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The transmuted GEV distribution: properties and application

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Abstract

The transmuted generalized extreme value (TGEV) distribution was first introduced by Aryal and Tsokos (*Nonlinear Analysis: Theory, Methods & Applications*, **71**, 401–407, 2009) and applied by Nascimento *et al.* (*Hacettepe Journal of Mathematics and Statistics*, **45**, 1847–1864, 2016). However, they did not give explicit expressions for all the moments, tail behaviour, quantiles, survival and risk functions and order statistics. The TGEV distribution is a more flexible model than the simple GEV distribution to model extreme or rare events because the right tail of the TGEV is heavier than the GEV. In addition the TGEV distribution can adjusted various forms of asymmetry. In this article, explicit expressions for these measures of the TGEV are obtained. The tail behavior and the survival and risk functions were determined for positive gamma, the moments for nonzero gamma and the moment generating function for zero gamma. The performance of the maximum likelihood estimators (MLEs) of the TGEV parameters were tested through a series of Monte Carlo simulation experiments. In addition, the model was used to fit three real data sets related to financial returns.

Keywords: TGEV, moments, extreme events, MLEs

1. Introduction

The generalized extreme value (GEV) distribution is widely used in several areas to model data from extreme events that occur infrequently. For example, Lettenmainer *et al.* (1987), Hewa *et al.* (2007), Morrison and Smith (2002) used it in hydrology to treat return periods of flood frequency or high wind speeds. In finance, Embrechts *et al.* (1997), presents how to calculate the value at risk (VaR) of maximum financial returns, and in actuarial science how to calculate the probability of ruin as consequence of extreme events. Extreme events are more suitably modeled with heavy tails and the GEV distribution has this characteristic. However, there are extreme event data that do not follow GEV distribution, because they require a more asymmetric distribution or with a heavier tail than GEV distribution. Thus, new classes of probability distributions have been developed that are more general than the GEV distribution such as: dual gamma GEV distribution (GGEV), exponentiated GEV distribution (EGEV) studied by Nascimento *et al.* (2016), transmuted GEV (TGEV) distribution defined by Aryal and Tsokos (2009), and q-GEV given by Provost *et al.* (2018). The advantage of TGEV distribution in relation to other generalized distributions is that it has a heavier tail than GEV distribution as shown in Section 2. Moments, moment generating function, hazard rate function and order statistics have a simple closed form. Therefore, TGEV distribution becomes flexible to

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model extreme events in several areas. According to Jenkinson (1955), the GEV distribution is the limit distribution of properly normalized maximum (or minimum) of a sequence of independent and identically distributed (iid) random variables. That is, if $X_1, X_2, \ldots, X_n$ are iid random variables with cumulative distribution function (cdf) $F(x)$ and if there are sequences of constants $a_n > 0$ and b_n such that

$$P\left(\frac{X_{(n)} - b_n}{a_n} \leq x\right) \to G(x), \quad \text{as } n \to \infty,$$

where $\mathbf{X_{(1)}}, \mathbf{X_{(2)}}, \ldots, \mathbf{X_{(n)}}$ denote the order statistics and $G(x)$ is a non-degenerate cdf, then $G(x)$ belongs to one of the following three distribution families:

$$\text{Gumbel: } \Lambda(x) = \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}, \quad -\infty < x < \infty;$$

$$\text{Fréchet: } \Phi(x) = \begin{cases} 0, & x < \mu, \\ \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right\}, & x \geq \mu; \end{cases}$$

$$\text{Weibull: } \Psi(x) = \begin{cases} \exp\left\{-\left[-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right]\right\}, & x < \mu, \\ 1, & x > \mu, \end{cases}$$

where $\sigma > 0$, $\mu \in \mathbb{R}$, and $\alpha > 0$. Jenkinson (1955) introduced the GEV distribution that contemplates the three previous distributions. A random variable X follows the GEV distribution if its cdf is given by

$$G(x) = \begin{cases} \exp\left\{-\left[1 + \gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}\right\}, & \gamma \neq 0, \\ \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}, & \gamma = 0, \end{cases} \tag{1.1}$$

where it is defined in the set $\{x : 1+\gamma((x-\mu)/\sigma) > 0\}$, μ is a location parameter, σ is a scale parameter and γ is a shape parameter. For $\gamma = 0$ the expression (1.1) is interpreted by taking the limit as $\gamma > \mathbf{0}$. The γ parameter governs the tail behavior with an important impact on the shape of the distribution that is called the tail index directly related to the shape parameter α. The Gumbel distribution is a special case for $\gamma = 0$. Also note that the cases for $\gamma = \mathbf{1}/\alpha$ and $\gamma = -\mathbf{1}/\alpha$ we have that the GEV distribution is of the Fréchet type and the Weibull type respectively.

The transmutation map is a technique developed by Shaw and Buckley (2007) and consists of introducing skewness or kurtosis in a symmetric or other (asymmetrical) distribution. It is a relatively new technique; however, it has already been applied to several distribution functions. Some examples are the transmuted extreme value distribution introduced by Aryal and Tsokos (2009), Aryal and Tsokos (2011), Aryal (2013) obtained the transmuted Weibull distribution and the transmuted log-logistic distribution, and Merovci (2013) introduced the transmuted exponentiated exponential distribution. With respect to the Weibull distribution and some extensions, Khan and King (2013a, 2016) developed its transmutation. Khan and King (2013b, 2014) and Mahmoud and Mandouh (2013) developed the transmutation for the Weibull inverse distribution and some extensions. Khan and King also obtained the transmutation of inverse Rayleigh distributions (2015), Khan *et al.* (2016a, 2016b, 2017) obtained the transmutation of Kumaraswamy distribution, new generalized Weibull distribution, and new generalized inverse Weibull distribution. Elgarhy *et al.* (2017) introduced the transmuted generalized quasi Lindley distribution. Khan (2018) and Nassar *et al.* (2019) obtained the transmuted generalized power Weibull distribution and transmuted Weibull Logistic Distribution, respectively.

Aryal and Tsokos (2009) defined TGEV distribution and discussed some properties about the transmuted Gumbel distribution. Recently, Nascimento *et al.* (2016) applied the TGEV distribution to environmental data with the parameter estimation of this distribution was done under the Bayesian model. This work investigated the tail behavior and the main mathematical measures of the TGEV distribution. The parameters are also estimated by maximum likelihood that included an application to illustrate the model. This paper is organized as follows. Section 2 deals with some mathematical properties of the TGEV distribution, such as the tail behavior, the moments, the hazard rate function, and the order statistics. In Section 3, the inference procedure is performed by maximum likelihood and some simulations are used to test the efficiency of the estimators. An application for extreme data is presented in Section 4.

2. Tail behavior and properties of the TGEV

In this section we present the behavior of the right tail of the TGEV distribution. We show that the right tail of the TGEV distribution is heavier than the right tail of the GEV distribution. Expressions of moments, moments gerating function, hazard rate function, quantile function, and order statistics were also obtained.

The transmutation map, proposed by Shaw and Buckley (2007), consists of a powerful technique that considers some perturbations of the symmetry and manage kurtosis adjustments. Given one base distribution function, say $G(x)$, the transmuted distribution function F is defined by

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2, \quad |\lambda| < 1. \tag{2.1}$$

Aryal and Tsokos (2009) introduced the TGEV distribution and studied basic mathematical characteristics just of the transmuted Gumbel distribution. However, it is important to study moments, order statistics and other statistical properties in the modeling of extreme events because GEV distribution can occur with $\gamma \neq 0$. A random variable X is said to be TGEV distributed, say $X \sim F_T(\,\cdot\,; \mu, \sigma, \gamma, \lambda)$ distribution, if its cdf can be expressed as

$$F_T(x; \mu, \sigma, \gamma, \lambda) = \begin{cases} e^{\left\{-[w]^{-\frac{1}{\gamma}}\right\}}\left[(1+\lambda) - \lambda e^{\left\{-[w]^{-\frac{1}{\gamma}}\right\}}\right], & \gamma \neq 0, \\ e^{-e^{-\frac{(x-\mu)}{\sigma}}}\left[(1+\lambda) - \lambda e^{-e^{-\frac{(x-\mu)}{\sigma}}}\right], & \gamma = 0, \end{cases} \tag{2.2}$$

by replacing (1.1) in (2.1). The function (2.2) is well defined for x such that $\{x : w = 1+\gamma((x-\mu)/\sigma) > 0\}$, μ is a location parameter, σ is a scale parameter, γ is a shape parameter (tail index), and λ is the shape parameter. Note that to $\lambda < \mathbf{0}$ the model (2.2) corresponds to a mixture of a GEV distribution and a skew GEV distribution.

