + ||f(\cdot,x) - f(\cdot,0)|| \\
&\le \left(\int_0^T |\int_0^t k(t-s)g(s,0)ds|^2 dt\right)^{1/2} \\
&\quad + \left(\int_0^T |\int_0^t k(t-s)\{g(s,x(s)) - g(s,0)\}ds|^2 dt\right)^{1/2} \\
&\le L_0||k||_{L^2(0,T)} T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}||g(\cdot,x) - g(\cdot,0)||_{L^2(0,T;H)} \\
&\le L_0||k||_{L^2(0,T)} T/\sqrt{2} + ||k||_{L^2(0,T)} L_1\sqrt{T}||x||_{L^2(0,T;V)}.
\end{aligned}$$

The proof of (2.11) is similar. □

Theorem 2.1. *Under the assumptions (F1), and (F2) for the nonlinear mapping f, as given by*

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds,$$

there exists a unique solution x of (1.1) such that

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(|x_0| + ||u||_{L^2(0,T;U)}). \tag{2.12}$$

Proof. Let us fix $T_0 > 0$ satisfying

$$C_2 L_1 T_0 ||k||_{L^2(0,T)} < 1 \tag{2.13}$$

with the constant C_2 in Lemma 2.3. Let y be the solution of

$$y(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s,x(s)) + Bu(s)\}ds.$$

We are going to show that $x \mapsto y$ is strictly contractive from $L^2(0,T_0;V)$ to itself. Let y, $\hat{y}$ belong to V with the same initial condition in $[0,T_0]$. Then from assumption (2.9), (2.11) and

$$y(t) - \hat{y}(t) = \int_0^t S(t-s)\{f(s,x(s)) - f(s,\hat{x}(s))\}ds$$

we have

$$\begin{aligned}
||y-\hat{y}||_{L^2(0,T_0;V)} &\le C_2\sqrt{T_0}||f(\cdot,x) - f(\cdot,\hat{x})||_{L^2(0,T_0;H)} \\
&\le C_2 L_1 T_0 ||k||_{L^2(0,T_0)}||x(\cdot) - \hat{x}(\cdot)||_{L^2(0,T_0;V)}.
\end{aligned}$$

So by virtue of the condition (2.13) the contraction mapping principle gives that the solution of (1.1) exists uniquely in $[0, T_0]$. Let x be a solution of (1.1) and $x_0 \in H$. Then there exists a constant C_1 such that

$$||S(t)x_0||_{L^2(0,T_0;V)} \le C_1|x_0| \tag{2.14}$$

in view of Proposition 2.1. Let

$$x_1(t) = \int_0^t S(t-s)\{f(s,x(s)) + Bu(s)\}ds.$$

Then from (2.11), it follows

$$\begin{aligned} ||x_1||_{L^2(0,T_0;V)} &\le C_2\sqrt{T_0}||f(\cdot,x) + Bu||_{L^2(0,T_0;H)} \\ &\le C_2\sqrt{T}_0(L_1\sqrt{T_0}||k||_{L^2(0,T_0)}||x||_{L^2(0,T_0;V)} + ||f(\cdot,0) + Bu||_{L^2(0,T_0;H)}). \end{aligned} \tag{2.15}$$

Thus, combining (2.14) with (2.15) we have

$$\begin{aligned} ||x||_{L^2(0,T_0;V)} \le &(1 - C_2L_1T_0||k||_{L^2(0,T_0)})^{-1}(C_1|x_0| \\ &+ C_2\sqrt{T_0}||f(\cdot,0) + Bu||_{L^2(0,T_0;H)}). \end{aligned}$$

Hence, (2.12) holds. Now from

$$\begin{aligned} |x(T_0)| = &|S(T_0)x_0 + \int_0^{T_0} S(T_0 - s)\{f(s,x(s)) + Bu(s)\}ds| \\ \le &M|x_0| + ML_1\sqrt{T_0}||k||_{L^2(0,T_0)}||x||_{L^2(0,T_0;V)} \\ &+ M\sqrt{T_0}||f(\cdot,0) + Bu||_{L^2(0,T_0;H)}, \end{aligned}$$

since the condition (2.13) is independent of initial values, the solution of (1.1) can be extended to the interval $[0, nT_0]$ for every natural number n. An analogous estimate to (2.12) holds for the solution in $[0, nT_0]$, and hence for the initial value x_{nT_0} in the interval $[nT_0, (n+1)T_0]$. □

