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An Application of Absolute Matrix Summability using Almost Increasing and δ-quasi-monotone Sequences

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Abstract. In the present paper, absolute matrix summability of infinite series is studied. A new theorem concerning absolute matrix summability factors, which generalizes a known theorem dealing with absolute Riesz summability factors of infinite series, is proved using almost increasing and δ-quasi-monotone sequences. Also, a result dealing with absolute Cesàro summability is given.

1. Introduction

A positive sequence (v_n) is said to be *almost increasing* if there exists a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leq v_n \leq Lc_n$ (see [1]). A sequence (y_n) is said to be *δ-quasi-monotone*, if $y_n \to 0$, $y_n > 0$ ultimately and $\Delta y_n \geq -\delta_n$, where $\Delta y_n = y_n - y_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C,1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be $|C,1|_k$ summable, $k \geq 1$, if (see [6], [8])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.2}$$

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The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (z_n) of the Riesz mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be $\left|\bar{N}, p_n\right|_k$ summable, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\Delta z_{n-1}\right|^k < \infty, \tag{1.4}$$

where

$$\Delta z_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{i=0}^{n} a_{ni} s_i, \quad n = 0, 1, ... \tag{1.5}$$

The series $\sum a_n$ is said to be $\left|A, p_n\right|_k$ summable, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.6}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.7}$$

When we take $a_{nv} = \frac{p_v}{P_n}$, then $\left|A, p_n\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability. Also, when we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\left|A, p_n\right|_k$ reduces to $\left|C, 1\right|_k$ summability.

Let $A = (a_{nv})$ be a normal matrix. Lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.8}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.9}$$

$\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we write

$$A_n(s) = \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} \bar{a}_{ni} a_i \tag{1.10}$$

and

$$\bar{\Delta} A_n(s) = \sum_{i=0}^{n} \hat{a}_{ni} a_i. \tag{1.11}$$

2. Known Result

In [4, 5], the following theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series has been proved by Bor.

Theorem 2.1. *Let (X_n) be an almost increasing sequence such that $|\Delta X_n| = O(X_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \le |A_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n}|\lambda_n| = O(1) \quad as \qquad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(X_m) \quad as \qquad m \to \infty, \tag{2.2}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m) \quad as \qquad m \to \infty, \tag{2.3}$$

then the series $\sum a_n\lambda_n$ is $|\bar{N}, p_n|_k$ summable, $k \ge 1$.

3. Main Result

The aim of this paper is to prove following more general theorem dealing with $|A, p_n|_k$ summability.

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{3.1}$$

$$a_{n-1,v} \ge a_{nv}, \quad for \quad n \ge v + 1, \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right). \tag{3.3}$$

If all conditions of Theorem 2.1 are satisfied, then the series $\sum a_n\lambda_n$ *is* $|A, p_n|_k$ *summable,* $k \geq 1$.

Lemma 3.2.([4]) *Under the conditions of Theorem 3.1, we have*

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{3.4}$$

Lemma 3.3.([5]) *Let* (X_n) *be an almost increasing sequence such that* $n \mid \Delta X_n \mid = O(X_n)$. *If* (A_n) *is a* δ*-quasi monotone with* $\sum nX_n\delta_n < \infty$, *and* $\sum A_nX_n$ *is convergent, then*

$$nA_nX_n = O(1) \quad as \quad n \to \infty, \tag{3.5}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty. \tag{3.6}$$

4. Proof of Theorem 3.1

Let (M_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (1.10) and (1.11), we have

$$\bar{\Delta} M_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v.$$

Applying Abel's transformation to above sum, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \frac{n+1}{n} a_{nn}\lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv})\lambda_v t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1} \frac{t_v}{v} \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

To prove Theorem 3.1, we will show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |M_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, by using (2.3), (3.3) and (3.4), we have