The probability density function (pdf) corresponding to (2.2) is given by

$$f_T(x; \mu, \sigma, \gamma, \lambda) = \begin{cases} \left[(w)^{-\frac{1}{\gamma}-1}\,\dfrac{1}{\sigma} e^{\left\{-[w]^{-\frac{1}{\gamma}}\right\}}\right]\left[(1+\lambda) - 2\lambda e^{\left\{-[w]^{-\frac{1}{\gamma}}\right\}}\right], & \gamma \neq 0, \\ \dfrac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} e^{-e^{-\frac{(x-\mu)}{\sigma}}}\left[(1+\lambda) - 2\lambda e^{-e^{-\frac{(x-\mu)}{\sigma}}}\right], & \gamma = 0. \end{cases} \tag{2.3}$$

Figures 1–4 show how the TGEV density function is influenced by the parameter λ. Note that at $\lambda = 0$ we have the particular case of base GEV distribution. In the four figures $\mu = 0$, $\sigma = 1$ are fixed and

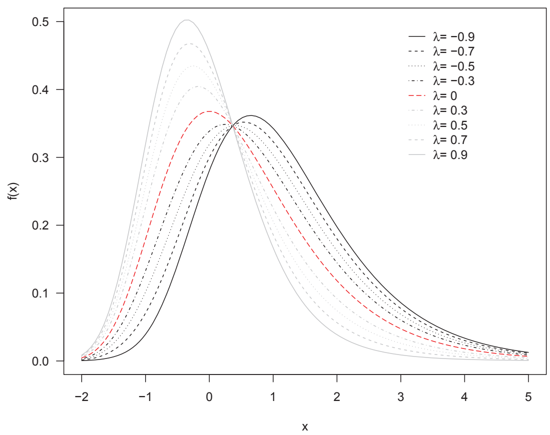

Figure 1: *Plots for the transmuted generalized extreme value density* $\mu = 0, \sigma = 1, \gamma = 0$, *and* λ *varying as shown in the caption.*

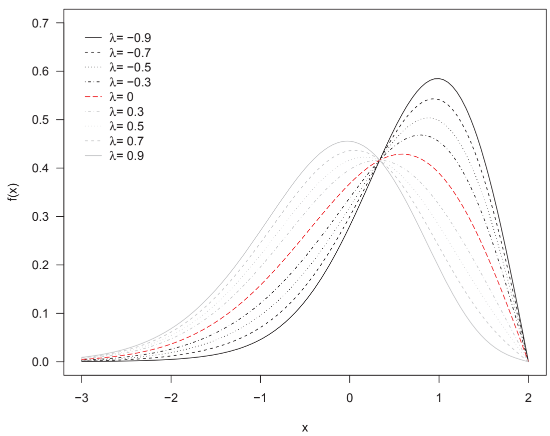

Figure 2: *Plots for the transmuted generalized extreme value density for* $\mu = 0, \sigma = 1, \gamma = -0.5$, *and* λ *varying as shown in the caption.*

λ varies according to the values of the legend of the figures, but the tail index $\gamma = 0, -0.5, 0.5, 0.5$ in Figures 1–4, respectively.

Note that the parameter λ can modify the distribution according to the signal of γ. For $\gamma < 0$, the greater the absolute value of λ, the larger the maximum value of the pdf and the heavier its tail. For negative values of λ, the density curves are larger than those for the positive values of λ. For $\gamma = 0$, positive values of λ produce higher maximum values of the pdf. However, when $\gamma > 0$, we have that the greater the value of λ, the larger the maximum value of the pdf and its tail will be heavier. Extreme event data usually follows a heavy tail distribution and the GEV distribution has this property for $\gamma > 0$. Therefore, the study the tail behavior of the TGEV density for $\gamma > 0$ is important for modeling extreme events.

2.1. Tail behavior

By Embrechts *et al.* (1997) showed that a probability distribution is said to have a heavy tail if its

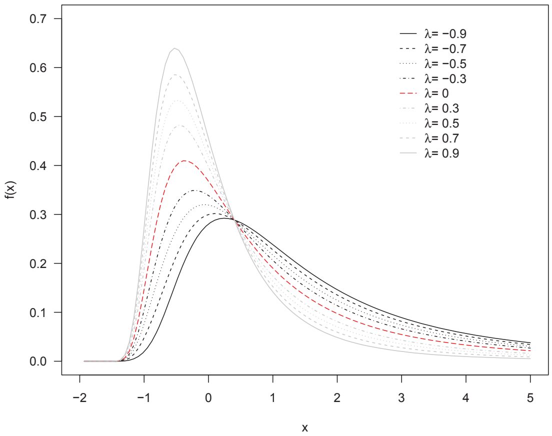

Figure 3: *Plots for the transmuted generalized extreme value density for $\mu = 0, \sigma = 1, \gamma = 0.5$, and λ varying as show in the caption.*

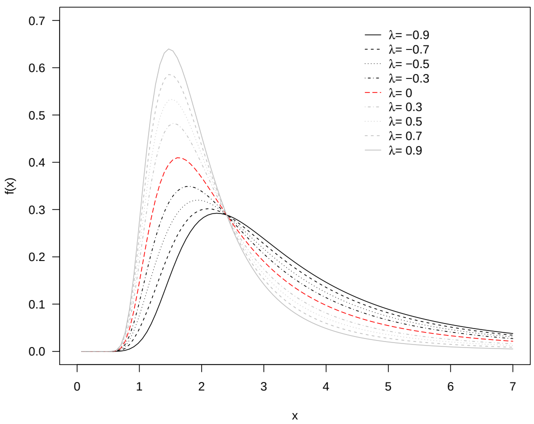

Figure 4: *Plots for the transmuted generalized extreme value density for $\mu = 2, \sigma = 1, \gamma = 0.5$, and λ varying as show in the caption.*

reliability function is regularly varying. Thus, in this section we analyze the tail behavior of the TGEV distribution via regular variation property at infinity.

Definition 1. *A positive measurable function f defined on some neighbourhood $[x_o, \infty)$ is called regularly varying (at ∞) with index $\alpha \in R$ if*

$$\lim_{x\to\infty} \frac{f(tx)}{f(t)} = x^{\alpha}. \tag{2.4}$$

If $\alpha = 0$ f is said to be slowly varying (at ∞).

From Equation (2.4) it is easy to see that every regularly varying function f of index α has representation

$$f(x) = x^{\alpha} L(x), \tag{2.5}$$

where L is some slowly varying function.

Proposition 1. *Let X be a random variable with cdf $F_T(\cdot\,;\mu,\sigma,\gamma,\lambda)$, $\gamma>0$, then*

$$\bar{F}_T(x) = 2(1+\lambda)x^{-\frac{1}{\gamma}}L_1(x) + \lambda x^{-\frac{2}{\gamma}}L_2(x), \quad x\to\infty, \tag{2.6}$$

where $\bar{F}_T = 1-F_T$ is the reliability function of the TGEV distribution and L_1 and L_2 are any two slowly varying functions (at ∞).

Proof: The tail behavior of a TGEV distribution at infinity is determined by considering the reliability function of the GEV distribution as $\bar{G} = 1-G$ and replacing it in (2.4). Therefore, we have

$$\lim_{x\to\infty}\frac{\bar{G}(tx)}{\bar{G}(t)} = x^{-\frac{1}{\gamma}},$$

where $\bar{G}$ is regularly varying with index $-1/\gamma$. Thus, by (2.5), the reliability function G can be represented by

$$\bar{G}(x) = x^{-\frac{1}{\gamma}}L_1(x), \tag{2.7}$$

where $L_1(x)$ is slowly varying function (at ∞). We obtain (2.6) simply by replacing (2.7) in (2.1). Then from Equation (2.6), we can conclude that the tail behavior of the TGEV distribution is the same as that of a mixture of regular varying functions and, therefore, a mixture of heavy tail functions whose right tail weight is influenced by the parameters γ and λ. □

2.2. Moments

The moments of a transmuted Gumbel random variable $X\sim F_T(\cdot\,;\mu,\sigma,\gamma,\lambda)$ have already been obtained by Aryal and Tsokos (2009). Therefore, in this section, we compute the moments of a random variable $X\sim F_T$ with $\gamma\neq 0$.

Proposition 2. *If $X\sim F_T(\cdot\,;\mu,\sigma,\gamma,\lambda)$ with $\gamma\neq 0$, then the k^{th} moment (non-negative integer) of X is given by*

$$E\left(X^k\right) = \sum_{i=0}^{k} C_k^i\left(\mu-\frac{\sigma}{\gamma}\right)^{k-i}\left(\frac{\sigma}{\gamma}\right)^{i}\left[\Gamma(1-\gamma i)\left(1+\lambda-2^{\gamma i}\lambda\right)\right], \tag{2.8}$$

where $\mu-\sigma/\gamma>0$.

