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MONODROMY GROUPOID OF A LOCAL TOPOLOGICAL GROUP-GROUPOID

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Abstract. In this paper, we define a local topological group-groupoid and prove that if G is a local topological group-groupoid, then the monodromy groupoid $\mathrm{Mon}(G)$ of G is a local group-groupoid.

Introduction

The general idea of the monodromy principle was stated in Chevalley [10] for a topological structure G and also for a topological group and developed by Douady and Lazard in [11] for Lie groups, generalized to topological groupoid case in [3] and [15].

The notion of *monodromy groupoid* was described by J. Pradines [21] in the early 1960s. Let G be a topological groupoid such that each star G_x has a universal cover and as a set, $\mathrm{Mon}(G)$ be the union of the stars $(\pi_1 G_x)_{1_x}$. Then there is a groupoid structure on $\mathrm{Mon}(G)$ whose object set X is the same as that of G and groupoid composition is defined by the concatenation composition of the paths in the stars G_x. In [3], in the smooth groupoid case including topological groupoids, the star topological groupoid and topological groupoid structures of $\mathrm{Mon}(G)$ are studied under some suitable local conditions. We call $\mathrm{Mon}(G)$, the monodromy groupoid of G.

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In Mackenzie [12], it was given a non-trivial direct construction of the topology on $\mathrm{Mon}(G)$ and proved also that $\mathrm{Mon}(G)$ satisfies the monodromy principle on the globalisation of continuous local morphisms on G.

A group-groupoid is an internal groupoid in the category of groups [2,15]. In [14] it is proved that if G is a topological group-groupoid, then the monodromy groupoid $\mathrm{Mon}(G)$ becomes a group-groupoid, which is defined in [2,4] as the group object in the category of groupoids. In [19], this result was generalized to internal groupoids in topological groups with operations. So it is proved in [19] that if G is a topological group with operations, then $\mathrm{Mon}(G)$ becomes an internal groupoid in the category of groups with operations. Also in [8], the monodromy groupoid of a Lie groupoid is constructed.

The properties an examples of local subgroupoids are given in [9]. on the other hand the notion of local topological group-groupoid which is the group object in the category of local topological groupoids is given in [20]. Also it is proved that the category $\mathsf{LTGpCov}/L$ of covering morphisms $p\colon \widetilde{L} \to L$ of local topological groups in which $\widetilde{L}$ has also a universal cover and the category $\mathsf{LTGpGdCov}/\pi_1(L)$ of covering morphisms $q\colon \widetilde{G} \to \pi_1(L)$ of local topological group-groupoids based on $\pi_1(L)$ are equivalent [20].

In this paper we prove that if G is a local topological group-groupoid, then the monodromy groupoid $\mathrm{Mon}(G)$ becomes a local group-groupoid.

1. Prelimineries

A *groupoid* is a (small) category in which each morphism is an isomorphism [1, p.205]. So a groupoid G has a set G of morphisms , which we call just *elements* of G, a set $\mathrm{Ob}(G)$ of *objects* together with maps $s, t\colon G \to \mathrm{Ob}(G)$ and $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $s\epsilon = t\epsilon = 1_{\mathrm{Ob}(G)}$. The maps s, t are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion.* If $g, h \in G$ and $t(g) = s(h)$, then the *composite* gh exists such that $s(gh) = s(g)$ and $t(gh) = t(h)$. So there exists a partial composition defined by $G_t \times_s G \to G, (g, h) \mapsto gh$, where $G_t \times_s G$ is the pullback of t and s. Further, this partial composition is associative, for $x \in \mathrm{Ob}(G)$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element g has an inverse g^{-1} such that $s(g^{-1}) = t(g)$,

$t(g^{-1}) = s(g)$, $gg^{-1} = (\epsilon s)(g)$, $g^{-1}g = (\epsilon t)(g)$. The map $G \to G$, $a \mapsto g^{-1}$ is called the *inversion*.

In a groupoid G for $x, y \in \mathrm{Ob}(G)$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *transitive* if for all $x, y \in \mathrm{Ob}(G)$, the set $G(x, y)$ is not empty. For $x \in \mathrm{Ob}(G)$ we denote the star $\{g \in G \mid s(g) = x\}$ of x by G_x.

A star topological groupoid is a groupoid in which the stars G_x's have topologies such that for each $g \in G(x, y)$ the left (and hence right) translation

$$L_g \colon G_y \longrightarrow G_x, h \mapsto gh$$

is a homeomorphism and G is the topological sum of the G_x's.