3. Null Controllability of Semilinear Systems

Let $S(t)$ be the analytic semigroup generated by the principal operator A. We define the linear operator $\hat{S}$ from $L^2(0,T;H)$ to H by

$$\hat{S}^T p = \int_0^T S(T-s)p(s)ds$$

for $p \in L^2(0,T;H)$. Let $x(T; f, u)$ be a state value of the system (1.1) at time T corresponding to the nonlinear term f and the control u. Then the solution $x(T; f, u)$ of (1.1) is represented by

$$x(T; f, u) = S(T)x_0 + \hat{S}^T f(\cdot, x) + \hat{S}^T Bu.$$

Definition 3.1. Equation (1.1) is said to be *null controllable* at time $T > 0$ if for a given $x_0 \in H$ there exists a control $u \in L^2(0, T; U)$ such that $x(T; f, u) = 0$.

Let $G^T = \hat{S}^T B$. Here, we remark that G is a bounded linear operator(see Proposition 2.1 or Theorem 2.1) but necessary one-to-one. Denote the orthogonal complement in $L^2(0, T; U)$ by $[\ker G^T]^\perp$. Let $G : [\ker G^T]^\perp \to \operatorname{Im} G^T$ be the restriction of G^T to $[\ker G^T]^\perp$. Then we know that G is necessary a one-to-one operator.

For any $(x, h) \in H \times L^2(0, T; V^*)$, define

$$N(x, h) = S(T)x_0 + \int_0^T S(T-s)h(s)ds : H \times L^2(0, T; V^*) \to H,$$

$$W(x, h) \equiv (G)^{-1}N = (G)^{-1}\left(S(T)x_0 + \int_0^T S(T-s)h(s)ds\right).$$

First, we consider the following linear control equation with a general forcing term h:

$$\begin{cases} x'(t) &= Ax(t) + Bu(t) + h(t), \\ x(0) &= x_0. \end{cases} \tag{3.1}$$

The following is immediately seen from Definition 3.1.

Lemma 3.1. *The linear system (3.1) is null controllable at time $T > 0$ if*

$$\operatorname{Im} G (= \hat{S}^T B) \supset \operatorname{Im} N.$$

We need the following hypothesis:

(A) Let us assume the natural assumption that the embedding

$$D(A) \subset V \text{ is compact.}$$

(B) For any $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\hat{S}^T p = \hat{S}^T Bu.$$

Remak 3.1. Denote the kernel of the operator $\hat{S}^T$ by N, which is a closed subspace in $L^2(0, T; H)$, and its orthogonal space in $L^2(0, T; H)$ by $N^\perp$. Let $\mathcal{B}$ be defined by $\underline{(\mathcal{B}u)}(\cdot) = Bu(\cdot)$. Denote the range of the operator $\mathcal{B}$ by $R(\mathcal{B})$ and its closure by $\overline{R(\mathcal{B})}$ in $L^2(0, T; H)$. As seen in [9, 16], it is easily known that the hypothesis (B) is equivalent to the following condition: $L^2(0, T; H) = \overline{R(\mathcal{B})} + N$.

Lemma 3.2. *Let us assume the hypothesis (B). Then we have*

$$D(A) \subset \operatorname{Im} G.$$

Proof. Let $x_0 \in D(A)$ and put $p(s) = (x_0 - sAx_0)/T$. Then $p \in L^2(0,T;H)$ and

$$x_0 = \int_0^T S(T-s)p(s)ds (= \hat{S}^T p).$$

Hence, from (B) we can choose a control $u_0 \in L^2(0,T;U)$ satisfying

$$\hat{S}^T p = \hat{S}^T B u_0,$$

which implies $D(A) \subset \operatorname{Im} G$. □

Theorem 3.1. *For $u \in L^2(0,T;U)$, let $x_u = G^T u$ with $x_u(0) = 0$. Under Assumption(A), we have the mapping $G^T : u \to x_u$ is compact from $L^2(0,T;U)$ to $L^2(0,T;V) \subset L^2(0,T;H)$.*