Proof: For $\gamma>0$, from (2.2) we obtain

$$\begin{aligned} E\left(X^k\right) &= \int_{\mu-\frac{\sigma}{\gamma}}^{\infty} x^k dF(x) \\ &= \frac{(1+\lambda)}{\sigma}\int_{\mu-\frac{\sigma}{\gamma}}^{\infty} x^k\left[\left(1+\gamma\frac{(x-\mu)}{\sigma}\right)^{-\frac{1}{\gamma}-1}\right]\left[e^{\left\{-\left[1+\gamma\frac{(x-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\right]dx \\ &\quad -\frac{2\lambda}{\sigma}\int_{\mu-\frac{\sigma}{\gamma}}^{\infty} x^k\left[\left(1+\gamma\frac{(x-\mu)}{\sigma}\right)^{-\frac{1}{\gamma}-1}\right]\left[\left(e^{\left\{-\left[1+\gamma\frac{(x-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\right)^2\right]dx. \end{aligned} \tag{2.9}$$

In order to solve the integrals in (2.9), we first replace w by $1 + \gamma(x - \mu)/\sigma$ and then use the Newton's formula,

$$x^k = \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma} y\right)^i,$$

then

$$\begin{aligned}
E\left(X^k\right) &= \frac{(1+\lambda)}{\sigma} \int_0^\infty \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma} w\right)^i w^{-\frac{1}{\gamma}-1} e^{-w^{-\frac{1}{\gamma}}} \frac{\sigma}{\gamma} dw \\
&\quad - \frac{2\lambda}{\sigma} \int_0^\infty \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma} w\right)^i w^{-\frac{1}{\gamma}-1} e^{-2w^{-\frac{1}{\gamma}}} \frac{\sigma}{\gamma} dw \\
&= \frac{(1+\lambda)}{\sigma} \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma}\right)^{i+1} \int_0^\infty w^{i-\frac{1}{\gamma}-1} e^{-w^{-\frac{1}{\gamma}}} dw \\
&\quad - \frac{2\lambda}{\sigma} \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma}\right)^{i+1} \int_0^\infty w^{i-\frac{1}{\gamma}-1} e^{-2w^{-\frac{1}{\gamma}}} dw.
\end{aligned} \tag{2.10}$$

Now, taking $u = w^{-1}$ in (2.10), we have

$$\begin{aligned}
E\left(X^k\right) &= \left[\frac{(1+\lambda)}{\sigma} \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma}\right)^{i+1}\right] \int_0^\infty u^{-i+\frac{1}{\gamma}-1} e^{-u^{\frac{1}{\gamma}}} du \\
&\quad - \left[\frac{2\lambda}{\sigma} \sum_{i=0}^{k} C_k^i \left(\mu - \frac{\sigma}{\gamma}\right)^{k-i} \left(\frac{\sigma}{\gamma}\right)^{i+1}\right] \int_0^\infty u^{-i+\frac{1}{\gamma}-1} e^{-2u^{\frac{1}{\gamma}}} du.
\end{aligned} \tag{2.11}$$

Note that the integrals in (2.11) are Gamma functions, then (2.8) is obtained.

Analogously, we compute $E(X^k)$ for $\gamma < 0$, where the integration domain in this case is $[-\infty, \mu - \sigma/\gamma]$, and we obtain the same result (2.8). □

The mean and variance of the TGEV distribution, also obtained by Nascimento *et al.* (2016), can be deduced directly from Equation (2.8).

For $k = 1$,

$$E(X) = \left(\mu - \frac{\sigma}{\gamma}\right) + \left(\frac{\sigma}{\gamma}\right)\left[\Gamma(1-\gamma)\left(1 + \lambda - 2^\gamma \lambda\right)\right]$$

and for $k = 2$,

$$E\left(X^2\right) = \left(\mu - \frac{\sigma}{\gamma}\right)^2 + 2\left(\mu - \frac{\sigma}{\gamma}\right)\frac{\sigma}{\gamma}\left[\Gamma(1-\gamma)\left(1+\lambda-2^\gamma\lambda\right)\right] + \left(\frac{\sigma}{\gamma}\right)^2\left[\Gamma(1-2\gamma)\left(1+\lambda-2^{2\gamma}\lambda\right)\right],$$

then

$$\mathrm{Var}(X) = \left(\frac{\sigma}{\gamma}\right)^2\left[\Gamma(1-2\gamma)\left(1+\lambda-2^{2\gamma}\lambda\right) - \Gamma^2(1-\gamma)\left(1+\lambda-2^\gamma\lambda\right)^2\right].$$

In the case of the transmuted Gumbel distribution, $X \sim F_T(\,\cdot\,;\mu,\sigma,\gamma,\lambda)$, the moment generating function of X, say $M(t) = E(X^k)$, is of great importance to obtain moments of distribution, since a general expression of $E(X^k)$ is not simple to calculate.

Proposition 3. *If $X \sim F_T(\,\cdot\,;\mu,\sigma,0,\lambda)$, then*

$$M_X(t) = E\left(e^{tX}\right) = e^{t\mu}\Gamma(1-t\sigma)\left[(1+\lambda) - 2^{t\sigma}\lambda\right], \quad t < \frac{1}{\sigma}. \tag{2.12}$$

Proof: By definition,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\frac{e^{-\frac{(x-\mu)}{\sigma}}e^{-e^{-\frac{(x-\mu)}{\sigma}}}}{\sigma}\left[(1+\lambda) - 2\lambda e^{-e^{-\frac{(x-\mu)}{\sigma}}}\right]dx. \tag{2.13}$$

Setting $u = e^{-(x-\mu)/\sigma}$, we can rewrite (2.13) as

$$\begin{aligned}
M_X(t) &= \int_{\infty}^{0} e^{t(\mu-\sigma\log(u))}\frac{ue^{-u}}{\sigma}\left[1+\lambda-2\lambda e^{-u}\right]\frac{-\sigma}{u}du \\
&= \int_{0}^{\infty} e^{t\mu}e^{\log(u^{-t\sigma})}e^{-u}\left[(1+\lambda)-2\lambda e^{-u}\right]du \\
&= e^{t\mu}\left[(1+\lambda)\int_0^{\infty}u^{-t\sigma}e^{-u}du - 2\lambda\int_0^{\infty}u^{-t\sigma}e^{-2u}du\right].
\end{aligned} \tag{2.14}$$

The expression (2.12) is obtained by solving the integrals in (2.14), considering $t < 1/\sigma$ and using the Gamma function. □

The same expressions of $E(X)$ and $\text{Var}(X)$, obtained by Aryal and Tsokos (2009), we now obtain with the first and second derivatives of $M_X(t)$ at $t = 0$,

$$E(X) = \mu + \sigma C - \lambda\sigma\log(2) \tag{2.15}$$

$$\text{Var}(X) = \sigma^2\left[\frac{\pi^2}{6} - \lambda(1+\lambda)\log^2(2)\right]. \tag{2.16}$$

However, the moment generating function (2.12) can also be useful for analyzing data on the sum of TGEV random variables.

2.3. Reliability measures

If we consider a random variable $T \sim F_T(\,\cdot\,;\mu,\sigma,0,\lambda)$ with $\mu > 0$ whose distribution support is a subset of non-negative real numbers, T can represent the failure time of an event of interest. In this sense, this random variable can be characterized by the survival function, $R(t) = \bar{F}_T(t) = 1 - F_T(t;\mu,\sigma,0,\lambda)$, or by its hazard rate function, $h(t) = f_T(t)/R(t)$. The survival and hazard rate functions for T are given, respectively, by

$$R(t) = 1 - \left[e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\left(1+\lambda-\lambda e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\right)\right] \tag{2.17}$$

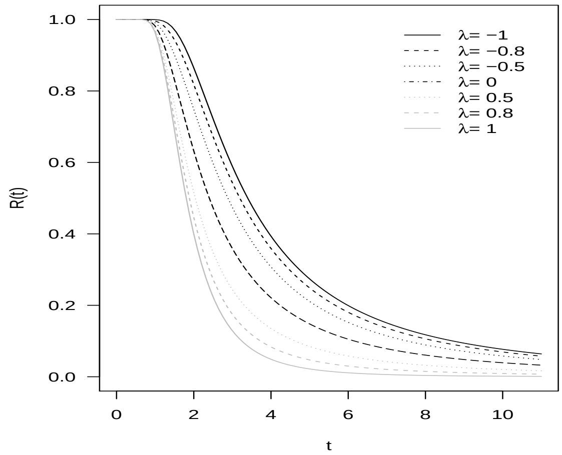

Figure 5: *Reliability function of TGEV distribution for* $\mu = 2$, $\sigma = 1$, $\gamma = 0.5$, *and* λ *varying in* $[-1, 1]$.

and

$$h(t) = \frac{\left[\left(1+\gamma\frac{(t-\mu)}{\sigma}\right)^{-\frac{1}{\gamma}-1}\frac{e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}}{\sigma}\right]\left[(1+\lambda)-2\lambda e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\right]}{1-\left[e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\left(1+\lambda-\lambda e^{\left\{-\left[1+\gamma\frac{(t-\mu)}{\sigma}\right]^{-\frac{1}{\gamma}}\right\}}\right)\right]}. \tag{2.18}$$

Figure 5 shows the behavior of the reliability function (2.17) for λ taking values from -1 to 1. Note that for smaller values of λ the function R decays more slowly, i.e., the parameters λ and γ influence the tail weight of the TGEV distribution. This is illustrated by Proposition 1. Also notice the hazard rate function (2.18), shown in Figure 6, presents unimodal behavior. We observe that the mode changes as the parameter λ varies.

