

An Efficient Model Based on Smoothed ℓ_0 Norm for Sparse Signal Reconstruction

Yangyang Li, Guiling Sun*, Zhouzhou Li and Tianyu Geng

College of Electronic Information and Optical Engineering, Nankai University, Tianjin 300350, P.R.China
[e-mail: liyangyang@mail.nankai.edu.cn, sungl@nankai.edu.cn]

*Corresponding author: Guiling Sun

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Abstract

Compressed sensing (CS) is a new theory. With regard to the sparse signal, an exact reconstruction can be obtained with sufficient CS measurements. Nevertheless, in practical applications, the transform coefficients of many signals usually have weak sparsity and suffer from a variety of noise disturbances. What's worse, most existing classical algorithms are not able to effectively solve this issue. So we proposed an efficient algorithm based on smoothed ℓ_0 norm for sparse signal reconstruction. The direct ℓ_0 norm problem is *NP* hard, but it is unrealistic to directly solve the ℓ_0 norm problem for the reconstruction of the sparse signal. To select a suitable sequence of smoothed function and solve the ℓ_0 norm optimization problem effectively, we come up with a generalized approximate function model as the objective function to calculate the original signal. The proposed model preserves sharper edges, which is better than any other existing norm based algorithm. As a result, following this model, extensive simulations show that the proposed algorithm is superior to the similar algorithms used for solving the same problem.

Keywords: Compressed sensing, smoothed ℓ_0 norm, generalized approximate function, reconstruction algorithm.

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1. Introduction

In recent years, the research of compressed sensing(CS) [1-2] has received more attention as a mean to process the sparse signal (i.e., the number of nonzero elements in the vector is small). CS is a new signal processing technology, by solving underdetermined linear systems, to effectively acquire and reconstruct the signal. It was proposed by Donoho, Candes and Tao Zhexuan in 2006. It is very magical that CS theory can exactly recover the sparse signal or compressible signals by a sampling rate which does not satisfy the Nyquist–Shannon sampling theorem. But many papers have demonstrated that CS can effectively get the key information from sample value with fewer non-correlative measurements. The key advantage of CS is that it allows both compression and sampling to run simultaneously. CS technique can reduce the hardware requirements, further reduce the sampling rate, improve the signal quality, save on signal processing and transmission costs. Currently, CS has been widely used in wireless sensor networks, information theory, signal processing, medical image, optical/microwave imaging, SAR image, wireless communications, atmosphere, geology and other fields [3-5].

The research of CS theory is mainly divided into three aspects: 1) the sparse representation of signals; 2) the uncorrelated sampling[6]; 3) sparse reconstruction [7]. The design of sparse reconstruction algorithm is the most important one. It is the huge challenge for the researchers to propose an efficient reconstruction algorithm with reliable accuracy.

Theoretically, under the condition of sparse assumption, one hope to reconstruct the signal $\mathbf{x} \in \mathbf{R}^N$, for example, $\mathbf{y} \in \mathbf{R}^M$ is a known vector. For the reconstruction of the sparse signal \mathbf{x} , which can be solved through the following non-convex problem:

$$\min \|\mathbf{x}\|_0 \quad s.t. \quad \mathbf{y} = \Phi \mathbf{x} . \quad (1)$$

Where $\|\mathbf{x}\|_0$ is the zero-norm of the \mathbf{x} . $\Phi \in \mathbf{R}^{M \times N}$ is the measurement matrix. The above formula (1) is NP-hard. To solve the issue of formula (1), we have to get a solution with the least non-zero elements in all solutions. It is not practical to directly solve that problem. This paper has proved that if the measurement matrix obeys a constraint known as the Restricted Isometry Property(RIP) [8-10], an equivalent solution can be get for the optimization problem (1) based on ℓ_1 norm. With regard to the measurement matrix Φ , it considers the sparse signal \mathbf{x} ($\|\mathbf{x}\|_0 = k$), when find a suitable constant δ_k satisfies:

$$(1 - \delta_k) \|\mathbf{x}\|_2^2 \leq \|\Phi \mathbf{x}\|_2^2 \leq (1 + \delta_k) \|\mathbf{x}\|_2^2 . \quad (2)$$

In (2), δ_k follows $0 < \delta_k \leq 1$. Generally speaking, if the δ_k is very closed to 1, then it is possible that the measurement \mathbf{y} can not preserve any information on \mathbf{x} when the $\|\Phi \mathbf{x}\|_2^2 \approx 0$. As a result, it is nearly impossible to reconstruct the sparse signal \mathbf{x} by using the greedy algorithms.

