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Coefficient Bounds for Several Subclasses of Analytic and Bi-univalent Functions

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Abstract. In the present paper, some generalizations of analytic functions have been considered and the bounds of the coefficients of these classes of bi-univalent functions have been investigated.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let S denote the subclass of $\mathcal{A}$ consisting of univalent functions in U. A function f in S is said to be starlike of order α $(0 \leq \alpha < 1)$ in U if $Re(zf'(z)/f(z)) > \alpha$. We denote by $S^*(\alpha)$, the class of all such functions. A function f in S is said to be convex of order α $(0 \leq \alpha < 1)$ in U if $Re\,(1 + zf''(z)/f'(z)) > \alpha$. Let $K(\alpha)$ denote the class of all those functions which are convex of order α in U. Mocanu [10] studied linear combinations of the representations of convex and starlike functions and defined the

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class of α-convex functions. In [9], it was shown that if

$$Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0, \quad z\in U$$

then f is in the class of starlike functions $S^*(0)$ for α real and is in the class of convex functions $K(0)$ for $\alpha \geq 1$.

A function $f(z) \in S$ is said to be in the class of strongly starlike functions of order α, denoted by $S_s^*(\alpha)$, $0 \leq \alpha < 1$, if it satisfies the following inequality

$$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\alpha\pi}{2}, \qquad z\in U,$$

and is said to be in the class of strongly convex functions of order α, $0 \leq \alpha < 1$, denoted by $K_c(\alpha)$, if it satisfies the following inequality

$$\left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| \leq \frac{\alpha\pi}{2}, \qquad z\in U.$$

For two functions f and g, analytic in U, we say that the function f is subordinate to g, and write $f(z) \prec g(z)$, if there exists an analytic function $\varphi(z)$ defined in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ such that $f(z) = g(\varphi(z))$.

For each $f \in S$, the Koebe one-quarter theorem [5] ensures the image of U under f contains a disk of radius $1/4$. Thus every univalent function $f \in S$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad z\in U$$

and

$$f(f^{-1}(w)) = w, \qquad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}.$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. If f given by (1.1) is bi-univalent, then a simple computation shows that $g = f^{-1}$ has the expansion

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{1.2}$$

Let Σ denote the class of all bi-univalent functions defined in the unit disk U.

Lewin [8] considered the class of bi-univalent functions Σ and obtained the following bound for the second coefficient in the Tylor-Maclaurin expansion as follows

$$|a_2| < 1.51,$$

for function $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \Sigma$. After that, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [11] showed that max $|a_2| = 4/3$ if $f \in \Sigma$. Brannan and Taha [4] studied some subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike,

starlike and convex functions. They introduced the class of bi-starlike functions and bi-convex functions and investigated estimates on the two initial coefficients. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N}\backslash\{1,2\}, \mathbb{N} = \{1,2,\ldots\},$$

is presumably still an open problem. The bi-univalent functions was actually revived in the recent years by the pioneering work by Srivastava et al. [15]. In [15], two new subclasses of the function class Σ has been introduced and the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ has been considered. After that, many authors investigated bounds for various subclasses of bi-univalent functions [1, 6, 12, 15, 16].

We notice that the class Σ is not empty. For example, the following functions are members of Σ:

$$z, \quad \frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\frac{1+z}{1-z}.$$

However, the Koebe function is not a member of Σ.

In the next section, motivated essentially by the recent work of Srivastava et al. [15], Frasin and Aouf [6], Xu et al. [16] and other authors, we investigte interesting subclasses of analytic and bi-univalent functions in the open unit dick U. So we estimate the first two Tylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for these classes.

2. Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk U, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. Assume also that

$$\varphi(z) = 1 + B_1 z + B_2 z + B_3 z + \ldots, \qquad B_1 > 0. \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$ and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n, \qquad |z| < 1. \tag{2.2}$$

It is well known that

$$|b_1| \le 1, \quad |b_2| \le 1 - |b_1|^2, \quad |c_1| \le 1, \quad |c_2| \le 1 - |c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \ldots, \tag{2.4}$$

$$\varphi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \ldots. \tag{2.5}$$

Definition 2.1. A function f given by (1.1) is said to be *in the class* $H_\Sigma(\alpha,\beta,\varphi)$, $0\le\alpha\le 1$, $\beta\ge 0$, if $f\in\Sigma$ and satisfies the following conditions $f\in\mathcal{A}$ is said to be in the class

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]+\beta\frac{zf'(z)+\alpha z^2f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)}\prec\varphi(z),$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]+\beta\frac{wg'(w)+\alpha w^2g''(w)}{(1-\alpha)g(w)+\alpha wg'(w)}\prec\varphi(w),$$

where $g(w)=f^{-1}(w)$ is given by (1.2).