2.4. Order statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, X_3, \ldots, X_n$ from a population $X \sim F_T(\cdot\,; \mu, \sigma, \gamma, \lambda)$. Then we have that the pdf of the j^{th} order statistic $X_{(j)}$, for $\gamma \neq 0$, is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\left[\left(w^{-\frac{1}{\gamma}-1}\frac{e^{-w^{-\frac{1}{\gamma}}}}{\sigma}\right)\left(1+\lambda-2\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right] \\ \times\left[e^{-w^{-\frac{1}{\gamma}}}\left(1+\lambda-\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]^{j-1}\left[1-e^{-w^{-\frac{1}{\gamma}}}\left(1+\lambda-\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]^{n-j}, \tag{2.19}$$

where $w = 1 + \gamma(x-\mu)/\sigma$. Therefore, the pfd of the n^{th} order statistic and the pdf of the 1^{st} order statistic are given, respectively, by

$$f_{X_{(n)}}(x) = n\left[\left(w^{-\frac{1}{\gamma}-1}\frac{e^{-w^{-\frac{1}{\gamma}}}}{\sigma}\right)\left(1+\lambda-2\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]\left[e^{-w^{-\frac{1}{\gamma}}}\left(1+\lambda-\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]^{n-1} \tag{2.20}$$

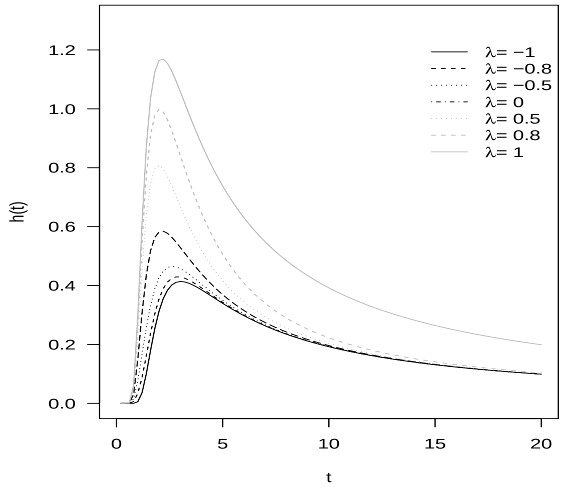

Figure 6: *Hazard rate function of transmuted generalized extreme value distribution for* $\mu = 2$, $\sigma = 1$, $\gamma = 0.5$, *and* λ *varying in* $[-1, 1]$.

and

$$f_{X_{(1)}}(x) = n\left[\left(w^{-\frac{1}{\gamma}-1}\frac{e^{-w^{-\frac{1}{\gamma}}}}{\sigma}\right)\left(1+\lambda-2\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]\left[1-e^{-w^{-\frac{1}{\gamma}}}\left(1+\lambda-\lambda e^{-w^{-\frac{1}{\gamma}}}\right)\right]^{n-1}. \quad (2.21)$$

For $\gamma = 0$, the pdfs of the j^{th}, n^{th}, and 1^{st} order statistics are given, respectively, by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{ve^{-v}}{\sigma}\left[(1+\lambda)-2\lambda e^{-v}\right]\left[(1+\lambda-\lambda e^{-v})^{j-1}\right]\left[1-(e^{-v}(1+\lambda-\lambda e^{-v}))\right]^{n-j}, \quad (2.22)$$

where $v = e^{-(x-\mu)/\sigma}$,

$$f_{X_{(n)}}(x) = n\frac{ve^{-nv}}{\sigma}\left[(1+\lambda)-2\lambda e^{-v}\right]\left[(1+\lambda-\lambda e^{-v})^{n-1}\right] \quad (2.23)$$

and

$$f_{X_{(1)}}(x) = n\frac{ve^{-v}}{\sigma}\left[(1+\lambda)-2\lambda e^{-v}\right]. \quad (2.24)$$

3. Estimation and results

Nascimento *et al.* (2016) used Bayesian inference to obtain parameter estimators of the TGEV distribution. In this section we present the system to be solved to obtain the MLEs estimators, then simulation experiments were run in order to test the performance of these estimators.

3.1. Estimation

The parameters of the TGEV distribution are estimated by the method of maximum likelihood. Let $X_1, X_2, \ldots, X_n$ be a sample of size n from $X \sim F_T(\cdot\,; \mu, \sigma, \gamma, \lambda)$. Let $\theta = (\mu, \sigma, \gamma, \lambda)$ be the parametric vector. The log-likelihood function for θ with $\gamma = 0$ and $\gamma \neq 0$ can be expressed, respectively, as

$$l(\theta) = -\sum_{i=1}^{n}\left[\log(\sigma) + \frac{x_i - \mu}{\sigma} + e^{-\frac{(x_i-\mu)}{\sigma}}\right] + \sum_{i=1}^{n}\log\left(1 + \lambda - 2\lambda e^{-e^{-\frac{(x_i-\mu)}{\sigma}}}\right) \tag{3.1}$$

and

$$\begin{aligned} l(\theta) = &-n\log(\sigma) - \left(\frac{1}{\gamma} + 1\right)\sum_{i=1}^{n}\log\left(1 + \frac{\gamma}{\sigma}(x_i - \mu)\right) \\ &- \sum_{i=1}^{n}\left(1 + \frac{\gamma}{\sigma}(x_i - \mu)\right)^{-\frac{1}{\gamma}} + \sum_{i=1}^{n}\log\left(1 + \lambda - 2\lambda e^{\left\{-\left(1+\frac{\gamma}{\sigma}(x_i-\mu)\right)^{-\frac{1}{\gamma}}\right\}}\right). \end{aligned} \tag{3.2}$$

Thus, for $\gamma = 0$, the MLEs of μ, σ, λ which maximize $l(\theta)$), given by (3.1), must satisfy the equations

$$\begin{aligned} \frac{\partial l}{\partial \mu} &= n - \sum_{i=1}^{n} e^{-y} + 2\lambda\sum_{i=1}^{n}\frac{e^{-y}e^{-e^{-y}}}{1 + \lambda - 2\lambda e^{-e^{-y}}} = 0, \\ \frac{\partial l}{\partial \sigma} &= -n + \sum_{i=1}^{n}(x_i - \mu)\left[1 - e^{-y}\right] + \frac{2\lambda}{\sigma}\sum_{i=1}^{n}\frac{(x_i - \mu)e^{-y}e^{-e^{-y}}}{1 + \lambda - 2\lambda e^{-e^{-y}}} = 0, \\ \frac{\partial l}{\partial \lambda} &= \sum_{i=1}^{n}\frac{1 - 2e^{-e^{-y}}}{1 + \lambda - 2\lambda e^{-e^{-y}}} = 0, \end{aligned} \tag{3.3}$$

where $y = (x_i - \mu)/\sigma$ and for $\gamma \neq 0$ the MLEs of $\mu, \sigma, \mu, \lambda$ which maximize $l(\theta)$, given by (3.2), must satisfy the equations

$$\begin{aligned} \frac{\partial l}{\partial \mu} &= \sum_{i=1}^{n}\frac{1+\gamma}{\sigma + \gamma(x_i - \mu)} - \sum_{i=1}^{n}\frac{w^{-\frac{1}{\gamma}-1}}{\sigma} + \sum_{i=1}^{n}\frac{w^{-\frac{1}{\gamma}-1}2\lambda e^{-w^{-\frac{1}{\gamma}}}}{\sigma\left(1 + \lambda - 2\lambda e^{-w^{-\frac{1}{\gamma}}}\right)} = 0, \\ \frac{\partial l}{\partial \sigma} &= -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(1+\gamma)(x_i - \mu)}{\sigma[\sigma + \gamma(x_i - \mu)]} - \sum_{i=1}^{n}\frac{(x_i - \mu)w^{-\frac{1}{\gamma}-1}}{\sigma^2} + \sum_{i=1}^{n}\frac{(x_i - \mu)w^{-\frac{1}{\gamma}-1}2\lambda e^{-w^{-\frac{1}{\gamma}}}}{\sigma^2[1 + \lambda - 2\lambda e^{-w^{-\frac{1}{\gamma}}}]} = 0, \\ \frac{\partial l}{\partial \gamma} &= \sum_{i=1}^{n}\left[\frac{\log(w)}{\gamma^2} + \frac{\left(\frac{1}{\gamma}\right)(x_i - \mu)}{\sigma w}\right] - \sum_{i=1}^{n} w^{-\frac{1}{\gamma}}\left[\frac{\log(w)}{\gamma^2} - \frac{(x_i - \mu)}{\gamma\sigma w}\right] \\ &\quad + \sum_{i=1}^{n} w^{-\frac{1}{\gamma}}\left[\frac{\log(w)}{\gamma^2} - \frac{(x_i - \mu)}{\gamma\sigma w}\right]\left[\frac{2\lambda e^{-w^{-\frac{1}{\gamma}}}}{1 + \lambda - 2\lambda e^{-w^{-\frac{1}{\gamma}}}}\right] = 0, \\ \frac{\partial l}{\partial \lambda} &= \sum_{i=1}^{n}\frac{1 - 2e^{-w^{-\frac{1}{\gamma}}}}{1 + \lambda - 2\lambda e^{-w^{-\frac{1}{\gamma}}}} = 0, \end{aligned} \tag{3.4}$$

where $w = 1 + (\gamma/\sigma)(x_i - \mu)$.