Proof. If $u \in L^2(0,T;U)$, with the aid of Proposition 2.1 (or Lemma 2.3), we have $x_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;V^*)$ and satisfy the following inequality:

$$||x_u||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \leq C_1 ||u||_{L^2(0,T;U)},$$

where C_1 is the constant in Proposition 2.1. Hence if u is bounded in $L^2(0,T;U)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ by the above inequality. Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is compact in view of Theorem 2 of J. P. Aubin [1].

Therefore, if we define the operator $x_u = G^T u$, then G^T is a compact mapping from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

The following lemma is obtained from the proof of Lemma 3 of [8].

Lemma 3.3. *Under Hypothesis (B), we have*

$$W(x,h) : H \times L^2(0,T;V^*) \to L^2(0,T;U).$$

is bounded and the control

$$u(t) = -W(x_0, h) \tag{3.2}$$

transfers the linear system (3.1) from $x_0 \in D(A)$ to 0.

Proof. By the definition of G, let

$$G : [\ker G^T]^\perp \to \operatorname{Im} G^T$$

be the restriction of G^T to $[\ker G^T]^\perp$. G is necessarily a one-to-one operator. Here, we remark that by Lemma 3.2, $S(T)x_0 \in \operatorname{Im} G$ since $S(T)x_0 \in D(A)$ for $x_0 \in D(A)$. Define

$$W(x,h) : H \times L^2(0,T;V^*) \to L^2(0,T;U)$$

by $W(x,h) \equiv (G)^{-1}N(x,h)$. From Theorem 3.1, it follows that $\mathrm{Im}G^T$ is closed and $[\ker G^T]^\perp$ is obviously closed. Hence, the inverse mapping theorem says that G^{-1} is a bounded linear operator, and so is W. □

Since the operator BW is bounded, for the sake of simplicity, we assume that

$$|BW(x_0,f)| \le ||BW||(|x_0|+|f|).$$

Lemma 3.4. *For $x \in L^2(0,T;V)$, we set*

$$K(s,x) = -BW(x_0, f(s,x(s)) + f(s,x(s)).$$

Then we obtain the following:

$$||K(\cdot,x)||_{L^2(0,T;H)} \le (1+||BW||)\,||k||_{L^2(0,T)} \left\{ L_0 \frac{T}{\sqrt{2}} + L_1\sqrt{T}||x||_{L^2(0,T;V)} \right\} + 2||BW||\,|x_0|$$

and

$$||K(\cdot,x_1) - K(\cdot,x_2)||_{L^2(0,T;H)} \le (1+||BW||)||k||_{L^2(0,T)} L_1\sqrt{T}||x_1-x_2||_{L^2(0,T;V)}.$$

Proof. From (2.10), (2.11) it is easily seen that

$$\begin{aligned}
&||K(\cdot,x)||_{L^2(0,T;H)} \le ||K(\cdot,0)|| + ||K(\cdot,x)-K(\cdot,0)|| \\
&\le || -BW(x_0,f(\cdot,0)) + f(\cdot,0)|| \\
&\quad + || -BW(x_0,f(\cdot,x)) + BW(x_0,f(\cdot,0)) + f(\cdot,x) - f(\cdot,0)|| \\
&\le ||BW|| \left(|x_0| + L_0||k||_{L^2(0,T)} \frac{T}{\sqrt{2}} \right) + L_0||k||_{L^2(0,T)} \frac{T}{\sqrt{2}} \\
&\quad + ||BW|| \left(|x_0| + L_1||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)} \right) \\
&\quad + L_1||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)} \\
&\le (1+||BW||)\,||k||_{L^2(0,T)} \{ L_0 \frac{T}{\sqrt{2}} + L_1\sqrt{T}||x||_{L^2(0,T;V)} \} + 2||BW||\,|x_0|.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
&||K(\cdot,x_1) - K(\cdot,x_2)||_{L^2(0,T;H)} \\
&= || -BW(x_0,f(s,x_1(s))) + f(s,x_1(s)) + BW(x_0,f(s,x_2(s)) - f(s,x_2(s))||_{L^2(0,T;H)} \\
&\le ||BW||||f(s,x_1(s)) - f(s,x_2(s))||_{L^2(0,T;H)} + ||f(s,x_1(s)) - f(s,x_2(s))||_{L^2(0,T;H)} \\
&\le (1+||BW||)||k||_{L^2(0,T)} L_1\sqrt{T}||x_1-x_2||_{L^2(0,T;V)}.
\end{aligned}$$

