KYUNGPOOK Math. J. 59(2019), 203-207
https://doi.org/10.5666/KMJ.2019.59.2.203
pISSN 1225-6951 eISSN 0454-8124

The Commutativity Degree in the Class of Nonabelian Groups of Same Order

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Abstract. The commutativity degree of a finite group is the probability that two randomly chosen group elements commute. In this paper we give a sharp upper bound of commutativity degree of nonabelial groups in terms of their order.

1. Introduction

The commutativity degree of a finite group is the probability that two randomly chosen group elements commute. In the other words, the commutativity degree of a finite group G is the ratio

$$cp(G) = \frac{|\{(x,y) \in G \times G : [x,y] = 1\}|}{|G|^2}.$$

It is easy to see that $cp(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|^2}$ where $C_G(x)$ is the centralizer of x in G. Also G is abelian if and only if $cp(G) = 1$.

During the last few decades, there has been a growing interest in the study of finite groups in terms of their commutativity degree. This ratio has been investigated by many authors. For example, Gustafson in [4] showed $cp(G) \leq \frac{5}{8}$ for all non-abelian groups G. Lescot, in [6], determined all groups G with $cp(G) \geq \frac{1}{2}$. Guranlik and G. R. Robinson considered all groups with $cp(G) > \frac{3}{40}$ and proved that such groups are either solvable or are isomorphic to $A_5 \times C_2^n$ where $n \geq 1$ (see [3]). Another results about $cp(G)$ can be found in [1] and [9].

Our result is as follows.

Theorem 1.1. *Let G be a non-abelian group of order $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ where $n_i > 0$ for each i and $p_1, \cdots, p_r$ are distinct primes. Then*

Received July 30, 2017; revised March 29, 2019; accepted April 23, 2019.
2010 Mathematics Subject Classification: 20D60.
Key words and phrases: finite group, commutativity degree, nonnilpotent.

(1) *$cp(G) \leq \frac{p^2+p-1}{p^3}$ and the equality holds if and only if $\frac{G}{Z(G)} \cong Z_p \times Z_p$ where p is the smallest prime divisor of $|\frac{G}{Z(G)}|$.*

(2) *Let G be a non-nilpotent group. Then*

$$cp(G) \leq max\{\frac{p_i^{t_i} + p_j^2 - 1}{p_i^{t_i} p_j^2} : \quad p_j | p_i^{t_i} - 1, 1 \leq i, j \leq r, 1 \leq t_i \leq n_i\}.$$

and the equality holds if and only if $\frac{G}{Z(G)} \cong Z_{p_i}^{t_i} \rtimes Z_{p_j}$ for integers i and j such that $p_j | p_i^{t_i} - 1$ and $1 \leq t_i \leq n_i$.

First part of above theorem generalizes Lemma 1.3 of [5] and the second part generalizes a result of Lescot which can be found in [5].

Throughout the paper all groups are finite and p is a prime. Also, for $g \in G$, we often use $\overline{g}$ to denote the coset $gZ(G)$, and use $\overline{H}$ for the factor group $\frac{HZ(G)}{Z(G)}$ where H is any subgroup of G. Other notation is standard and can be found in [8].

2. Results

A minimal non-abelin group is a non-abelian group all of whose subgroups are abelian. Following lemma gives an important property of minimal non-abelian p-groups. It will be used in the proof of Lemma 2.2.

Lemma 2.1. *Let G be a minimal non-abelian p- group. Then $\frac{G}{Z(G)} \cong Z_p \times Z_p$.*

Proof. Since G is non-abelian, there exist two elements x and y such that $xy \neq yx$. Therefore $C_G(x)$ and $C_G(y)$ are two distinct centralizers of G. By hypothesis these centralizers are abelian and have index p. On the other hand if $t \in (C_G(x) \bigcap C_G(y)) \setminus Z(G)$ then $C_G(t) = C_G(x) = C_G(y)$. This is a contradiction. Thus $C_G(x) \cap C_G(y) = Z(G)$ and so $\frac{G}{Z(G)} \cong Z_p \times Z_p$ as a desired. □

Using the above lemma and Theorem 10.1.7 of [8] and Exercise 9.1.11 of [8], we can say that every non-abelian group contains either a minimal non-abelian p-subgroup or a subgroup with Frobenius central factor. The following lemma play an important role in the proof of our main theorem.

Lemma 2.2. *Let G be a group. If G is non-abelian, then G contains a subgroup H such that either $\frac{H}{Z(H)} \cong Z_r^t \rtimes Z_q$ or $Z_p \times Z_p$ where p, q and r are distinct primes.*

Proof. Towards contradiction, let G be a counter-example of minimal order. If H is a non-abelian subgroup of G, then it, and consequently G, has a subgroup with the desired structure. So we may assume that G is a minimal non-abelian group. Using Theorem 10.1.7 of [8] and Exercise 9.1.11 of [8] and Lemma 2.1 we get a contradiction. □

The following lemma, which will be used in prove of Theorem 1.1, gives some relations between commutativity degree of certain groups and their sections and subgroups.