3.2. Simulation and results

In order to investigate the performance of the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}, \hat{\gamma}, \hat{\lambda})$ of $\theta = (\mu, \sigma, \gamma, \lambda)$, random samples of size $n = 500$ and 1,000 of the random variable $X \sim F_T(\cdot\,; \mu, \sigma, \gamma, \lambda)$ were simulated for 32

Table 1: Parameters and mean estimates for $\gamma > 0$

θ	n	μ	σ	γ	λ	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$
θ_1	500	0	1	0.50	−0.50	0.001	1.026	0.508	−0.500
	1000	0	1	0.50	−0.50	-0.025	1.006	0.504	−0.531
θ_2	500	0	1	0.50	−0.90	0.229	1.111	0.506	−0.614
	1000	0	1	0.50	−0.90	0.183	1.078	0.505	−0.681
θ_3	500	0	1	0.50	0.50	−0.368	0.704	0.469	−0.269
	1000	0	1	0.50	0.50	−0.388	0.697	0.462	−0.314
θ_4	500	0	1	0.50	0.90	−0.427	0.597	0.322	−0.145
	1000	0	1	0.50	0.90	−0.433	0.596	0.326	−0.140
θ_5	500	0	1	0.75	−0.50	0.020	1.027	0.756	−0.481
	1000	0	1	0.75	−0.50	0.003	1.023	0.754	−0.506
θ_6	500	0	1	0.75	−0.90	0.187	1.117	0.754	−0.677
	1000	0	1	0.75	−0.90	0.179	1.120	0.744	−0.701
θ_7	500	0	1	0.75	0.50	−0.271	0.730	0.700	−0.111
	1000	0	1	0.75	0.50	−0.306	0.685	0.681	−0.173
θ_8	500	0	1	0.75	0.90	−0.391	0.533	0.507	−0.091
	1000	0	1	0.75	0.90	−0.379	0.553	0.515	−0.063
θ_9	500	0	1	0.90	−0.50	0.034	1.044	0.919	−0.451
	1000	0	1	0.90	−0.50	0.002	1.011	0.908	−0.501
θ_{10}	500	0	1	0.90	−0.90	0.184	1.135	0.897	−0.677
	1000	0	1	0.90	−0.90	0.167	1.129	0.898	−0.701
θ_{11}	500	0	1	0.90	0.50	−0.252	0.708	0.832	−0.084
	1000	0	1	0.90	0.50	−0.270	0.692	0.832	−0.134
θ_{12}	500	0	1	0.90	0.90	−0.330	0.556	0.643	0.021
	1000	0	1	0.90	0.90	−0.314	0.576	0.650	0.084
θ_{13}	500	0	1	1.00	−0.50	−0.023	0.981	1.014	−0.549
	1000	0	1	1.00	−0.50	−0.004	1.011	1.013	−0.516
θ_{14}	500	0	1	1.00	−0.90	0.149	1.126	1.004	−0.728
	1000	0	1	1.00	−0.90	0.135	1.118	0.990	−0.751
θ_{15}	500	0	1	1.00	0.50	−0.248	0.701	0.936	−0.094
	1000	0	1	1.00	0.50	−0.235	0.707	0.918	−0.062
θ_{16}	500	0	1	1.00	0.90	−0.329	0.529	0.720	0.021
	1000	0	1	1.00	0.90	−0.352	0.511	0.719	−0.063

combinations of μ, σ, γ and λ. These parameters are presented in Tables 1 and 3 that can be seen in two different configurations. In configuration 1, the values of the parametric vector corresponding to $\gamma > 0$, $\mu = 0$, $\sigma = 1$, and $\lambda = -0.9, -0.5, 0.5, 0.9$ are presented in Table 1. In configuration 2, the values of the parametric vector corresponding to $\gamma \leq 0$, $\mu = 0$, $\sigma = 1$, and $\lambda = -0.9, -0.5, 0.5, 0.9$ are seen in Table 1.

The procedure to investigate the performance of the MLEs consists of:

(1) Generating $M = 100$ random samples of size $n = 500$ and 1,000 from the TGEV distribution by the method of inversion using the quantiles

$$x_Q = \begin{cases} \dfrac{\sigma\left\{-1 + \left[-\ln\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda Q}}{2\lambda}\right)\right]^{-\gamma}\right\}}{\gamma} + \mu, & \text{for } \gamma \neq 0, \\ x_Q = \sigma\left\{-\ln\left[-\ln\left(\dfrac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda Q}}{2\lambda}\right)\right]\right\} + \mu, & \text{for } \gamma = 0. \end{cases}$$

(2) Obtaining the maximum likelihood estimates of the parameters $\mu, \sigma, \gamma, \lambda$ by maximizing the log-

Table 2: Bias and MSE of $\hat{\theta}$ for $\gamma > 0$

θ	n	Bias				MSE			
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$
θ_1	500	−0.001	−0.026	−0.008	0.000	0.070	0.034	0.002	0.120
	1000	0.025	−0.006	−0.004	0.031	0.046	0.015	0.001	0.085
θ_2	500	−0.229	−0.111	−0.006	−0.286	0.121	0.058	0.002	0.165
	1000	−0.183	−0.078	−0.005	−0.219	0.065	0.032	0.002	0.091
θ_3	500	0.368	0.296	0.031	0.769	0.059	0.035	0.002	0.245
	1000	0.388	0.303	0.038	0.814	0.056	0.032	0.002	0.251
θ_4	500	0.427	0.403	0.178	1.045	0.032	0.019	0.003	0.216
	1000	0.433	0.404	0.174	1.040	0.040	0.020	0.002	0.263
θ_5	500	−0.020	−0.027	−0.006	−0.019	0.044	0.033	0.003	0.077
	1000	−0.003	−0.023	−0.004	0.006	0.045	0.039	0.001	0.077
θ_6	500	−0.187	−0.117	−0.004	−0.223	0.048	0.027	0.002	0.062
	1000	−0.179	−0.120	0.006	−0.199	0.047	0.027	0.001	0.058
θ_7	500	0.271	0.270	0.050	0.611	0.066	0.077	0.007	0.298
	1000	0.306	0.315	0.069	0.673	0.047	0.050	0.004	0.238
θ_8	500	0.391	0.467	0.243	0.991	0.023	0.020	0.006	0.195
	1000	0.379	0.447	0.235	0.963	0.030	0.032	0.008	0.229
θ_9	500	−0.034	−0.044	−0.019	−0.049	0.038	0.039	0.002	0.064
	1000	−0.002	−0.011	−0.008	0.001	0.029	0.027	0.002	0.053
θ_{10}	500	−0.184	−0.135	0.003	−0.223	0.036	0.027	0.003	0.049
	1000	−0.167	−0.129	0.002	−0.199	0.047	0.036	0.001	0.060
θ_{11}	500	0.252	0.292	0.068	0.584	0.052	0.075	0.010	0.290
	1000	0.270	0.308	0.068	0.634	0.053	0.070	0.007	0.311
θ_{12}	500	0.330	0.444	0.257	0.879	0.029	0.042	0.014	0.235
	1000	0.314	0.424	0.250	0.816	0.031	0.046	0.015	0.263
θ_{13}	500	0.023	0.019	−0.014	0.049	0.033	0.039	0.003	0.055
	1000	0.004	−0.011	−0.013	0.016	0.039	0.068	0.004	0.061
θ_{14}	500	−0.149	−0.126	−0.004	−0.172	0.037	0.033	0.003	0.047
	1000	−0.135	−0.118	0.010	−0.149	0.030	0.023	0.002	0.036
θ_{15}	500	0.248	0.299	0.064	0.594	0.050	0.089	0.013	0.306
	1000	0.235	0.293	0.082	0.562	0.050	0.079	0.008	0.304
θ_{16}	500	0.329	0.471	0.280	0.879	0.025	0.041	0.017	0.251
	1000	0.352	0.489	0.281	0.963	0.030	0.047	0.015	0.254

likelihood function, (3.1) or (3.2), through of the "optim function" of the R Core Team software 2015. In this function the method for optimization is a derivative-free optimization routine called the Nelder-Mead simplex algorithm. This method already provides the Hessian matrix.