If RIP is satisfied, the solution based on the ℓ_1 norm problem is the convex relaxation of the ℓ_0 norm:

$$\min \|\mathbf{x}\|_1 \quad s.t. \quad \mathbf{y} = \Phi \mathbf{x} . \quad (3)$$

For (3), Many existing methods can solve this issue. Equation (3) is a convex problem, and the methods to solve equation (3) is called a convex optimization algorithm [11], such as the basis

pursuit algorithm (BP) [12], and linear programming algorithm. But this kind of algorithm has high computational complexity.

A series of greedy algorithms has been receiving great interest due to their low complexity and simple geometric interpretation, such as the Orthogonal Matching Pursuit (OMP)[13], the Stagewise OMP (StOMP)[14] algorithms, and stagewise weak gradient pursuits (SWOMP)[15]. The feature of those algorithms is to seek the sparse position of the unknown signal step by step. And the smoothed ℓ_0 norm algorithms have also received significant attention, such as the SL0 algorithm, Thresholded SL0(TSL0)[16]. The simulation for typical reconstruction problems, including one-dimensional and two-dimensional signal reconstruction, show that the proposed model is superior to the existing advanced reconstruction algorithms. But Anti-noise performance of the greedy algorithms is poor. Even small additive noise is likely to lead to the bad signal recovery effect.

In a sense, the ℓ_0 norm is robust to noise, it gives the highest possibility of sparse reconstruction with fewer measurements. That also motivates the use of continuous approximate function to solve (1).

We introduce an efficient algorithm based on smoothed ℓ_0 norm for sparse signal reconstruction in this paper. To design an suitable iterative sequence of smoothed function and get an optimized solution of the ℓ_0 norm problem, we also come up with a generalized approximate function model as the objective function to calculate the original signal. The proposed model preserves sharper edges, which is better than any other existing norm regularized algorithm. The results of experiment verified that the new algorithm based on the generalized approximate function model is better than other similar algorithms used for solving the same problem.

Other parts of this paper is arranged as follows. Part 2 introduces the basic ideas of the proposed algorithm. Part 3, we discuss the procedure of the new algorithm. Part 4 is simulation result and analysis, and the last part is the conclusion.

2. Main Idea

The fundamental idea of CS theory is to extract vector \mathbf{x} from \mathbf{y} . To solve the issue of ℓ_0 norm of discontinuity, the existing idea is to approximate this discontinuous function by a suitable continuous one. And there are many kinds of approximations smooth function. For example, the most classic Gaussian Function. But this paper proposed a generalized approximate function model, and it has a more accurate approximation effect.

$$f_{\sigma}(x) = \frac{e^{\beta x^2/\sigma^2} - e^{-\beta x^2/\sigma^2}}{e^{\beta x^2/\sigma^2} + e^{-\beta x^2/\sigma^2}}. \quad (4)$$

In (4), β represents a positive number. The parameter σ determines the reconstructed quality of the sparse signal \mathbf{x} by the approximations smooth function. The smaller σ , the better approximation, and the larger σ , the smoother approximation. And note that:

$$\lim_{\sigma \rightarrow 0} f_{\sigma}(x) = \begin{cases} 1; & \text{if } x \neq 0 \\ 0; & \text{if } x = 0 \end{cases}. \quad (5)$$

Or approximately that:

$$f_{\sigma}(x) = \begin{cases} 1; & \text{if } x \gg \sigma \\ 0; & \text{if } x \ll \sigma \end{cases}. \quad (6)$$

Then we can define:

$$\lim_{\sigma \rightarrow 0} F_{\sigma}(x) = \sum_{i=1}^N f_{\sigma}(x_i) . \tag{7}$$

It can be learned from the above formula that $\|x\|_0 = F_{\sigma}(x)$. In this paper [17], it considers the continuous Gaussian function for the smoothed approximations.