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,1}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{n+1}{n}a_{nn}\lambda_n t_n\right|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k|\lambda_n|^k|t_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n|^{k-1}|\lambda_n||t_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||t_n|^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\frac{p_r}{P_r}|t_r|^k + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k \\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{n=1}^{m-1}\mid A_n\mid X_n + O(1)|\lambda_m|X_m \\
&= O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Now, as in $M_{n,1}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|\,|t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|t_v|^k\right) \\
&\quad\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}.
\end{aligned}$$

Since

$$\begin{aligned}
\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv} - \hat{a}_{n,v+1} \\
&= \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} \\
&= a_{nv} - a_{n-1,v} \qquad (4.1)
\end{aligned}$$

by (1.8) and (1.9), we have

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v} - a_{nv}) \le a_{nn}$$

by using (1.8), (3.1) and (3.2). Hence, we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k = O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left|\Delta_v(\hat{a}_{nv})\right|$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|\,|t_v|^k\sum_{n=v+1}^{m+1}\left|\Delta_v(\hat{a}_{nv})\right|.$$

Now, using (3.2) and (4.1), we obtain

$$\sum_{n=v+1}^{m+1}\left|\Delta_v(\hat{a}_{nv})\right| = \sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv}) \le a_{vv},$$

then

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k = O(1)\sum_{v=1}^{m}|\lambda_v|\,|t_v|^k\,a_{vv}$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|\,|t_v|^k\,\frac{p_v}{P_v}$$
$$= O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Also, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_v t_v\right|^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v||t_v|\right)^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v||t_v|^k\right)$$
$$\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v|\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}|A_v||t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.$$

By (1.8), (1.9), (3.1) and (3.2), we obtain

$$|\hat{a}_{n,v+1}| = \sum_{i=0}^{v}(a_{n-1,i}-a_{ni}).$$

Thus, using (1.8) and (3.1), we have

$$\sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| = \sum_{n=v+1}^{m+1} \sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \leq 1,$$

then we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |M_{n,3}|^k &= O(1) \sum_{v=1}^{m} |A_v||t_v|^k = O(1) \sum_{v=1}^{m} v|A_v| \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|A_v|) \sum_{r=1}^{v} \frac{1}{r} |t_r|^k + O(1) m|A_m| \sum_{v=1}^{m} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta A_v| X_v + O(1) \sum_{v=1}^{m-1} |A_v| X_v + O(1) m|A_m| X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Again, operating Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |M_{n,4}|^k &\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v}\right)^k \\
&\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v}\right) \\
&\quad \times \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_{v+1}| \sum_{r=1}^{v} \frac{1}{r} |t_r|^k + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
&= O(1) \sum_{v=1}^{m-1} |A_{v+1}| X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (2.1), (2.2), (3.3) and (3.4). This completes the proof of Theorem 3.1. □

If we take $a_{nv} = \frac{p_v}{P_n}$ in this theorem, then we get Theorem 2.1. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

References

[1] N. K. Bari and S. B. Stečkin, *Best approximations and differential proprerties of two conjugate functions*, Tr. Mosk. Mat. Obs., **5**(1956), 483–522.

[2] R. P. Boas, *Quasi-positive sequences and trigonometric series*, Proc. London Math. Soc. (3), **14a**(1965), 38–46.

[3] H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc., **97**(1985), 147–149.

[4] H. Bor, *An application of almost increasing and δ-quasi-monotone sequences*, JIPAM. J. Inequal. Pure Appl. Math., **1(2)**(2000), Article 18, 6 pp.

[5] H. Bor, *Corrigendum on the paper "An application of almost increasing and δ-quasi-monotone sequences"*, JIPAM. J. Inequal. Pure Appl. Math., **3(1)**(2002), Article 16, 2 pp.

[6] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc., **7**(1957), 113–141.

[7] G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1949.

[8] E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Math., **49**(1925), 234–256.

[9] W. T. Sulaiman, *Inclusion theorems for absolute matrix summability methods of an infinite series. IV*, Indian J. Pure Appl. Math., **34(11)**(2003), 1547–1557.