Results about mean estimates of each parameter and their corresponding mean square errors (MSEs) were calculated via Monte Carlo simulation with $M = 100$ samples of size $n = 500$ and 1,000. The results are presented in Tables 1–4.

In Section 2.2 we have that for $\gamma > 0$ the TGEV moments are defined in $[1/2, 1]$, so the values chosen for $\gamma > 0$ (Configuration 1) were $0.5, 0.75, 0.9, 1$. Table 2 shows that the bias and MSE of the mean estimates are small. The algorithm has obtained good estimates. In configuration 2, Table 4 shows estimates of the TGEV for $\gamma \leq 0$. We have the convergence of the algorithm for γ in $[-0.5, 0]$, the bias is significant in some cases, but the MSE values are low.

Illustrations graphical of the fitted density $f(x, \hat{\theta})$ together the theoretical density $f(x, \theta)$ for several cases of θ_1 to θ_{32} are shown in Appendix in Figures A.1, A.2, and A.3. These figures also show that the method yields satisfactory results.

Table 3: Parameters and mean estimates for $\gamma \leq 0$

θ	n	μ	σ	γ	λ	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$
θ_{17}	500	0	1	−0.10	0.50	−0.052	1.024	−0.105	−0.546
	1000	0	1	−0.10	0.50	−0.165	1.047	−0.112	−0.700
θ_{18}	500	0	1	−0.10	−0.90	0.164	0.992	−0.094	−0.673
	1000	0	1	−0.10	−0.90	0.173	0.978	−0.101	−0.649
θ_{19}	500	0	1	−0.10	0.50	−0.650	0.861	−0.051	−0.632
	1000	0	1	−0.10	0.50	−0.691	0.867	−0.049	−0.691
θ_{20}	500	0	1	−0.10	0.90	−0.757	0.763	−0.085	−0.598
	1000	0	1	−0.10	0.90	−0.781	0.761	−0.083	−0.644
θ_{21}	500	0	1	−0.25	−0.50	−0.110	1.059	−0.263	−0.625
	1000	0	1	−0.25	−0.50	−0.150	1.066	−0.262	−0.668
θ_{22}	500	0	1	−0.25	−0.90	0.156	0.966	−0.256	−0.652
	1000	0	1	−0.25	−0.90	0.109	0.965	−0.245	−0.744
θ_{23}	500	0	1	−0.25	0.50	−0.704	0.956	−0.179	−0.656
	1000	0	1	−0.25	0.50	−0.784	0.980	−0.186	−0.771
θ_{24}	500	0	1	−0.25	0.90	−0.791	0.850	−0.188	−0.579
	1000	0	1	−0.25	0.90	−0.873	0.859	−0.184	−0.713
θ_{25}	500	0	1	−0.50	−0.50	−0.217	1.142	−0.519	−0.748
	1000	0	1	−0.50	−0.50	−0.210	1.140	−0.518	−0.743
θ_{26}	500	0	1	−0.50	−0.90	0.140	0.941	−0.510	−0.663
	1000	0	1	−0.50	−0.90	0.095	0.945	−0.497	−0.754
θ_{27}	500	0	1	−0.50	0.50	−0.793	1.165	−0.411	−0.679
	1000	0	1	−0.50	0.50	−0.754	1.135	−0.402	−0.639
θ_{28}	500	0	1	−0.50	0.90	−0.957	1.045	−0.362	−0.680
	1000	0	1	−0.50	0.90	−0.934	1.030	−0.356	−0.665
θ_{29}	500	0	1	0.00	−0.50	0.440	1.187	0.000	0.135
	1000	0	1	0.00	−0.50	0.412	1.170	0.000	0.098
θ_{30}	500	0	1	0.00	−0.90	0.729	1.137	0.000	0.167
	1000	0	1	0.00	−0.90	0.715	1.148	0.000	0.121
θ_{31}	500	0	1	0.00	0.50	−0.084	0.962	0.000	0.340
	1000	0	1	0.00	0.50	−0.095	0.963	0.000	0.324
θ_{32}	500	0	1	0.00	0.90	−0.360	0.853	0.000	0.157
	1000	0	1	0.00	0.90	−0.362	0.846	0.000	0.163

4. Application

In order to apply the model TGEV, we use three sets of real financial data: Ibovespa, S&P 500, and Dow Jones. The data were obtained from the website http://br.investing.com/indices. For each data set we use daily log-returns from June 3, 2006 to October 31, 2016. The log-returns are given by $r_t = \log(P_t) - \log(P_{t-1})$, where P_t is the opening price at day t. For the Ibovespa were used 2,707 observations, for the S&P 500 were 2,768 and for the Dow Jones 2,795. The data modeled by the TGEV distribution correspond to the maximum values of the returns in blocks of size 7. The histograms for these values are shown in Figure 7.

Table 5 shows the main descriptive statistics for the three data sets in question. The MLEs of the parameters $\mu, \sigma, \gamma, \lambda$ are obtained from Equations (3.3) and (3.4). Table 6 shows the estimates obtained for the three data sets.

Figures 8–10 present the histogram of the returns versus the fitted density for each data set, whose parameters are shown in Table 6 along with QQplot that compares the theoretical quantiles on the vertical axis with the empirical quantiles on the horizontal axis. From the results, we can observe that the estimates of λ are different from zero. This indicates that TGEV distribution is more appropriate for these databases than the GEV distribution often considered by many authors.

Table 4: Bias and MSE of $\hat{\theta}$ for $\gamma \leq 0$

θ	n	Bias				MSE			
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$
θ_{17}	500	0.052	−0.024	0.005	0.046	0.057	0.004	0.001	0.107
	1000	0.165	−0.047	0.012	0.200	0.046	0.004	0.001	0.076
θ_{18}	500	−0.164	0.008	−0.006	−0.227	0.077	0.007	0.004	0.168
	1000	−0.173	0.022	0.001	−0.251	0.060	0.002	0.001	0.126
θ_{19}	500	0.650	0.139	−0.049	1.132	0.039	0.002	0.001	0.096
	1000	0.691	0.133	−0.051	1.191	0.027	0.002	0.000	0.071
θ_{20}	500	0.757	0.237	−0.015	1.498	0.031	0.003	0.001	0.105
	1000	0.781	0.239	−0.017	1.544	0.015	0.001	0.001	0.046
θ_{21}	500	0.110	−0.059	0.013	0.125	0.058	0.007	0.001	0.096
	1000	0.150	−0.066	0.012	0.168	0.049	0.006	0.000	0.083
θ_{22}	500	−0.156	0.034	0.006	−0.248	0.070	0.010	0.001	0.128
	1000	−0.109	0.035	−0.005	−0.156	0.035	0.005	0.001	0.071
θ_{23}	500	0.704	0.044	−0.071	1.156	0.044	0.004	0.001	0.092
	1000	0.784	0.020	−0.064	1.271	0.029	0.004	0.000	0.055
θ_{24}	500	0.791	0.150	−0.062	1.479	0.031	0.003	0.002	0.079
	1000	0.873	0.141	−0.066	1.613	0.016	0.002	0.001	0.032
θ_{25}	500	0.217	−0.142	0.019	0.248	0.038	0.016	0.001	0.048
	1000	0.210	−0.140	0.018	0.243	0.043	0.015	0.001	0.069
θ_{26}	500	−0.140	0.059	0.010	−0.237	0.056	0.014	0.002	0.149
	1000	−0.095	0.055	−0.003	−0.146	0.029	0.010	0.001	0.069
θ_{27}	500	0.793	−0.165	−0.089	1.179	0.068	0.014	0.001	0.116
	1000	0.754	−0.135	−0.098	1.139	0.068	0.011	0.001	0.114
θ_{28}	500	0.957	−0.045	−0.138	1.580	0.030	0.008	0.001	0.049
	1000	0.934	−0.030	−0.144	1.565	0.022	0.005	0.000	0.038
θ_{29}	500	−0.440	−0.187	0.000	−0.635	0.200	0.041	0.000	0.393
	1000	−0.412	−0.170	0.000	−0.598	0.191	0.036	0.000	0.390
θ_{30}	500	−0.729	−0.137	0.000	−1.067	0.142	0.026	0.000	0.307
	1000	−0.715	−0.148	0.000	−1.021	0.191	0.027	0.000	0.410
θ_{31}	500	0.084	0.038	0.000	0.160	0.041	0.008	0.000	0.117
	1000	0.095	0.037	0.000	0.176	0.043	0.007	0.000	0.135
θ_{32}	500	0.360	0.147	0.000	0.743	0.139	0.022	0.000	0.581
	1000	0.362	0.154	0.000	0.737	0.134	0.022	0.000	0.571