$$g_{\sigma}(x) = e^{-x^2/2\sigma^2} . \tag{8}$$

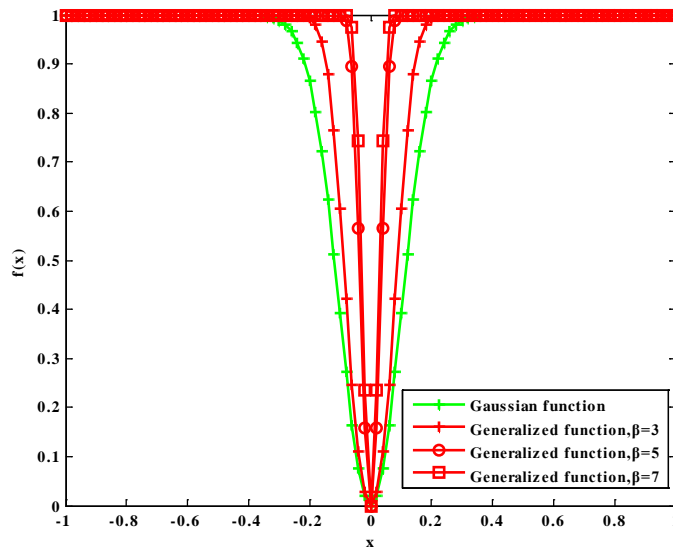


Fig. 1. Comparison of the approximate ℓ_0 norm functions

For (4) (8), the approximation performance of ℓ_0 norm, however, is different. To further prove the superiority of the proposed generalized approximation model as a smooth continuous approximation function, we have experimentally compared the distribution of the proposed generalized approximate function with the standard Gaussian function for different parameters β at interval $[-1, 1]$ when $\sigma = 0.1$. The comparison results are shown in Fig. 1.

As can be seen from the Fig. 1, the proposed generalized approximate function model has steeper properties. Therefore, it would be more precise to estimate the ℓ_0 norm.

Remark 1. When $\beta > 0.5$, the generalized approximation function model proposed in this paper has a better approximation than other models. It can be proved by comparison with standard gaussian function.

Proof of remark 1: Take $u(x) = f(x) - (1 - g(x))$, when $\beta > 0.5$, $u(x) \geq 0$. To simplify the proof, let $\beta = \alpha / 2$.

$$\begin{aligned}
u(x) &= \frac{e^{\alpha x^2/2\sigma^2} - e^{-\alpha x^2/2\sigma^2}}{e^{\alpha x^2/2\sigma^2} + e^{-\alpha x^2/2\sigma^2}} - 1 + e^{-x^2/2\sigma^2} \\
&= \left(1 - \frac{2e^{-\alpha x^2/2\sigma^2}}{e^{\alpha x^2/2\sigma^2} + e^{-\alpha x^2/2\sigma^2}} \right) - 1 + e^{-x^2/2\sigma^2} \\
&= e^{-x^2/2\sigma^2} - \frac{2e^{-\alpha x^2/2\sigma^2}}{e^{\alpha x^2/2\sigma^2} + e^{-\alpha x^2/2\sigma^2}} \\
&= \frac{e^{(\alpha-1)x^2/2\sigma^2} + e^{-(\alpha+1)x^2/2\sigma^2} - 2e^{-\alpha x^2/2\sigma^2}}{e^{\alpha x^2/2\sigma^2} + e^{-\alpha x^2/2\sigma^2}} \\
&= \frac{e^{-\alpha x^2/2\sigma^2} \left(e^{(2\alpha-1)x^2/2\sigma^2} + e^{-x^2/2\sigma^2} - 2 \right)}{e^{\alpha x^2/2\sigma^2} + e^{-\alpha x^2/2\sigma^2}}
\end{aligned} \tag{9}$$

Introduce auxiliary function $h(x)$:

$$\begin{aligned}
h(x) &= e^{(2\alpha-1)x^2/2\sigma^2} + e^{-x^2/2\sigma^2} - 2 \\
&\geq 2\sqrt{e^{(\alpha-1)x^2/\sigma^2}} - 2 \geq 2(e^{(\alpha-1)x^2/2\sigma^2} - 1) .
\end{aligned} \tag{10}$$

In (10), when $\alpha > 1$ (i.e. $\beta > 0.5$), $u(x) > 0$, which completes the proof. In summary, if $\beta > 0.5$, then the proposed generalized approximate function model is steeper between the -0.2 and 0.2. So the approximation of the ℓ_0 norm is more efficient. Furthermore, it's easy to see that when β gets bigger, the better the effect of function approximation will be. But β is not infinitely large, and it usually achieves the perfect result between 1 and 10.

3. The Proposed Algorithm

In this part, we introduce the proposed generalized approximations model to solve the sparse signal reconstruction problem and give the mathematical analysis. At the same time, the improved quasi-newton method is used as the target search direction to strongly accelerate the convergence speed. Finally, we design a novel reconstruction algorithm to recover the original signal.