□

Theorem 3.2. *Assume the assumptions (F1-2), (A), and (B) be satisfied. Then for the initial data $x_0 \in D(A)$ the system (1.1) is null controllable at time $T > 0$.*

Proof. Define the operator F on $L^2(0,T;V)$ by

$$Fx(t) = \begin{cases} S(t)x_0 + \int_0^t S(t-s)\{-BW(x_0,f) + f(s,x)\}ds, & 0 < t \le T, \\ x_0, \quad \text{if} \quad t = 0. \end{cases}$$

Let us fix $T_0 > 0$ so that

$$C_0(\frac{T_0^3}{\sqrt{2}})^{1/2} L_1 M(1 + ||BW|| \, ||k||_{L^2(0,T)}) < 1, \tag{3.3}$$

where C_0 is constant in (2.3). We are going to show that $x \to Fx$ is strictly contractive from $L^2(0,T_0;V)$ to itself if the condition (3.3) is satisfied. Let Fx_1, Fx_2 be the solutions of the above equation with x replaced by $x_1, x_2 \in L^2(0,T_0;V)$ respectively. From (3.3) it follows that

$$\begin{aligned} &||Fx_1(t) - Fx_2(t)||_{L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;H)} \\ &\le || \int_0^{T_0} S(T_0 - s)\{K(s,x_1) - K(s,x_2)\}ds|| \\ &\le M\sqrt{T}_0 ||K(\cdot,x_1) - K(\cdot,x_2)||_{L^2(0,T_0;H)} \\ &\le (1 + ||BW||) M T_0 L_1 ||k||_{L^2(0,T_0)} ||x_1 - x_2||_{L^2(0,T_0;V)} \end{aligned}$$

and hence in view of (2.3) we have

$$\begin{aligned} &||Fx_1(t) - Fx_2(t)||_{L^2(0,T_0;V)} \\ &\le C_0 ||Fx_1(t) - Fx_2(t)||^{1/2}_{L^2(0,T_0;D(A))} ||Fx_1(t) - Fx_2(t)||^{1/2}_{L^2(0,T_0;H)} \\ &\le C_0 ||Fx_1(t) - Fx_2(t)||^{1/2}_{L^2(0,T_0;D(A))} (\frac{T_0}{\sqrt{2}})^{1/2} ||Fx_1(t) - Fx_2(t)||^{1/2}_{W^{1,2}(0,T_0;H)} \\ &\le C_0 (\frac{T_0}{\sqrt{2}})^{1/2} ||Fx_1(t) - Fx_2(t)||_{L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;H)} \\ &\le C_0 (\frac{T_0}{\sqrt{2}})^{1/2} M T_0 L_1 (1 + ||BW||) ||k||_{L^2(0,T_0)} ||x_1 - x_2||_{L^2(0,T_0;V)}. \end{aligned} \tag{3.4}$$

Here we used the following inequality

$$\begin{aligned} &||Fx_1(t) - Fx_2(t)||_{L^2(0,T_0;H)} = \{\int_0^{T_0} |Fx_1(t) - Fx_2(t)|^2 dt\}^{1/2} \\ &= \{\int_0^{T_0} | \int_0^t (\dot{F}x_1(\tau) - \dot{F}x_2(\tau)) d\tau|^2 dt\}^{1/2} \le \{\int_0^{T_0} t \int_0^t |\dot{F}x_1(\tau) - \dot{F}x_2(\tau)|^2 d\tau dt\}^{1/2} \\ &\le \frac{T_0}{\sqrt{2}} ||Fx_1(t) - Fx_2(t)||_{W^{1,2}(0,T_0;H)}. \end{aligned}$$