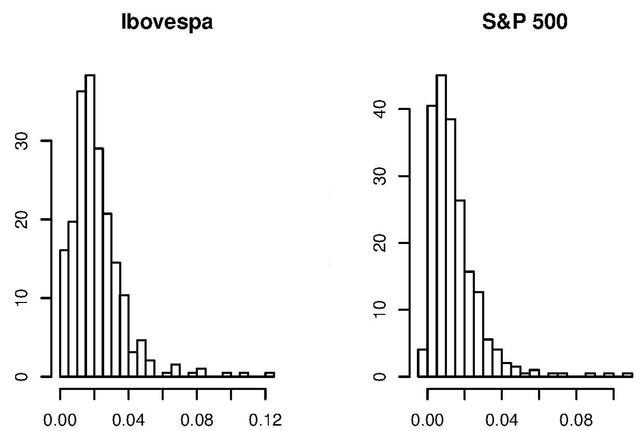

Figure 7: *Histogram of maximums of blocks of size 7 of the returns.*

Table 5: Descriptive statistics of data

Index	$E(X)$	Var(X)	max(X)	min(X)	median(X)
Ibovespa	0.0214	0.0002	0.1208	3×10^{-6}	0.0186
S&P 500	0.0144	0.0001	0.1055	-7×10^{-4}	0.0112
Dow Jones	0.0130	0.0001	0.0966	-0.0011	0.0099

Table 6: Parameter estimates

Index	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\lambda}$
Ibovespa	0.02098	0.01500	0.28645	0.95497
S&P 500	0.01222	0.01080	0.43309	0.80260
Dow Jones	0.00782	0.00651	0.30470	0.14741

Table 7: Model selection

Dados	Model	log-likelihood	AIC	BIC
Bovespa	GEV	−1159.070	2326.141	2341.964
	GEVT	−1159.584	2327.168	2342.991
S&P500	GEV	−1284.209	2576.417	2592.333
	GEVT	−1283.709	2575.418	2591.333
DowJones	GEV	−1336.870	2681.739	2697.695
	GEVT	−1336.921	2681.843	2697.799

AIC = Akaike information criteria; BIC = Bayesian information criteria; GEV = generalized extreme value.

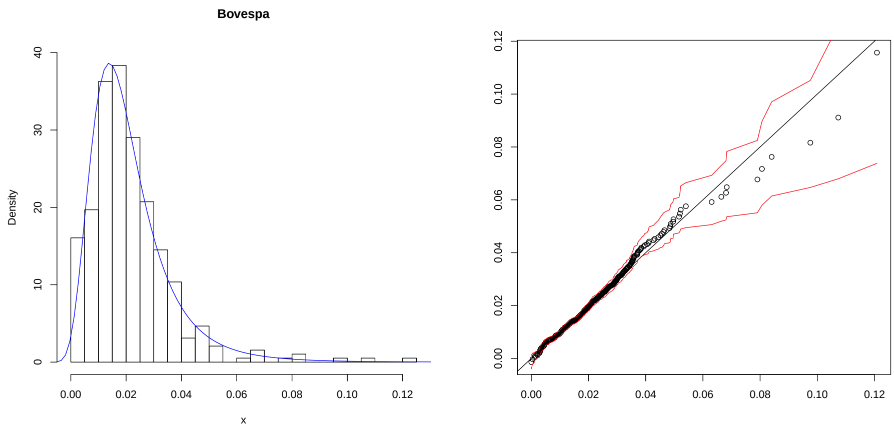

Figure 8: *Fitted Ibovespa data by TGEV (left) and QQplot empirical versus theoretical (right). TGEV = transmuted generalized extreme value.*

Analyzing the QQplots, we can see that the TGEV distribution fits very well as an estimated distribution for the data used. The Kolmogorov-Smirnov (KS) test and the Anderson Darling (AD) test were performed to verify the adjustment of the estimated distribution to the data. Both test the following hypotheses:

$$\begin{cases} H_0 : \text{The data follow the TGEV distribution;} \\ H_A : \text{The data does not follow the TGEV distribution.} \end{cases}$$

Table 8 therefore presents the results for the KS test statistic and its p-value, as well as the AD test statistic and its p-value.

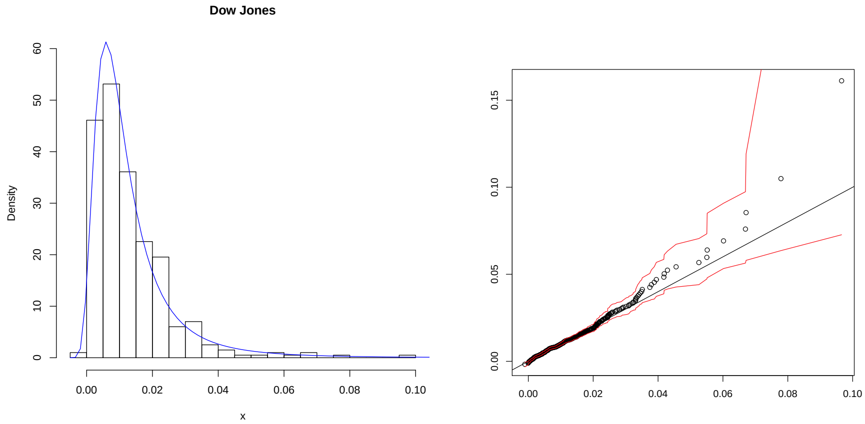

Figure 9: *Fitted S&P 500 data by TGEV (left) and QQplot empirical versus theoretical (right).*

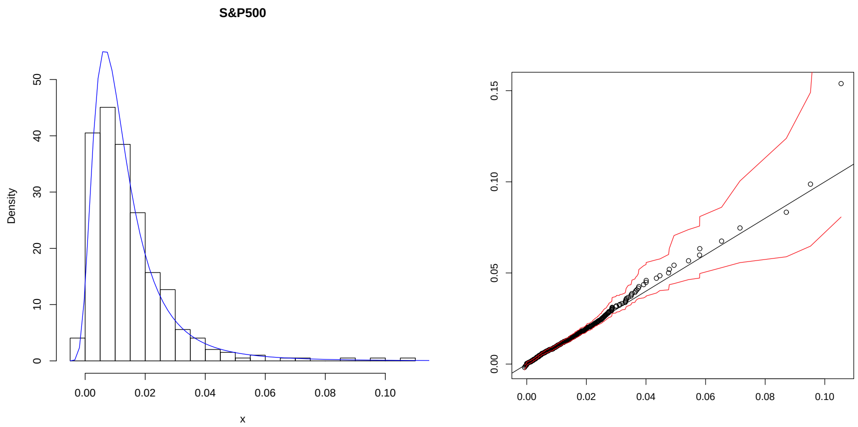

Figure 10: *Fitted Dow Jones data by TGEV (left) and QQplot empirical versus theoretical (right).*

Table 8: Hypothesis test result

Index	KS	p-valor	AD	p-valor
Ibovespa	0.03363	0.7752	0.4488	0.7993
S&P 500	0.03613	0.6810	0.5901	0.6573
Dow Jones	0.03558	0.6935	0.7058	0.5539

KS = Kolmogorov-Smirnov test; AD = Anderson Darling text.

It is possible to conclude from the KS and AD tests and the analysis of the p-values obtained that there is no statistical evidence against the hypothesis that the data follow the TGEV distributions estimated here.

5. Conclusion

In this paper, we present important properties of the TGEV distribution. TGEV distribution is a more flexible model than GEV distribution to model extreme event data. The estimation of the parameters is approached by the maximum likelihood method. Applications of TGEV for three data sets show that the new distribution can be used to effectively provide better adjustments than GEV distribution.

Appendix:

We added the Figures A.1.A.3 in order to illustrate the fit of the simulated data in Section 3. In almost all cases, the adjusted densities were close to the theoretical densities.