From (7), the minimization of the ℓ_0 norm is equivalent to the minimization of $F_\sigma(x)$ for sufficiently small σ .

$$\min F_\sigma(x), \quad s.t. \quad y = \Phi x . \tag{11}$$

Many algorithms can be used to solve equation (11), the most representative of which is the steepest descent method. The steepest descent, however, has a severe notched effect, which can seriously affect the convergence speed of the algorithm. Therefore, we use the improved Newton method to solve this problem. First, the Newton direction is calculated according to the generalized approximation function model

$$d = -\nabla^2 F_\sigma(x)^{-1} \nabla F_\sigma(x) . \tag{12}$$

Where

$$\nabla F_{\sigma}(\mathbf{x}) = \left[\begin{array}{c} \frac{8\beta x_1}{\sigma^2 (e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2})^2}, \\ \frac{8\beta x_2}{\sigma^2 (e^{\beta x_2^2/\sigma^2} + e^{-\beta x_2^2/\sigma^2})^2}, \dots, \\ \frac{8\beta x_N}{\sigma^2 (e^{\beta x_N^2/\sigma^2} + e^{-\beta x_N^2/\sigma^2})^2} \end{array} \right]^T \quad (13)$$

$$\begin{aligned} \nabla^2 F_{\sigma}(x_1) &= \frac{8\beta\sigma^2 (e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2}) - 32\beta^2 x_1^2 (e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2})}{\sigma^4 (e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2})^3} \\ &= \frac{\frac{8\beta}{\sigma^2} \left[(e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2}) - \frac{4\beta x_1^2}{\sigma^2} (e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2}) \right]}{(e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2})^3} \quad (14) \\ &= \frac{\frac{8\beta}{\sigma^2} \left[\left(1 - \frac{4\beta x_1^2}{\sigma^2}\right) e^{\beta x_1^2/\sigma^2} + \left(1 + \frac{4\beta x_1^2}{\sigma^2}\right) e^{-\beta x_1^2/\sigma^2} \right]}{(e^{\beta x_1^2/\sigma^2} + e^{-\beta x_1^2/\sigma^2})^3} \end{aligned}$$

In order to make sure that Newton direction is a descent direction, the matrix $\nabla^2 F_{\sigma}(\mathbf{x})$ must be a positive definite matrix. So the matrix $\nabla^2 F_{\sigma}(\mathbf{x})$ should be improved. Then, we can set up a new matrix:

$$\mathbf{G} = \nabla^2 F_{\sigma}(\mathbf{x}) + \boldsymbol{\varepsilon} \mathbf{I} \quad (15)$$

Where \mathbf{I} is the identity matrix, $\boldsymbol{\varepsilon}$ is a suitable set of improvement coefficients, and the diagonal elements are positive in matrix \mathbf{G} . For example, from (14), we can choose $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \frac{8\beta \left[\frac{5\beta x^2}{\sigma^2} e^{\beta x^2/\sigma^2} - \frac{3\beta x^2}{\sigma^2} e^{-\beta x^2/\sigma^2} \right]}{\sigma^2 (e^{\beta x^2/\sigma^2} + e^{-\beta x^2/\sigma^2})^3} \quad (16)$$

as the improvement coefficients. Then matrix \mathbf{G} can be shown as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}(x_1) & 0 & \dots & 0 \\ 0 & \mathbf{G}(x_2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \mathbf{G}(x_N) \end{bmatrix}, \quad (17)$$

$$\mathbf{G}(x_i) = \frac{8\beta \left(1 + \frac{\beta x_i^2}{\sigma^2}\right)}{\sigma^2 (e^{\beta x_i^2/\sigma^2} + e^{-\beta x_i^2/\sigma^2})} \quad (18)$$

So, it can be obtained that

$$\begin{aligned} \mathbf{d} &= -\mathbf{G}^{-1}\nabla F_{\sigma}(x) \\ &= \left[\frac{-\sigma^2 x_1}{\sigma^2 + \beta x_1^2}, \frac{-\sigma^2 x_2}{\sigma^2 + \beta x_2^2}, \dots, \frac{-\sigma^2 x_N}{\sigma^2 + \beta x_N^2} \right]^T. \end{aligned} \quad (19)$$

And in general, parameter σ is chosen as $\sigma_k = \varphi\sigma_{k-1}$, $k=2, 3, \dots, K$, and $\varphi \in (0.4, 1)$. Let $\sigma_1 = \max(\Phi_x^\dagger y)$. Φ_x^\dagger is the Moore-Penrose Pseudoinverse [18] of Φ .

$$\Phi_x^\dagger = (\Phi_x' \Phi_x)^{-1} \Phi_x'. \quad (20)$$

Using the above derivation, the main steps of using the generalized approximate function model proposed in this paper to reconstruct sparse signals are shown in Table 1. The corresponding algorithm is called gSLO. In the following section, we give a detailed comparison and description between our new algorithm and existing excellent algorithms.