Hence, by virtue of (3.4) the contraction mapping principle gives that the operator F has unique solution in $[0, T_0]$, that is, x is the solution of the following equation:

$$\begin{cases} x(t) = S(t)x_0 + \int_0^t S(t-s)\{-BW(x_0, f) + f(s, x)\}ds, & t \le T_0, \\ x(0) = x_0, \quad \text{if} \quad t = 0. \end{cases} \tag{3.5}$$

Next we establish the estimates of solution. Let $x(\cdot)$ be the solution of (3.5) in the $(0, T_0)$ and $y(\cdot)$ be the solution of (3.1) with the control $u(t) = -W(x_0, k)$ as in (3.2), i.e., the solution y of (3.1) is represented by

$$y(t) = S(t)x_0 - \int_0^t S(t-s)BW(x_0, f)ds, \quad t \ge 0.$$

Thus, the arguing as in the proof of Lemmas 2.3, 2,4, we have

$$\begin{aligned}
&||x-y||_{L^2(0,T_0;V)} \\
&= || \int_0^{T_0} S(t-s)\{-BW(x_0, f) + f(s, x)\} - Bu(t)\}ds||_{L^2(0,T_0;V)} \\
&\le || \int_0^{T_0} S(t-s)\{f(\cdot, x) - f(\cdot, 0) + f(\cdot, 0)\}ds||_{L^2(0,T_0;V)} \\
&\le C_2\sqrt{T}_0||f(\cdot, x) - f(\cdot, 0)||_{L^2(0,T_0;H)} + C_2\sqrt{T}_0||f(\cdot, 0)||_{L^2(0,T_0;H)} \\
&\le C_2T_0||k||_{L^2(0,T_0)}L_1||x||_{L^2(0,T_0;V)} + C_2\frac{T_0^{3/2}}{\sqrt{2}}L_0||k||_{L^2(0,T_0)} \\
&\le C_2T_0L_1||k||_{L^2(0,T_0)}||x-y||_{L^2(0,T_0;V)} + C_2T_0L_1||k||_{L^2(0,T_0)}||y||_{L^2(0,T_0;V)} \\
&\quad + C_2\frac{T_0^{3/2}}{\sqrt{2}}L_0||k||_{L^2(0,T_0)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&||x-y||_{L^2(0,T_0;V)} \\
&\le \frac{C_2T_0L_1||k||_{L^2(0,T_0)}}{1 - C_2T_0L_1||k||_{L^2(0,T_0)}}\{||y||_{L^2(0,T_0;V)} + \frac{\sqrt{2T_0}L_0}{2L_1}\}
\end{aligned}$$

and hence with the aid of Lemma 2.3, or Theorem 2.1

$$\begin{aligned}
&||x||_{L^2(0,T_0;V)} \\
&\le \frac{1}{1 - C_2T_0L_1||k||_{L^2(0,T_0)}}\{||y||_{L^2(0,T_0;V)} + \frac{\sqrt{2T_0}L_0}{2L_1}\} \\
&\le \frac{1}{1 - C_2T_0L_1||k||_{L^2(0,T_0)}}\{C_3(|x_0| + ||u||_{L^2(0,T_0;U)}) + \frac{\sqrt{2T_0}L_0}{2L_1}\}.
\end{aligned}$$

Thus, there exists a constant C_4 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_4(1+|x_0|+||u||_{L^2(0,T;U)}).$$

Now from

$$\begin{aligned}|x(T_0)| &= |S(T_0)x_0| + |\int_0^{T_0} S(T_0-s)\{-BW(x_0,f)+f(s,x)\}ds| \\ &\le (M+2||BW||)|x_0| + M(1+||BW||)\{L_0||k||_{L^2(0,T)}T/\sqrt{2} \\ &\quad + ||k||_{L^2(0,T)}L_1\sqrt{T}||x||_{L^2(0,T;V)}\},\end{aligned}$$

since the condition (3.3) is independent of initial values, the solution of (1.1) can be extended to the interval $[0, nT_0]$ for every natural number n. That is, an analogous estimate to (3.6) holds for the solution in $[0, nT_0]$, and hence for the initial value x_{nT_0} in the interval $[nT_0, (n+1)T_0]$, which means that the system (1.1) is null controllable at time $T > 0$ with the control $u = -W(x_0, f)$. □