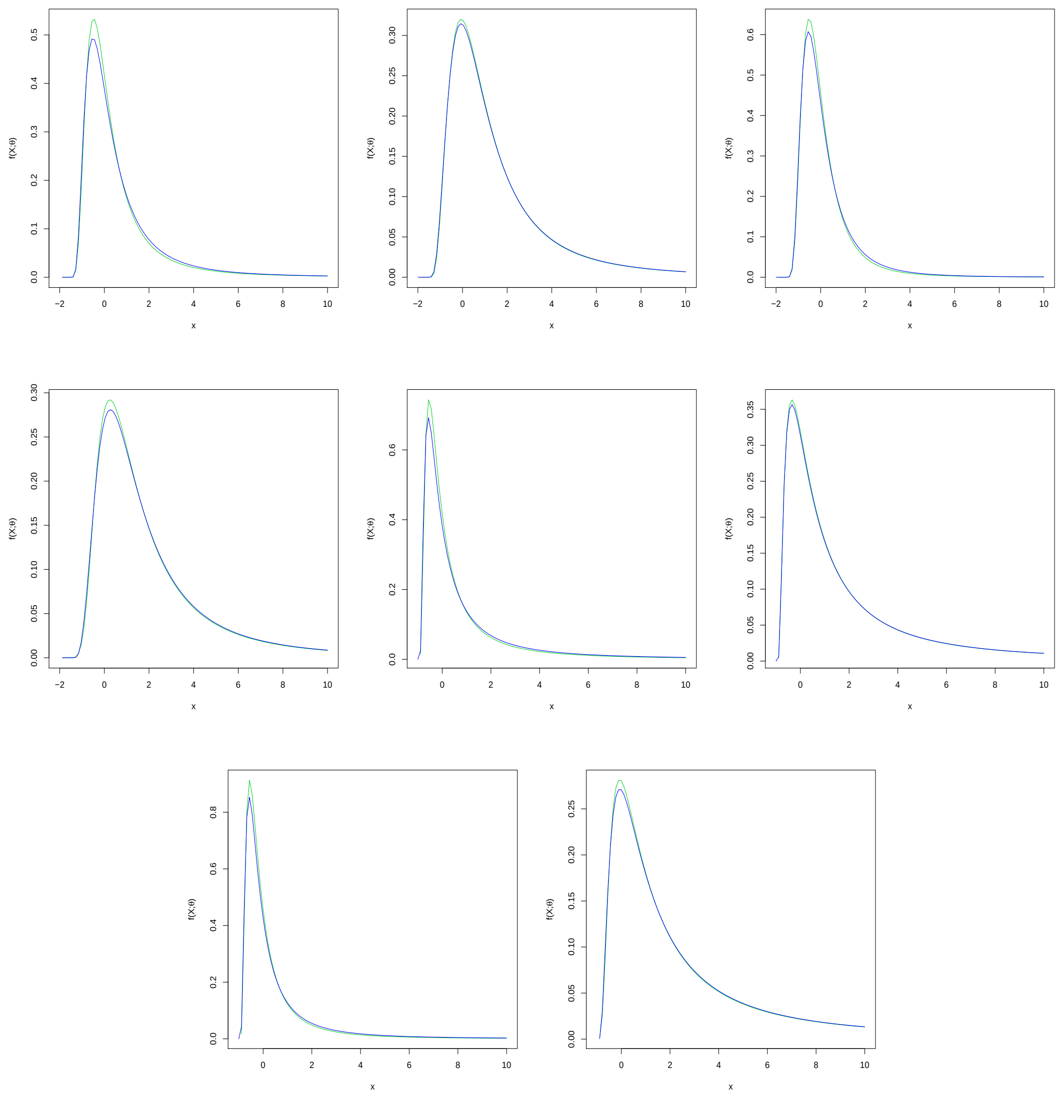

Figure A.1: *Fitted density $f(x, \hat{\theta})$ (blue) and the theoretical density $f(x, \theta)$ (red) for θ_1–θ_4 and θ_{13}–θ_{16}, according to Table 1.*

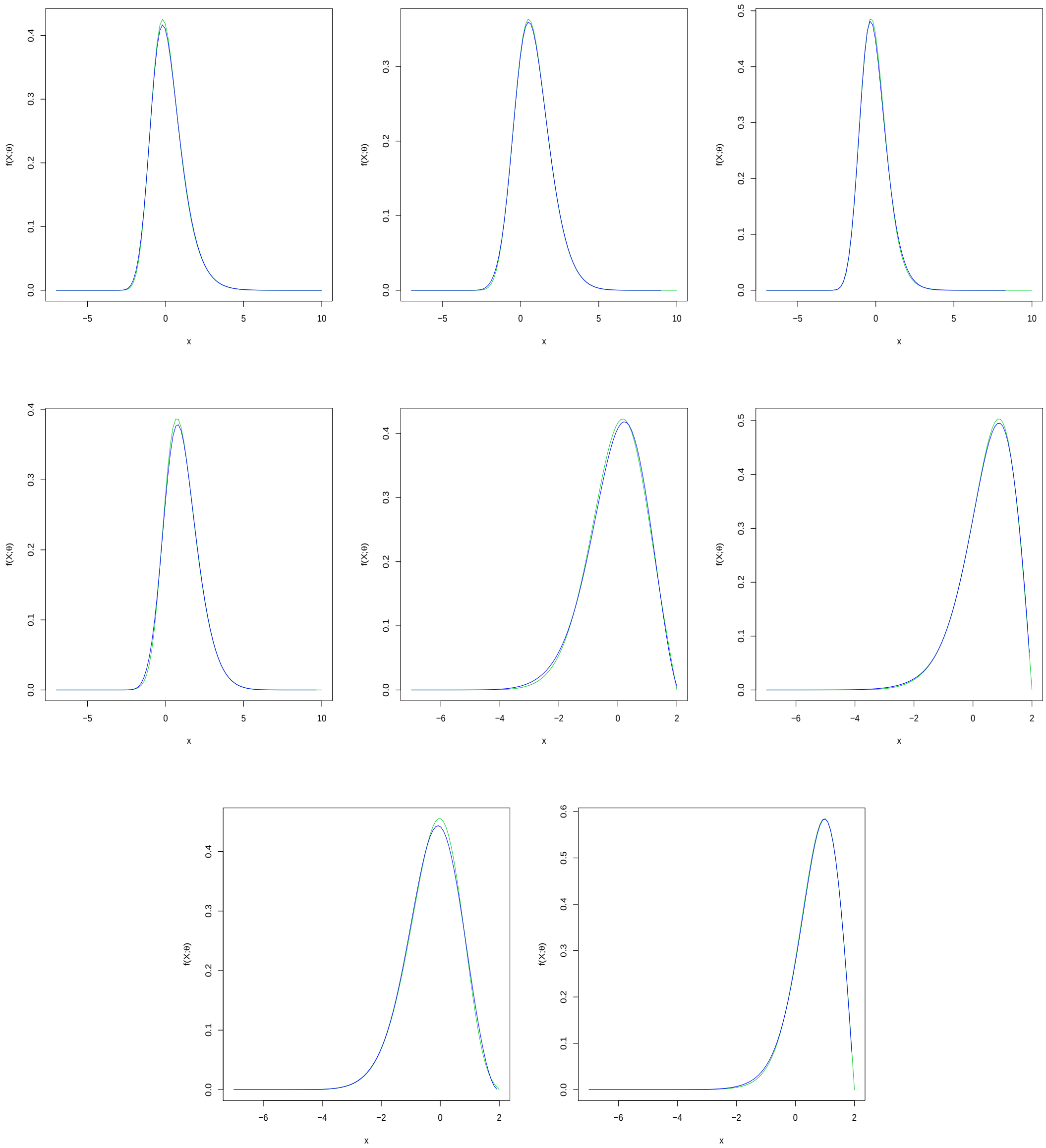

Figure A.2: *Fitted density* $f(x,\hat{\theta})$ *(blue) and the theoretical density* $f(x,\theta)$ *(red) for* θ_{17}*–*θ_{20} *and* θ_{25}*–*θ_{30}*, according to Table 3.*

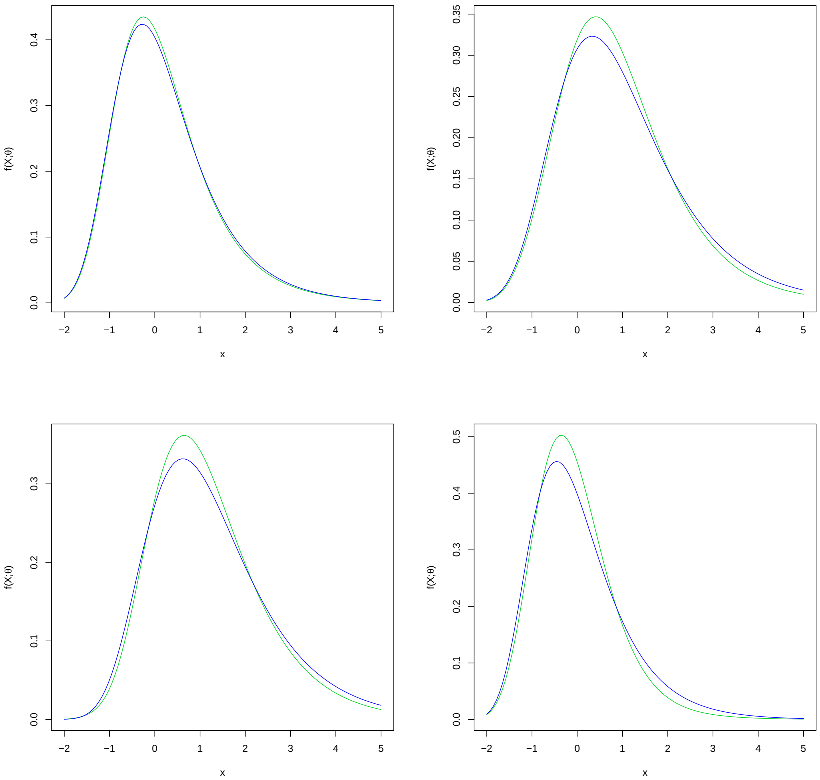

Figure A.3: *Fitted density $f(x,\hat{\theta})$ (blue) and the theoretical density $f(x,\theta)$ (red) for θ_{10}–θ_{14}, according to Table 3.*

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