For special choices for the number α, β and function φ, we can obtain several important classes of analytic functions. For example, $H_\Sigma(0,0,\frac{1+(1-2\alpha)z}{1-z})$ is the class $S^*(\alpha)$, $H_\Sigma(1,0,\frac{1+(1-2\alpha)z}{1-z})$ is the class $K(\alpha)$, $H_\Sigma(0,0,(\frac{1+z}{1-z})^\alpha)$ is the class $S^*_s(\alpha)$ and $H_\Sigma(1,0,(\frac{1+z}{1-z})^\alpha)$ is the class $K_c(\alpha)$. These classes of analytic functions were well investigated by many authors [2, 7, 9, 10, 13, 14].

For proving our results we used the subordination concept which have been used by many authors (for example see [1]).

Theorem 2.2. *If $f\in H_\Sigma(\alpha,\beta,\varphi)$ is in $\mathcal{A}$, then*

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|+(\alpha+1)^2B_1}},\tag{2.6}$$

and

$$|a_3|\le\begin{cases}\dfrac{B_1}{2(1+2\alpha)}, & B_1\le\dfrac{(1+\alpha)^2}{2(1+2\alpha)},\\[2ex] \dfrac{\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|+2(1+2\alpha)B_1^3}{2(1+2\alpha)\left[(\alpha+1)^2B_1+\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|\right]}, & B_1\ge\dfrac{(1+\alpha)^2}{2(1+2\alpha)}.\end{cases}\tag{2.7}$$

Proof. Let $f\in H_\Sigma(\alpha,\beta,\varphi)$ and $g=f^{-1}$. Then there exist two functions u and v; analytic in U; with $|u(z)|<1$, $|v(w)|<1$, $u(0)=v(0)=0$, $z,w\in U$, given by (2.2), such that

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]+\beta\frac{zf'(z)+\alpha z^2f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)}=\varphi(u(z)),\tag{2.8}$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]+\beta\frac{wg'(w)+\alpha w^2g''(w)}{(1-\alpha)g(w)+\alpha wg'(w)}=\varphi(v(w)).\tag{2.9}$$

Since

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]+\beta\frac{zf'(z)+\alpha z^2f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)}$$
$$=1+(1+\alpha)a_2z+[2(1+2\alpha)a_3-(1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2]z^2+\dots$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]+\beta\frac{wg'(w)+\alpha w^2g''(w)}{(1-\alpha)g(w)+\alpha wg'(w)}$$
$$=1-(1+\alpha)a_2w-[(1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2-2(1+2\alpha)(2a_2^2-a_3)]w^2+\dots,$$

it follows from (2.4), (2.5), (2.8) and (2.9) that

$$(\alpha+1)a_2=B_1b_1, \tag{2.10}$$

$$2(1+2\alpha)a_3-(1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2=B_1b_2+B_2b_1^2, \tag{2.11}$$

$$-(\alpha+1)a_2=B_1c_1, \tag{2.12}$$

$$2(1+2\alpha)(2a_2^2-a_3)-(1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2=B_1c_2+B_2c_1^2. \tag{2.13}$$

From (2.10) and (2.12) we get

$$c_1=-b_1, \tag{2.14}$$

and

$$2(\alpha+1)^2a_2^2=B_1^2(b_1^2+c_1^2). \tag{2.15}$$

By adding the relation (2.11) with the relation (2.13) and use (2.15) we can get

$$2\big[(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\big]a_2^2=B_1^3(b_2+c_2)$$

Now, in view of (2.3), (2.10) and (2.14) we have

$$\begin{aligned}2\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|\,|a_2|^2&\le B_1^3(|b_2|+|c_2|)\\&\le B_1^3\big((1-|b_1|^2)+(1-|c_1|^2)\big)\\&\le 2B_1^3(1-|b_1|^2)\\&=2B_1^3-2(\alpha+1)^2B_1|a_2|^2.\end{aligned}$$

Then, we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|+(\alpha+1)^2B_1}}. \tag{2.16}$$

Next, from (2.11) and (2.13), we have

$$4(1+2\alpha)a_3 = B_1(b_2-c_2)+4(1+2\alpha)a_2^2.$$

Then in view of (2.3) and (2.14), we have

$$\begin{aligned}4(1+2\alpha)|a_3| &\le 4(1+2\alpha)|a_2|^2+B_1(|b_2|+|c_2|)\\ &\le 4(1+2\alpha)|a_2|^2+2B_1(1-|b_1|^2).\end{aligned}$$

