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Quasi 2-absorbing Submodules

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Abstract. In this paper, we introduce the notion of quasi 2-absorbing submodules of modules over a commutative ring and obtain some basic properties of this class of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, Z will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [6].

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [3]. A proper ideal I of R is called a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The authors in [5] and [12], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of an R-module M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

The purpose of this paper is to introduce the concepts of quasi 2-absorbing submodules as a generalization of 2-absorbing submodules and obtain some related results.

2. Main Results

Definition 2.1. We say that a proper submodule N of an R-module M is a *quasi 2-absorbing submodule* if $(N :_R M)$ is a 2-absorbing ideal of R.

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Example 2.2. By [12, 2.3], every 2-absorbing submodule is a quasi 2-absorbing submodule. But the converse is not true in general. For example, the submodules $\langle 1/p^2 + Z\rangle$ and $\langle 1/p + Z\rangle$ of the Z-module Z_{p^∞} are quasi 2-absorbing submodules which are not 2-absorbing submodules.

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [4].

Proposition 2.3. *Let M be a multiplication R-module. Then a submodule N of M is a 2-absorbing submodule of M if and only if it is a quasi 2-absorbing submodule of M.*

Proof. This follows from [2, Theorem 3.9]. □

Proposition 2.4. *Let M be an R-module and N_1, N_2 be two submodules of M with $(N_1 :_R M)$ and $(N_2 :_R M)$ prime ideals of R. Then $N_1 \cap N_2$ is a quasi 2-absorbing submodule of M.*

Proof. Since $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$, the result follows from [3]. □

Let N be a submodule of an R-module M. The intersection of all prime submodules of M containing N is said to be the (*prime*) *radical* of N and denote by $rad(N)$. In case N does not contained in any prime submodule, the radical of N is defined to be M [10].

An R-module M is said to be a *Laskerian module* if every proper submodule of M is a finite intersection of primary submodules of M [8]. We know that every Noetherian module is Laskerian.

Theorem 2.5. *Let M be an R-module and N be a quasi 2-absorbing submodule of M. Then we have the following:*

(a) *$(N :_M I)$ is a quasi 2-absorbing submodules of M for all ideals I of R with $I \not\subseteq (N :_R M)$.*

(b) *If I is an ideal of R, then $(N :_R I^n M) = (N :_R I^{n+1} M)$, for all $n \geq 2$.*

(c) *If M is a finitely generated Laskerian R-module, then $rad(N)$ is a quasi 2-absorbing submodule of M.*

Proof. (a) Let I be an ideal of R with $I \not\subseteq (N :_R M)$. Then $((N :_M I) :_R M)$ is a proper ideal of R. Now let $a, b, c \in R$ and $abcM \subseteq (N :_M I)$. Then $abcIM \subseteq N$. Thus either $acM \subseteq N$ or $cbIM \subseteq N$ or $abIM \subseteq N$. If $cbIM \subseteq N$ or $abIM \subseteq N$, then we are done. If $acM \subseteq N$, then $acM \subseteq (N :_M I)$, as needed.

(b) It is enough to show that $(N :_R I^2M) = (N :_R I^3M)$. It is clear that $(N :_R I^2M) \subseteq (N :_R I^3M)$. Since N is quasi 2-absorbing submodule, $(N :_R I^3M)I^3M \subseteq N$ implies that $(N :_R I^3M)I^2M \subseteq N$ or $I^2M \subseteq N$. If $(N :_R I^3M)I^2M \subseteq N$, then $(N :_R I^3M) \subseteq (N :_R I^2M)$. If $I^2M \subseteq N$, then $(N :_R I^2M) = R = (N :_R I^3M)$.

(c) Let M be a finitely generated Laskerian R-module. Then $\sqrt{(N :_R M)} = (rad(N) :_R M)$ by [9, Theorem 5]. Now the result follows from the fact that $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of R by [3, Theorem 2.1]. □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [1].

Corollary 2.6. *Let M be a comultiplication R-module such that the zero submodule of M is a quasi 2-absorbing submodule. Then every proper submodule of M is a quasi 2-absorbing submodule of M.*

Proof. This follows from Theorem 2.5 (a). □

Proposition 2.7. *Let M be an R-module and $\{K_i\}_{i\in I}$ be a chain of quasi 2-absorbing submodules of M. Then $\cap_{i\in I}K_i$ is a quasi 2-absorbing submodule of M.*

Proof. Clearly, $(\cap_{i\in I}K_i :_R M) \neq R$. Let $a, b, c \in R$ and $abc \in (\cap_{i\in I}K_i :_R M) = \cap_{i\in I}(K_i :_R M)$. Assume to the contrary that $ab \notin \cap_{i\in I}(K_i :_R M)$, $bc \notin \cap_{i\in I}(K_i :_R M)$, and $ac \notin \cap_{i\in I}(K_i :_R M)$. Then exist $m, n, t \in I$ such that $ab \notin (K_n :_R M)$, $bc \notin (K_m :_R M)$, and $ac \notin (K_t :_R M)$. Since $\{K_i\}_{i\in I}$ is a chain, we can assume without loss of generality that $K_m \subseteq K_n \subseteq K_t$. Then

$$(K_m :_R M) \subseteq (K_n :_R M) \subseteq (K_t :_R M).$$

As $abc \in (K_m :_R M)$, we have either $ab \in (K_m :_R M)$ or $ac \in (K_m :_R M)$ or $bc \in (K_m :_R M)$. In any case, we have a contradiction. □

Definition 2.8. We say that a quasi 2-absorbing submodule N of an R-module M is a *minimal quasi 2-absorbing submodule* of a submodule K of M, if $K \subseteq N$ and there does not exist a quasi 2-absorbing submodule T of M such that $K \subset T \subset N$.