Definition 1.1. [5,6] A topological groupoid is a groupoid such that the set G of morphisms and the set $\mathrm{Ob}(G)$ of objects are topological spaces and source, target, inclusion, inverse and product maps are continuous. □

Definition 1.2. [6] Let G and $\widetilde{G}$ be two topological groupoids. Then a groupoid morphism of topological groupoids is a morphism of groupoids $p \colon \widetilde{G} \to G$ such that the pair of maps $p \colon \widetilde{G} \to G$ and $O_p \colon \mathrm{Ob}(\widetilde{G}) \to \mathrm{Ob}(G)$ are continuous. □

Recall that a covering map $p \colon \widetilde{X} \longrightarrow X$ of connected spaces is called universal if it covers every covering of X in the sense that if $q \colon \widetilde{Y} \longrightarrow X$ is another covering of X then there exists a map $r \colon \widetilde{X} \longrightarrow \widetilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p \colon \widetilde{X} \longrightarrow X$ is called simply connected if $\widetilde{X}$ is simply connected. So a simply connected covering is a universal covering.

Let X be a topological space admitting a simply connected cover. A subset U of X is called liftable if U is open, path-connected and the inclusion $U \longrightarrow X$ maps each fundamental group of U trivially. If U is liftable, and $q \colon Y \longrightarrow X$ is a covering map, then for any $y \in Y$ and $x \in U$ such that $qy = x$, there is a unique map $i \colon U \longrightarrow Y$ such that $ix = y$ and qi is the inclusion $U \longrightarrow X$. A space X is called semi-locally simply connected if each point has a liftable neighborhood and locally simply connected if it has a base of simply connected sets. So a locally simply connected space is also semi-locally simply connected.

Let X be a topological space such that each path component of X admits a simply connected covering space. It is standard that if the fundamental groupoid $\pi_1 X$ is denoted with the topology as in [7], and $x \in X$, then the target map $t\colon (\pi_1 X)_x \longrightarrow X$ is the universal covering map of X based at x (see also Brown [1, 10.5.8]).

The following theorem is proved in [7, Theorem 1]. We give a sketch proof since we need some details of the proof in Theorem 3.11. An alternative but equivalent construction of the topology is in [1, 10.5.8].

THEOREM 1.3. *If X is a locally path connected and semi-locally simply connected space, then the fundamental groupoid $\pi_1 X$ may be given a topology making it a topological groupoid.*

Proof. Let $\mathcal{U}$ be the open cover of X consisting of all liftable subsets. For each U in $\mathcal{U}$ and $x \in U$ define a map $\lambda_x\colon U \to \pi X$ by choosing for each $x' \in U$, a path in U from x to x' and letting $\lambda_x(x')$ be the homotopy class of this path. By the condition on U the map λ_x is well defined. Let $\widetilde{U}_x = \lambda_x(U)$. Then the sets $\widetilde{U}_x^{-1}\alpha\widetilde{V}_y$ for all $\alpha \in \pi X(x, y)$ form a base for a topology such that πX is a topological groupoid with this topology.□

2. Monodromy Groupoids

In this section we give a review of the constructions of the monodromy groupoid from [3].

Let G be a star topological groupoid. The groupoid $\mathrm{Mon}(G)$ is defined from the universal covers of stars G_x's at the base points identities as follows: As a set, $\mathrm{Mon}(G)$ is the union of the stars $(\pi_1 G_x)_{1_x}$. The object set X of $\mathrm{Mon}(G)$ is the same as that of G. The initial point map $s\colon \mathrm{Mon}(G) \to X$ maps all of $(\pi_1 G_x)_{1_x}$ to x, while the target point map $t\colon \mathrm{Mon}(G) \to X$ is defined on each $(\pi_1 G_x)_{1_x}$ as the composition of the two target maps

$$(\pi_1 G_x)_{1_x} \xrightarrow{t} G_x \xrightarrow{t} X.$$

As expounded in Mackenzie [12] it is seen a multiplication on $\mathrm{Mon}(G)$ defined by

$$[a] \bullet [b] = [a \star (a(1)b)],$$

where $\star$ inside the bracket denotes the usual composition of paths. So the path $a \star (a(1)b)$ is defined by

$$(a \star (a(1)b))(t) = \begin{cases} a(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ a(1)b(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Here assume that a is a path in G_x from 1_x to $a(1)$, where $t(a(1)) = y$, say, and b is a path in G_x from 1_y to $b(1)$, then for each $t \in [0,1]$ the composition $a(1)b(t)$ is defined in G_y, running a path $a(1)b$ from $a(1)$ to $a(1)b(1)$. It is straightforward to prove that in this way a groupoid is defined on $\mathrm{Mon}(G)$ and that the composition of the final maps of paths induces a morphism of groupoids $p\colon \mathrm{Mon}(G) \to G$.

If each star G_x admits a simply connected cover at 1_x, then we may topologise each star $(\mathrm{Mon}(G))_x$ so that it is the universal cover of G_x based at 1_x, and then $\mathrm{Mon}(G)$ becomes a star topological groupoid. We call $\mathrm{Mon}(G)$ the *monodromy groupoid* or *star universal cover* of G.

Let Gpd be the category of groupoids and TGd the category of topological groupoids. Let STGd be the full subgroupoid of TGd on those topological groupoids whose stars have universal covers. Then we have a functor

$$\mathrm{Mon}\colon \mathsf{STGd} \to \mathsf{Gpd}$$

assigning each topological groupoid G such that the stars have universal covers, to the monodromy groupoid $\mathrm{Mon}(G)$.