We first initialize the following parameters: σ_{\min} (the minimum value of σ_{\min} that should be a very small positive number), L (the number of iterations for decreasing σ_k), \hat{x} (the initial solution that can be obtained using pseudo-inverse which has the minimum ℓ_2 norm and corresponds to $\sigma \rightarrow \infty$).

Table 1. The proposed gSLO Algorithm

Input: Sensing matrix Φ , measurement vector y , $k = 1$.

Initialization parameter σ_{\min} , $\hat{x} = \Phi_x^\dagger y$;

While $\sigma_k > \sigma_{\min}$

- 1) Let $\sigma_k = \varphi\sigma_{k-1}$ ($\varphi \in (0.4, 1)$);
- 2) Minimize the generalized approximate function model $F_{\sigma}(x)$ on the feasible set \hat{x}, v_1 . using L inner iterations of the improved Newton direction method (followed by projection onto the feasible set).
- 3) $k = k + 1$:

Inner iteration:

--- Initialization $x = v_{k-1}$;

---for $i = 1 \dots L$ (loop L times):

a) Let: $\Delta x = d$, $x \leftarrow x + \eta\Delta x$ ($\eta \in (0, 1)$);

b) Project x back onto the feasible set \hat{x} :

$$x \leftarrow x - \Phi^T (\Phi\Phi^T)^{-1} (\Phi x - y)$$

- 4) Set $v_k = x$.

Final answer is $\tilde{x} = v_k$.

4. Experimental Results and Analysis

In this part, some experiments are carried out for illustrating the performance of the proposed gSLO algorithm. The measurement matrix Φ is acquired by extracting the random matrix M

rows of the $N \times N$ dimension. The proposed gSL0 model is contrasted with the latest greedy algorithms such as OMP, StOMP, and SWOMP and with the smooth ℓ_0 norm algorithm such as SL0, ASL0, and TSL0 in the aspects of signal reconstruction and image reconstruction. With no exception, the whole experiments are realized, in Matlab 2014a of PC which contain 3.2 GHZ Intel Core i5 processor and 8.0 GB running memory Windows 7 system.

Experiment 1: One-dimensional Signal.

In this experiment, the Gaussian random sparse signal length and noise interference are respectively set to $N=256$ and Gaussian white noise. A large number of simulations are proceeded to compare the reconstruction capabilities among different reconstruction algorithms. In view of randomness of the proposed model, all the simulation results are gained from averaged results of 1,000 independent tests results. For checking the proposed algorithm's performances for signal reconstruction, the reconstruction effects are estimated by exact reconstruction probability, averaged running time, and reconstructed relative error(Re).

In the first simulation, we fix sparsity $K=20$ and $N=256$ (signal length) and the measurement number M varies between 40 and 100. According to the exact reconstruction rate, the 7 different algorithms are estimated in the aspect of reconstruction which is presented in Fig. 2. We can find the proposed algorithm gSL0 in this paper which possesses a better reconstructed rate with regard to the different measurements. gSL0 also obviously precedes the most advanced greedy algorithms while the measurement number is greater then 60.