Theorem 3.3. *Let the assumption (F1), (F2) be satisfied and $(x_0, u) \in V \times L^2(0,T;U)$, Then the solution x of the equation (1.1) belongs to $x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ and the mapping*

$$V \times L^2(0,T;U) \ni (x_0,u) \mapsto x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

is Lipschitz continuous.

Proof. It is easy to show that if $x_0 \in V$ and $f(\cdot,x) \in L^2(0,T;H)$, then x belongs to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Let $(x_0^i, u_i) \in H \times L^2(0,T;U)$ and x_i be the solution of (1.1) with (x_0, u) in place of (x_0^i, u_i) for $i = 1, 2$. Then

$$\begin{cases}(x_1-x_2)'(t) &= A(x_1-x_2)(t) + f(t,x_1(t)) - f(t,x_2(t)) + B(u_1-u_2)(t),\ t>0,\\ (x_1-x_2)(0) &= x_0^1 - x_0^2.\end{cases}$$

Hence in view of proposition 2.1 and lemma 2.4, we have

(3.6)

$$\begin{aligned}&||x_1-x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}\\ &\le C_1\{||x_0^1-x_0^2|| + ||u_1-u_2||_{L^2(0,T;U)} + ||f(\cdot,x_1)-f(\cdot,x_2)||_{L^2(0,T;H)}\}\\ &\le C_1\{||x_0^1-x_0^2|| + ||u_1-u_2||_{L^2(0,T;U)} + ||k||_{L^2(0,T)}L_1\sqrt{T}||x_1-x_2||_{L^2(0,T;V)}\}\end{aligned}$$

Since

$$x_1(t)-x_2(t) = x_0^1 - x_0^2 + \int_0^t (\dot{x}_1(s)-\dot{x}_2(s))ds$$

We get

$$||x_1-x_2||_{L^2(0,T;H)} \le \sqrt{T}|x_0^1-x_0^2| + \frac{T}{\sqrt{2}}||x_1-x_2||_{W^{1,2}(0,T;H)}$$

Hence arguing as in (3.4) we get

(3.7)

$$\begin{aligned}
&||x_1 - x_2||_{L^2(0,T;V)} \\
&\le C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))}||x_1 - x_2||^{1/2}_{L^2(0,T;H)} \\
&\le C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))}\{T^{1/4}|x_0^1 - x_0^2|^{1/2} + (\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\
&\le C_0T^{1/4}||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))}|x_0^1 - x_0^2|^{1/2} \\
&\quad + C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\
&\le 2^{-7/4}C_0|x_0^1 - x_0^2| + 2C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.
\end{aligned}$$

Combining (3.6) and (3.7) we obtain

(3.8)
$$\begin{aligned}
&||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\
&\le C_1\{||x_0^1 - x_0^2|| + ||k||_{L^2(0,T)}L_1\sqrt{T}(2^{-7/4}C_0||x_0^1 - x_0^2|| \\
&\quad + 2C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}) + ||u_1 - u_2||_{L^2(0,T;U)}\} \\
&= (C_1 + 2^{-7/4}C_0||k||_{L^2(0,T)}L_1\sqrt{T})||x_0^1 - x_0^2|| + C_1||u_1 - u_2||_{L^2(0,T;U)} \\
&\quad + 2^{3/4}C_0TL_1||k||_{L^2(0,T)}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.
\end{aligned}$$