It follows from (2.10) that

$$2B_1(1+2\alpha)|a_3| \le \left[2B_1(1+2\lambda)-(1+\alpha)^2\right]|a_2|^2+B_1^2.$$

Now from (2.16) we have

$$|a_3| \le \begin{cases} \dfrac{B_1}{2(1+2\alpha)}, & B_1 \le \dfrac{(1+\alpha)^2}{2(1+2\alpha)},\\[2mm] \dfrac{\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|+2(1+2\alpha)B_1^3}{2(1+2\alpha)\left[(\alpha+1)^2B_1+\left|(1+\alpha)(B_1^2-(1+\alpha)B_2)+B_1^2\alpha\beta(1-\alpha)\right|\right]}, & B_1 \ge \dfrac{(1+\alpha)^2}{2(1+2\alpha)}.\end{cases}$$

This completes the proof. □

Remark 2.3. Let

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+\dots,\qquad 0<\gamma\le 1$$

then inequalities (2.6) and (2.7) become

$$|a_2| \le \frac{2\sqrt{\gamma}}{\sqrt{(1-\alpha)(1+\alpha+2\alpha\beta)}},$$

and

$$|a_3| \le \begin{cases} \dfrac{\gamma}{1+2\alpha}, & \gamma \le \dfrac{(1+\alpha)^2}{4(1+2\alpha)},\\[2mm] \dfrac{(1-\alpha)(1+\alpha+2\alpha\beta)+16\gamma(1+2\alpha)}{\gamma(1+2\alpha)(1-\alpha)(1+\alpha+2\alpha\beta)}, & \gamma \ge \dfrac{(1+\alpha)^2}{4(1+2\alpha)}.\end{cases}$$

Definition 2.4. A function f given by (1.1) is said to be *in the class* $C_\Sigma(\lambda, \varphi)$, $0 \le \lambda < 1$, if $f \in \Sigma$ and satisfies the following conditions

$$\frac{\lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + z f'(z)}{\lambda z^2 f''(z) + z f'(z)} \prec \varphi(z),$$

and

$$\frac{\lambda w^3 g'''(w) + (2\lambda+1) w^2 g''(w) + w g'(z)}{\lambda w^2 g''(w) + w g'(w)} \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$ is given by (1.2).

Theorem 2.5. *If* $f \in C_\Sigma(\lambda, \varphi)$ *is in* $\mathcal{A}$, *then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|2(2\lambda+1-2\lambda^2)B_1^2 - 4(\lambda+1)^2 B_2\right| + 4(\lambda+1)^2 B_1}}, \tag{2.17}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{6(1+2\lambda)}, & B_1 \le \dfrac{2(1+\lambda)^2}{3(1+2\lambda)}, \\ \dfrac{\left|(2\lambda+1-2\lambda^2)B_1^2 - 2(\lambda+1)^2 B_2\right| + 3(1+2\lambda)B_1^3}{3(1+2\lambda)\left[\left|2(2\lambda+1-2\lambda^2)B_1^2 - 4(\lambda+1)^2 B_2\right| + 4(\lambda+1)^2 B_1\right]}, & B_1 \ge \dfrac{2(1+\lambda)^2}{3(1+2\lambda)}. \end{cases} \tag{2.18}$$

Proof. Let $f \in C_\Sigma(\lambda, \varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : U \longrightarrow U$ given by (2.2) such that

$$\frac{\lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + z f'(z)}{\lambda z^2 f''(z) + z f'(z)} = \varphi(u(z)), \tag{2.19}$$

and

$$\frac{\lambda w^3 g'''(w) + (2\lambda+1) w^2 g''(w) + w g'(z)}{\lambda w^2 g''(w) + w g'(w)} = \varphi(v(w)). \tag{2.20}$$