It should be noted that a minimal quasi 2-absorbing submodule of M means that a minimal quasi 2-absorbing submodule of the submodule 0 of M.

Lemma 2.9. *Let M be an R-module. Then every quasi 2-absorbing submodule of M contains a minimal quasi 2-absorbing submodule of M.*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.7. □

Theorem 2.10. *Let M be a Noetherian R-module. Then M contains a finite number of minimal quasi 2-absorbing submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of all proper submodules N of M such that the module M/N has an infinite number of minimal quasi 2-absorbing submodules. Since $0 \in \Sigma$, we have $\Sigma \neq \emptyset$. Therefore Σ has a maximal member T, since M is a Noetherian R-module. Clearly, T is not a quasi 2-absorbing submodule. Therefore, there exist $a, b, c \in R$ such that $abc(M/T) = 0$ but $ab(M/T) \neq 0$, $ac(M/T) \neq 0$, and $bc(M/T) \neq 0$. The maximality of T implies that $M/(T + abM)$, $M/(T + acM)$, and $M/(T + bcM)$ have only finitely many minimal quasi 2-absorbing submodules. Suppose P/T is a minimal quasi 2-absorbing submodule of M/T. So $abcM \subseteq T \subseteq P$, which implies that either

$abM \subseteq P$ or $acM \subseteq P$ or $bcM \subseteq P$. Thus either $P/(T + abM)$ is a minimal quasi 2-absorbing submodule of $M/(T + abM)$ or $P/(T + bcM)$ is a minimal quasi 2-absorbing submodule of $M/(T + bcM)$ or $P/(T + acM)$ is a minimal quasi 2-absorbing submodule of $M/(T + acM)$. Therefore, there are only a finite number of possibilities for the submodule P. This is a contradiction. □

Recall that $Z(R)$ denotes the set of zero divisors of R.

Proposition 2.11. *Let N be a submodule of a finitely generated R-module M and S be a multiplicatively closed subset of R. If N is a quasi 2-absorbing submodule and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a quasi 2-absorbing $S^{-1}R$-submodule of $S^{-1}M$. Furthermore, if $S^{-1}N$ is a quasi 2-absorbing $S^{-1}R$-submodule and $S \cap Z(R/(N :_R M)) = \emptyset$, then N is a quasi 2-absorbing submodule of M.*

Proof. As M is a finitely generated R-module,

$$(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}((N :_R M))$$

by [13, Lemma 9.12]. Now the result follows from [11, Theorem 1.3]. □

Lemma 2.12. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then N is a quasi 2-absorbing submodule of M if and only if $f(N)$ is a quasi 2-absorbing submodule of $f(M)$.*

Proof. This follows from the fact that $(N :_R M) = (f(N) :_R f(M))$. □

Lemma 2.13.([7, Corollary 2.11]) *Let N be a submodule of a multiplication R-module M. Then N is a prime submodule of M if and only if $(N :_R M)$ is a prime ideal of R.*

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.14. *Let $R = R_1 \times R_2$ be a decomposable ring and let $M = M_1 \times M_2$ be an R-module, where M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module. Suppose that $N = N_1 \times N_2$ is a proper submodule of M. Then the following conditions are equivalent:*

(a) *N is a quasi 2-absorbing submodule of M;*

(b) *Either $N_1 = M_1$ and N_2 is a quasi 2-absorbing submodule of M_2 or $N_2 = M_2$ and N_1 is a quasi 2-absorbing submodule of M_1 or N_1, N_2 are prime submodules of M_1, M_2, respectively.*

Proof. Since $(N_1 \times N_2 :_{R_1 \times R_2} M_1 \times M_2) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$, the result follows from [11, Theorem 1.2] and Lemma 2.13. □

Theorem 2.15. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is a multiplication R_i-module, respectively. Then for a proper submodule N of M the following conditions are equivalent:*

(a) *N is a quasi 2-absorbing submodule of M;*

(b) *Either $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a quasi 2-absorbing submodule of M_k and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$ or $N = \times_{i=1}^{n} N_i$ such that for some $k, m \in \{1, 2, ..., n\}$, N_k is a prime submodule of M_k, N_m is a prime submodule of M_m, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 2.14. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem **??**, N is a quasi 2-absorbing submodule of M if and only if either $N = L \times M_n$ for some quasi 2-absorbing submodule L of K or $N = K \times L_n$ for some quasi 2-absorbing submodule L_n of M_n or $N = L \times L_n$ for some prime submodule L of K and some prime submodule L_n of M_n. Notice that a proper submodule L of K is a prime submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, ..., n-1\}$, N_k is a prime submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n-1\} \setminus \{k\}$. Consequently we reach the claim. □

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