Suppose that $x_0^n \to x_0$ in V and let x_n and x be the solution (1.1) with x_0^n and x_0 respectively. Let $0 < T_1 \le T$ be such that

$$2^{3/4}C_0TL_1||k||_{L^2(0,T)} < 1.$$

Then by virtue of (3.8) with T replaced by T_1 we see that

$$x_n \to x \quad \text{in} \quad L^2(0,T;D(A)) \cap W^{1,2}(0,T;H).$$

This implies that $(x_n(T_1),(x_n)_{T_1}) \to (x(T_1), x_{T_1})$ in $V \times L^2(0,T;D(A))$. Hence the same argument shows that

$$x_n \to x \quad \text{in} \quad L^2(T_1,\min\{2T_1,T_1\};D(A)) \cap W^{1,2}(T_1,\min\{2T_1,T_1\};H).$$

Repeating this process we conclude that $x_n \to x$ in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ for any $T > 0$. □

Remark 3.2. Let us we assume the following hypothesis:
For any $\varepsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

(B1)
$$\begin{cases} |\hat{S}p - \hat{S}Bu| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1||p||_{L^2(0,t;H)}, \quad 0 \le t \le T, \end{cases}$$

where q_1 is a constant independent of p.

Then, as seen in [12], we note that for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;u)$ of (1.1) satisfies $|x(T;u) - x_1| < \epsilon$, i.e., the system (1.1) is said to be approximately controllable in the time interval $[0,T]$.

Example 3.1. We consider an application of the results obtained in the preceding sections to a class of partial functional integrodifferential systems with delay terms dealt with by Naito [16] and Zhou [23]:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) & = A(x,D_x)u(x,t) + \int_0^t k(t-s)g(s,u(x,s))ds + B_\alpha w(t), \\ & (x,t) \in \Omega \times (0,T),\ 0 < \alpha < T, \\ u(x,0) & = u_0(x), \end{cases} \tag{3.9}$$

The boundary condition attached to (3.9) is given by Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \quad 0 < t \leq T,$$

and k belongs to $L^2(0,T)$. Here, $\Omega \subset \mathcal{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. We set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Let $b(u,v)$ be the sesquilinear form in $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^n \beta_i \frac{\partial u}{\partial x_i} \bar{v} + cu\bar{v} \right\} dx.$$

Here, we assume that a_{ij} is a real-valued and smooth function for each $i,j = 1,\cdots,n$, and $a_{ij}(x) = a_{ji}(x)$ for each $x \in \bar{\Omega}$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, i.e., there exists a positive number c_0 such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2$$

for all $x \in \bar{\Omega}$ and all real vectors ξ. Let $b_i \in L^\infty(\Omega)$ and $c \in L^\infty(\Omega)$. As is well known this sesquilinear form $a(\cdot,\cdot)$ is bounded and satisfying the Gårding's inequality (2.2)(see e.g. Tanabe [20]. Let

$$\mathcal{A}(x,D_x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^n \beta_i(x)\frac{\partial}{\partial x_i} + c(x),\ x \in \Omega \tag{3.13}$$

be the associated uniformly elliptic differential operator of second order. Then the realization of $-\mathcal{A}(x,D_x)$ In $L^2(\Omega)$ under the Dirichlet boundary condition is exactly A, i.e.,

$$D(A) = W^{2,2}(\Omega) \cap H_0^1(\Omega),$$
$$Au = -\mathcal{A}(x,D_x)u, \quad \forall u \in D(A).$$

It is not difficult to verify that $Au = -\mathcal{A}(x, D_x)u$ for $u \in H_0^1(\Omega)$ in the sense of distribution and $u|_{\partial\Omega} = 0$ for $u \in H_0^1(\Omega)$ also in the sense of distribution(see Lions and Magenes [14]), and

$$(Au, v) = -a(u, v), \quad u,\ v \in H_0^1(\Omega).$$

We consider the nonlinear term g given by

$$g(t, u) = \gamma(t)\{||D_x u(x, t)|| + \phi(u(x, t))\}, \quad \gamma \in C([0, T]),\ \phi \in C(H).$$