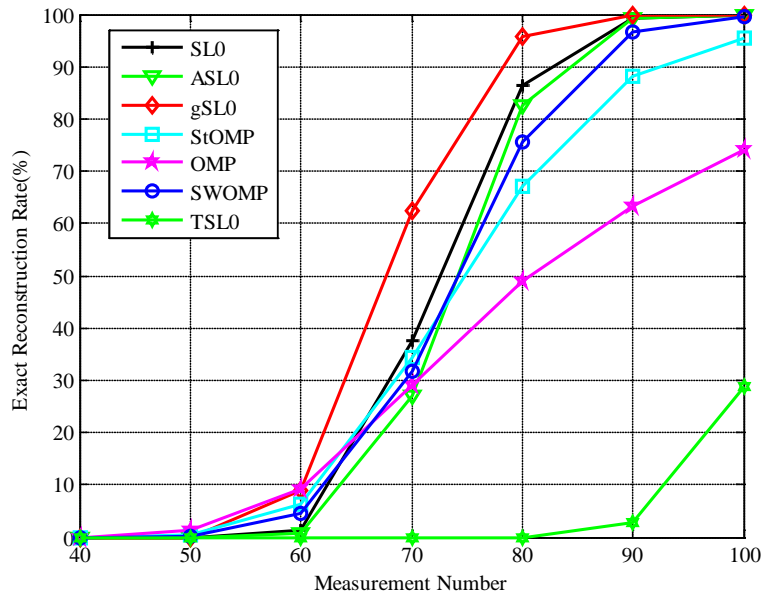


Fig. 2. In gaussian noise, reconstruction performance of gSL0, OMP, StOMP, SWOMP, SL0, TSL0 and ASL0 with the measurement number changing from 40 to 100.

In the second simulation, we fix $N=256$ and $M=80$ and the sparsity level K varies between 10 and 45. Fig. 3 shows the experimental results of different algorithms. It shows those algorithm's recovery probability under different sparsity level. We can find that the proposed algorithm can reconstruct the sparse signal with higher precision for different sparsity. And we can know that gSL0 performs much better than the other five algorithms in exact reconstruction rate in Fig. 3.

In the third simulation, we set $N=256$ and sparsity $K=20$ respectively and the measurement M varies between 40 and 100. From the Fig. 4, the simulation result shows that the average running time of gSL0 changes slowly with the increase of the sampling data. Considering the average running time of the different algorithms, we see that gSL0 performs relatively fast in the case of large measurement number. Since TSL0 introduces the mechanism of threshold selection to accelerate the inner iteration. We can also find that TSL0 is faster than the proposed gSL0 in Fig. 4. However, we can learn that the reconstruction effect of TSL0 under noisy is very bad from Fig. 2 and Fig. 3.

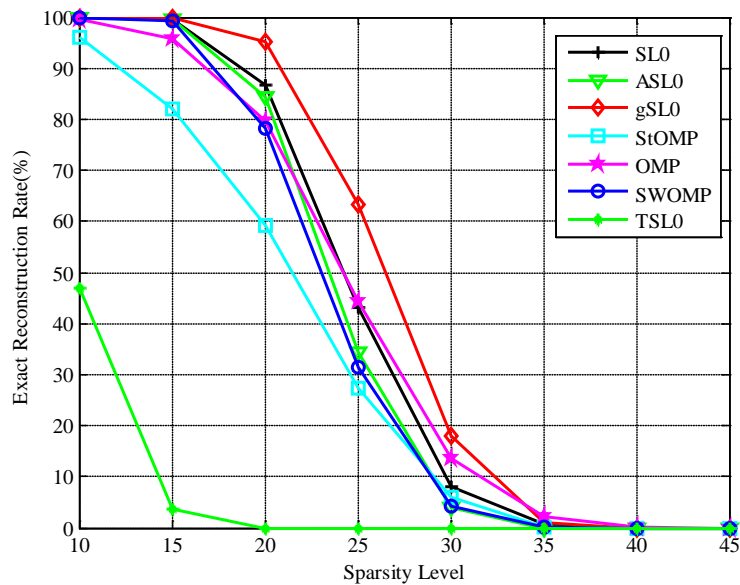


Fig. 3. Simulations for Gaussian sparse signals with Gaussian noise. The probability of exact reconstruction with different sparsity level.

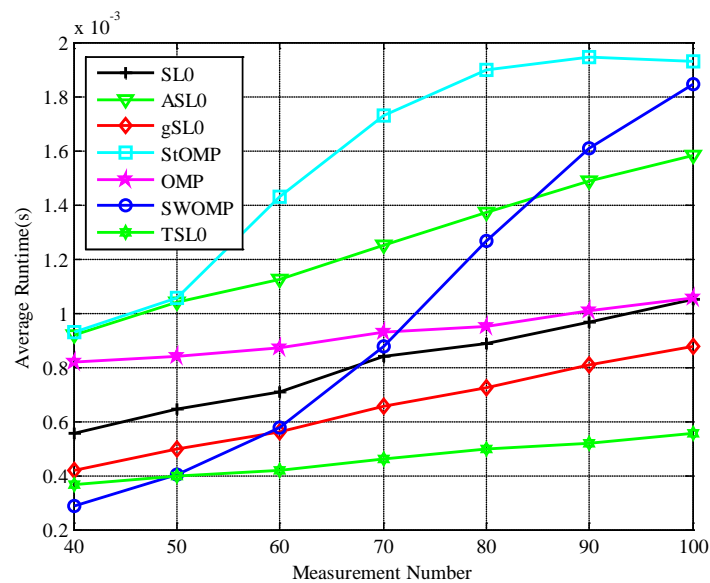


Fig. 4. In gaussian noise, average running time of gSL0, OMP, StOMP, SWOMP, SL0, TSL0 and ASL0 with the measurement number varies between 40 and 100 for the fixed $N=256$, $K=20$.