Lemma 2.3. *Let G be a finite group.*

(1) *For every proper subgroup H of G, we have $cp(G) \leq cp(H)$.*

(2) *Whenever $N \trianglelefteq G$, we have $cp(G) \leq cp(\frac{G}{N})$.*

(3) *For every section, X, of G, we have $cp(G) \leq cp(X)$.*

Proof. See proof of Lemma 2 of [3]. □

Lemma 2.4. *Let G be a non-abelian group.*

(1) *If $\frac{G}{Z(G)} \cong Z_p^t \rtimes Z_q$, then $cp(G) = \frac{p^t+q^2-1}{p^t q}$.*

(2) *If $\frac{G}{Z(G)} \cong Z_p \times Z_p$, then $cp(G) = \frac{p^2+p-1}{p^3}$.*

Proof. Let G be a non abelian group and $\frac{G}{Z(G)} \cong Z_p^t \rtimes Z_q$. Since $\frac{H}{Z(H)}$ has an abelian subgroup of index q, Theorem A of [2] tells us that $\overline{G}$ is a Frobenius group. Now, all centralizers of $\overline{G}$ are either of order q or p^t. By counting all of them and using the definition we get the result in first case.

Now let G be a group with central factor isomorphic to $Z_p \times Z_p$. It is clear that all centralizers of noncentral elements of G are abelian of index p. So as a result we have

$$cp(G) = \frac{(|G| - |Z(G)|(\frac{|G|}{p})) + \frac{|G|^2}{p^2}}{|G|^2} = \frac{p^2+p-1}{p^3}. \qquad \square$$

Now we recall a nilpotent number defined in [7].

Definition 2.5. A positive integer n is called a nilpotent number if every group of order n is nilpotent.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that there is at least one non-abelian group G of order n such that p is the smallest prime divisor of $|\frac{G}{Z(G)}|$. We denote some p-sylow subgroup of G by P. First we suppose that P is non-abelian and thus all Sylow p-subgroups P of G are non-abelian. By Lemma 2.2, P contains some minimal non abelian subgroup. Therefore by Lemma 2.3 and Lemma 2.4 we have the result in first case. Now let $cp(G) = \frac{p^2+p-1}{p^3}$ but $\frac{G}{Z(G)}$ be not isomorphic to $Z_p \times Z_p$. Then by Lemmas 2.2, 2.3 and 2.4 we have a contradiction. Now if $\frac{G}{Z(G)} \cong Z_p \times Z_p$, then Lemma 2.4 gives the result in first case.

Now let G be a non-nilpotent group. Thus n is not a nilpotent number and so there are positive integers r and s such that $r|s^i - 1$ for some integer i. Also let

$$\Upsilon = \{(r,s) | r \neq s \text{ are primes, } r|s^i - 1 \text{ but } r \nmid s^j - 1 \text{ for all integers } j < i\}.$$

Put also $\Gamma = \{\frac{r^t+s^2-1}{r^ts^2} | (r,s) \in \Upsilon\}$. We know by [7] that Υ and so Γ are non-empty. Now we choose $(p,q) \in \Upsilon$ such that $\frac{q^\alpha+p^2-1}{q^\alpha p^2}$ is a maximum element of Γ. We now define a group H.

It is well known that $Aut(Z_q^i) \cong PGL_i(q)$ and so

$$|Aut(Z_q^i)| = (q^i-q)(q^i-q)\cdots(q^i-q^{i-1}).$$

Therefore there is some $\theta \in Aut(Z_q^i)$ with $|\theta| = q$. Let

$$\iota : Z_p \to Aut(Z_q^i)$$

be a group homomorphism such that $\iota(a) = \theta$ and $Z_p = \langle a \rangle$. Now $Z_q^i \rtimes Z_p$ is a semidirect group with respect to ι and we denote $Z_q^i \rtimes Z_p \times A$ by H in which A is abelian group with $|A| = \frac{n}{q^ip}$. By Lemma 2.4, we have $cp(H) = \frac{q^i+p^2-1}{q^ip}$.

Now we prove that for all non-nilpotent groups G of order n, $cp(G) \leq \frac{p^i+q^2-1}{p^iq}$. If all proper subgroups of G are nilpotent, then using Theorem 10.1.7 of [8] and Exercise 9.1.11 of [8] one can see that $\frac{G}{Z(G)}$ is Frobenius group whose Frobenius kernel is elementary abelian and whose Frobenius complement is of prime order. Now assume that G contains at least one proper non-nilpotent subgroup. Therefore by Lemma 2.2, G has a proper subgroup whose central factor is a Frobenius group with elementary abelian Frobenius kernel and cyclic Frobenius complement of prime order. Anyway, we can assume that K is a subgroup of G (perhaps G itself) such that $\frac{K}{Z(K)} \cong Z_r^t \rtimes Z_s$ where r, s are distinct primes and t is an positive integer. By Lemma 2.3, $cp(G) \leq cp(\frac{K}{Z(K)}) = \frac{r^t+s^2-1}{r^ts^2}$. But by choosing $(p,q) \in \Upsilon$, we have $cp(G) \leq cp(H)$ and proof is complete. □

In the above theorem, if $|G|$ is even, then we have the following.

Corollary 2.6. *Let G be a non- nilpotent group of order n. Then $cp(G) \leq \frac{p+3}{4p}$ and equality holds if and only if $\frac{G}{Z(G)} \cong D_{2p}$.*

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