Since

$$\begin{aligned} &\frac{\lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + z f'(z)}{\lambda z^2 f''(z) + z f'(z)} \\ &= 1 + 2(\lambda+1)a_2 z + 2\Big((1+2\lambda)(3a_3 - 2a_2^2) - 2\lambda^2 a_2^2\Big) z^2 + \dots \end{aligned}$$

and

$$\begin{aligned} &\frac{\lambda w^3 g'''(w) + (2\lambda+1) w^2 g''(w) + w g'(z)}{\lambda w^2 g''(w) + w g'(w)} \\ &= 1 - 2(\lambda+1)a_2 w + \Big(2(1+2\lambda)(4a_2^2 - 3a_3) - 4\lambda^2 a_2^2\Big) w^2 + \dots, \end{aligned}$$

it follows from (2.4), (2.5), (2.19) and (2.20) that

$$2(\lambda+1)a_2 = B_1 b_1, \tag{2.21}$$

$$2(1+2\lambda)(3a_3 - 2a_2^2) - 4\lambda^2 a_2^2 = B_1 b_2 + B_2 b_1^2, \tag{2.22}$$

$$-2(\lambda+1)a_2 = B_1 c_1, \tag{2.23}$$

$$2(1+2\lambda)(4a_2^2 - 3a_3) - 4\lambda^2 a_2^2 = B_1 c_2 + B_2 c_1^2. \tag{2.24}$$

From (2.21) and (2.23) we get

$$c_1 = -b_1, \tag{2.25}$$

and

$$8(\lambda+1)^2 a_2^2 = B_1^2 (b_1^2 + c_1^2). \tag{2.26}$$

By adding the relation (2.22) with the relation (2.24) and use (2.26) we have

$$2\Big(B_1^2(4\lambda - 4\lambda^2 + 2) - 4(\lambda+1)^2 B_2\Big) a_2^2 = B_1^3 (b_2 + c_2).$$

Now, in view of (2.3), (2.21) and (2.25) we have

$$\begin{aligned} 2\Big|B_1^2(4\lambda - 4\lambda^2 + 2) - 4(\lambda+1)^2 B_2\Big|\ |a_2|^2 &\le B_1^3(|b_2| + |c_2|) \\ &\le B_1^3\big((1-|b_1|^2) + (1-|c_1|^2)\big) \\ &\le 2B_1^3(1-|b_1|^2) \\ &= 2B_1^3 - 8(\lambda+1)^2 B_1 |a_2|^2. \end{aligned}$$

Then, we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\big|2(2\lambda+1-2\lambda^2)B_1^2 - 4(\lambda+1)^2 B_2\big| + 4(\lambda+1)^2 B_1}}. \tag{2.27}$$

Next, from (2.22) and (2.24), we have

$$12(1+2\lambda)a_3 = B_1(b_2 - c_2) + 12(1+2\lambda)a_2^2.$$

Then in view of (2.3) and (2.25), we have

$$\begin{aligned} 12(1+2\lambda)|a_3| &\le 12(1+2\lambda)|a_2|^2 + B_1(|b_2| + |c_2|) \\ &\le 12(1+2\lambda)|a_2|^2 + 2B_1(1-|b_1|^2). \end{aligned}$$

It follows from (2.21) that

$$6B_1(1+2\lambda)|a_3| \leq \left[6B_1(1+2\lambda) - 4(1+\lambda)^2\right]|a_2|^2 + B_1^2.$$

Now from (2.27) we have

$$|a_3| \leq \begin{cases} \dfrac{B_1}{6(1+2\lambda)}, & B_1 \leq \dfrac{2(1+\lambda)^2}{3(1+2\lambda)}, \\ \dfrac{\left|(2\lambda+1-2\lambda^2)B_1^2 - 2(\lambda+1)^2B_2\right| + 3(1+2\lambda)B_1^3}{3(1+2\lambda)\left[\left|2(2\lambda+1-2\lambda^2)B_1^2 - 4(\lambda+1)^2B_2\right| + 4(\lambda+1)^2B_1\right]}, & B_1 \geq \dfrac{2(1+\lambda)^2}{3(1+2\lambda)}. \end{cases}$$

This completes the proof. □

Remark 2.6. If let

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots, \qquad 0 < \gamma \leq 1$$

in theorem 2.5, then inequalities (2.17) and (2.18) become

$$|a_2| \leq \frac{\alpha}{\sqrt{\lambda(2+\lambda+3\alpha\lambda)}},$$

and

$$|a_3| \leq \begin{cases} \dfrac{\alpha}{3(1+2\lambda)}, & \alpha \leq \dfrac{(1+\lambda)^2}{3(1+2\lambda)}, \\ \dfrac{\alpha\big(2\alpha + \lambda(3\alpha+\lambda)\big)}{2\lambda(1+2\lambda)(1-\alpha)(1+\lambda+3\alpha\lambda)}, & \gamma \geq \dfrac{(1+\alpha)^2}{4(1+2\alpha)}. \end{cases}$$

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