Then g is not uniformly bounded and satisfies hypotheses (F1)and (F2). Let $U = H$ be the space of control variables and let us define the intercept controller operator $B_\alpha (0 < \alpha < T)$ on $L^2(0, T; H)$ by

$$B_\alpha w(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ w(t), & \alpha \le t \le T \end{cases}$$

for $w \in L^2(0, T; H)$. For a given $p \in L^2(0, T; H)$ let us choose a control function w satisfying

$$w(t) = \begin{cases} 0, \quad 0 \le t < \alpha, \\ p(t) + \frac{\alpha}{T-\alpha} S(t - \frac{\alpha}{T-\alpha}(t - \alpha))p(\frac{\alpha}{T-\alpha}(t - \alpha)),\ \alpha \le t \le T. \end{cases}$$

Then $w \in L^2(0, T; H)$ and $\hat{S}p = \hat{S}B_\alpha w$, which is that the controller B_α satisfies Assumption (B). Hence, for the initial data $u_0 \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$ the system (3.9) is null controllable.

Acknowledgements. Authors would like to thank the referees for their useful suggestions which have significantly improved the paper.

References

[1] J. P. Aubin, *Un thèoréme de compacité*, C. R. Acad. Sci. Paris, **256**(1963), 5042–5044.

[2] K. Balachandran and J. P. Dauer, *Controllability of nonlinear systems in Banach spaces: a survey*, J. Optim. Theory Appl., **115**(2002), 7–28.

[3] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Nordhoff Leiden, Netherlands, 1976.

[4] G. Di Blasio, K. Kunisch and E. Sinestrari, *L^2−regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl., **102**(1984), 38–57.

[5] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Springer-verlag, Belin-Heidelberg-Newyork, 1967.

[6] A. Carrasco and H. Leiva, *Approximate controllability of a system of parabolic equations with delay*, J. Math. Anal. Appl., **345**(2008), 845–853.

[7] R. F. Curtain and H. Zwart, *An introduction to infinite dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[8] J. P. Dauer and N. I. Mahmudov, *Exact null controllability of semilinear integrodifferential systems in Hilbert spaces*, J. Math. Anal. Appl., **299**(2004), 322–332.

[9] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynam. Control Systems, **5(3)**(1999), 329–346.

[10] J. M. Jeong and H. H. Roh, *Approximate controllability for semilinear retarded systems*, J. Math. Anal. Appl., **321**(2006), 961–975.

[12] J. M. Jeong, J. E. Ju and K. Y. Lee, *Controllability for variational inequalities of parabolic type with nonlinear perturbation*, J. Inequal. Appl., (2010), Art. ID 768469, 16 pp.

[11] J. M. Jeong, J. R. Kim and H. G. Kim, *Regularity for solutions of nonlinear second order evolution equations*, J. Math. Anal. Appl., **338**(2008), 209–222.

[13] J. L. Lions, *Quelques méthodes de résolution des problems aux limites non-linéaires*, Paris, Dunnod, Gauthier-Villars, 1969.

[14] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problèmes and applications*, Springer-Verlag, Berlin-heidelberg-New York, 1972.

[15] N. I. Mahmudov, *Approximate controllability of evolution systems with nonlocal conditions*, Nonlinear Anal., **68**(2008), 536–546.

[16] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim., **25**(1987), 715–722.

[17] D. G. Park, J. M. Jeong and S. H. Park, *Regularity of parabolic hemivariational inequalities with boundary conditions*, J. Inequal. Appl., (2009), Art. ID 207873, 22 pp.

[18] J. Y. Park and S. H. Park, *On solutions for a hyperbolic system with differential inclusion and memory source term on the boundary,* Nonlinear Anal., **57**(2004), 459–472.

[19] R. Sakthivel, N. I. Mahmudov and J. H. Kim, *Approximate controllability of nonlinear impulsive differential systems*, Rep. Math. Phys., **60**(2007), 85–96.

[22] N. Sukavanam and N. K. Tomar, *Approximate controllability of semilinear delay control systems*, Nonlinear Funct. Anal. Appl., **12**(2007), 53–59.

[20] H. Tanabe, *Equations of evolution*, Pitman, Boston, Mass.-London, 1979.

[21] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, 1978.

[23] H. X. Zhou, *Approximate controllability for a class of semilinear abstract equations*, SIAM J. Control Optim., **21**(1983), 551–565.