Finally, the relative error of one-dimensional signal reconstruction is defined as follows:

$$\mathbf{Re} = \frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|_2}. \quad (21)$$

Obviously, the Re is lower, the effect of signal reconstruction will be better.

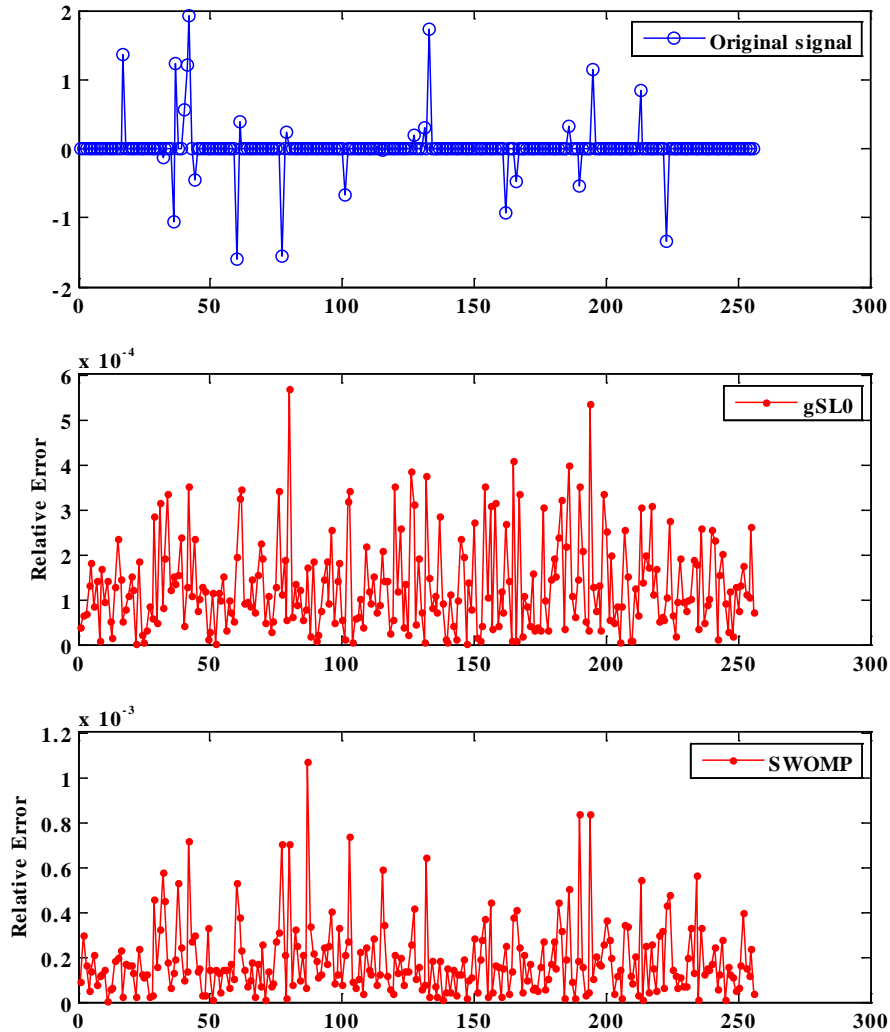


Fig. 5. reconstruction relative error of gSL0, SWOMP.

We can learn from **Fig. 5** that the Re of gSL0 is lower. It is to say that the proposed algorithm can more accurately reconstruct the original signal.

Experiment 2: Algorithm Performances Comparison of Image Reconstruction.

In this part, to verify the effectiveness of the proposed algorithm, three standard images, i.e. Lena image, Camera image and Boat image, are adopted as the input to conduct the comparison analysis of different algorithms. Furthermore, the peak signal-to-noise ratio (PSNR) is employed in this paper to evaluate the reconstruction performance of each algorithm, which measures the quality of reconstruction of sparse codes and is defined as

$$PSNR = 10 \log_{10} \left(\frac{M \times N \times 255^2}{\|x - \tilde{x}\|_2} \right). \quad (22)$$

The measurement matrix Φ is acquired by extracting the random matrix M rows of the $N \times N$ dimension. Sampling ratio is $M/N=0.42$. The size of each image is 256×256 . The reconstruction effects are estimated by PSNR, Re, averaged running time and visual effects of reconstruction. The mean values of the PSNR, Re and Time over 20 independent tests are given in [Table 2](#). We can see the reconstruction effects of different algorithms on two-dimensional images in [Fig. 6](#).



Fig. 6. The quality of the reconstruction with different algorithms in different images.

From [Fig. 6](#), we can learn that the proposed gSL0 is better than other algorithms (SL0, ASL0, OMP) in the PSNR of the reconstructed image. And we can see that the gSL0 algorithm has better reconstruction effect for different kinds of images. So the gSL0 algorithm can accurately recover the original signal.

The reconstruction relative error of two-dimensional signal is defined as follows:

$$\mathbf{Re} = \frac{\|x - \tilde{x}\|_2}{\|x\|_2}. \quad (23)$$

Table 2. Reconstruction effect of SL0, ASL0, OMP, SWOMP and the proposed gSL0 for different images, each image with the fixed measurement rate 0.4 (M/N).

Image	Algorithm	PSNR(dB)	Re	Time(s)
Lena	SL0	38.53	0.030	0.44
	ASL0	39.38	0.027	0.69
	OMP	17.20	0.348	7.81
	SWOMP	18.53	0.298	0.35
	gSL0	39.57	0.026	0.38
Camera	SL0	35.04	0.036	0.44
	ASL0	35.59	0.034	0.69
	OMP	12.70	0.472	8.08
	SWOMP	12.28	0.496	0.35
	gSL0	35.90	0.034	0.38
Boat	SL0	34.68	0.045	0.45
	ASL0	34.88	0.044	0.71
	OMP	14.46	0.465	8.92
	SWOMP	30.18	0.076	0.37
	gSL0	35.10	0.043	0.39

From **Table 2**, it shows that the proposed gSL0 algorithm can achieve the best reconstruction performance in PSNR and Re of all test images. However, we also can find that the speed of reconstruction is a little slower than SWOMP. As we can see from **Fig. 6** and **Table 2**, the proposed algorithm gSL0 precedes the most advanced algorithms in the aspect of image reconstruction.

5. Conclusion

In this paper, we proposed a generalized approximate function model to reconstruct ℓ_0 norm and design the gSL0 algorithm based on the proposed generalized model. To accelerate the convergence, we use the improved Newton direction as the search direction. Through the proof and simulation, we can get that the generalized approximate model has better “steep nature” and the estimation of the ℓ_0 norm is more precise. The extensive test results show that the proposed model has a good recovery effect not only for the one-dimensional signal but also for the two-dimensional image. The proposed gSL0 is compared with some known algorithms based on a smoothed ℓ_0 norm and existing excellent greedy algorithms, it has a high probability of reconstruction and faster reconstruction, even with Gaussian white noise. State-of-the-art reconstruction effect is achieved by the gSL0 algorithm for different images.

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Yangyang Lin received the M.S. degree of the Information and Communication Engineering in Nankai University in 2017. And he is pursuing his Ph.D. degree in Electronic Science and Technology in Nankai University, Tianjin, China. His research interests include wireless sensor networks, compressed sensing, and image processing.



Guiling sun received Ph.D from Nankai University in 2004 and has become the professor of Nankai University. She is also the vice director of the Tianjin Institute of Communications, the vice director of EDA Technology Research Society in North China and the leader of Electronic technology experiment and practice teaching team of Tianjin city. She has published more than 100 papers and 2 research monographs in a number of areas about the Wireless Sensor Networks, Internet of things, Compressed Sensing, Information detection, Intelligent control systems and Signal processing.



Zhouzhou Li, is pursuing her Ph.D. degree in Electronic Science and Technology in Nankai University, Tianjin, China. She received the B. S. degree in Electronic Science and Technology from Nankai University, Tianjin, China, in 2010. And Her research interests are sparse signal processing and wireless sensor network.



Tianyu Geng was born in Tianjin, China, in 1992. He received the B. S. degree in Electronic Science and Technology from Nankai University, Tianjin, China, in 2014. And he is pursuing his Ph.D. degree in Electronic Science and Technology in Nankai University, Tianjin, China. His research interests are in the areas of compressed sensing and image processing.