Theorem 2.1. [14, Theorem 2.1] *For the topological groupoids G and H such that the stars have universal covers, the monodromy groupoids* $\mathrm{Mon}(G \times H)$ *and* $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ *are isomorphic.*

Example 2.2. Let G be a topological group which can be thought as a topological groupoid with only one object. If G has a simply connected cover, then the monodromy groupoid $\mathrm{Mon}(G)$ of G is just the universal cover of G. □

Example 2.3. [3, Theorem 6.2] If X is a topological space, then $G = X \times X$ becomes a topological groupoid on X. Here a pair (x,y) is a morphism from x to y with inverse morphism (y,x). The groupoid composition is defined by $(x,y)(u,z) = (x,z)$ whenever $y = u$. If X has a simply connected cover, then the monodromy groupoid $\mathrm{Mon}(G)$ of G is isomorphic to the fundamental groupoid $\pi_1(X)$. □

3. Local group-groupoid structure of monodromy groupoids

In this section we introduce the notion of local topological group-groupoid and prove that if G is a local topological group-groupoid where each star G_x has a universal cover, then $\mathrm{Mon}(G)$ is a local group-groupoid.

Now we emphasis the definition given in [16, Definition 2].

Definition 3.1. Let L be a set. A *local group* is a quintuple $\mathbf{L} = (L, \mu, \mathcal{U}, i, V)$, where

(1) a distinguish element $e \in L$, the identity element,
(2) a multiplication $\mu\colon \mathcal{U} \to L, (x, y) \mapsto x \circ y$ defined on a subset $\mathcal{U}$ of $L \times L$ such that $(\{e\} \times L) \cup (L \times \{e\}) \subseteq \mathcal{U}$,
(3) an inversion map $i\colon V \to L, x \mapsto \overline{x}$ defined on a subset $e \in V \subseteq L$ such that $V \times i(V) \subseteq \mathcal{U}$ and $i(V) \times V \subseteq \mathcal{U}$,

all satisfying the following properties:

(i) Identity: $e \circ x = x = x \circ e$ for all $x \in L$
(ii) Inverse: $i(x) \circ x = e = x \circ i(x)$, for all $x \in V$
(iii) Associativity: If $(x, y), (y, z), (x \circ y, z)$ and $(x, y \circ z)$ all belong to $\mathcal{U}$, then

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

□

From now on we denote such a local group by $\mathbf{L}$.

Note that if $\mathcal{U} = L \times L$ and $V = L$, then a local group becomes a group. It means that the notion of local group generalizes that of group. Now we give the following definition (see [16, Definition 5]):

Definition 3.2. Let $(L, \mu, \mathcal{U}, i, V)$ and $(\widetilde{L}, \widetilde{\mu}, \widetilde{\mathcal{U}}, \widetilde{i}, \widetilde{V})$ be local groups. A map $f\colon L \to \widetilde{L}$ is called a *local group morphism* if

(i) $(f \times f)(\mathcal{U}) \subseteq \widetilde{\mathcal{U}}$, $f(V) \subseteq \widetilde{V}$, $f(e) = \widetilde{e}$
(ii) $f(x \circ y) = f(x) \circ f(y)$ for $(x, y) \in \mathcal{U}$
(iii) $f(i(x)) = \widetilde{i}(f(x))$ for $x \in V$.

□

We study on the topological version of Definition 3.1.

Definition 3.3. [16] Let L be a local group, if L has a topology structure such that $\mathcal{U}$ is open in $L \times L$, V is open in L, the maps μ and i are continuous, then $(L, \mu, \mathcal{U}, i, V)$ is called a *local topological group*.□

It is obvious that if $\mathcal{U} = L \times L$ and $V = L$, then a local topological group L becomes a topological group.

Example 3.4. [16, p.26] Let G be a topological group, L be an open neighbourhood of the identity element e. Then we obtain a local topological group taking $\mathcal{U} = (L \times L) \cap \mu^{-1}(L)$ and $V = L \cap \overline{L}$, where $\overline{L} = \{\overline{x} | x \in L\}$.

Here the group product μ and the inversion i on G are restricted to define a local group product and inverse maps on L.

Further if we choose $\mathcal{U}$ and V such that

$$(\{e\} \times L) \cup (L \times \{e\}) \subseteq \mathcal{U} \subseteq (L \times L) \cap \mu^{-1}(L)$$
$$\{e\} \subseteq V \subseteq L \cap i^{-1}(L)$$

and

$$V \times i(V)) \cup (i(V) \times V) \subseteq \mathcal{U}$$

then we have a local topological group.

□

Definition 3.5. [17, Definition 3.3] Let $(L, \mu, \mathcal{U}, i, V)$ and $(\widetilde{L}, \widetilde{\mu}, \widetilde{\mathcal{U}}, \widetilde{i}, \widetilde{V})$ be local topological groups. A continuous map $f\colon L \to \widetilde{L}$ is called a *local topological group morphism* if

(i) $(f \times f)(\mathcal{U}) \subseteq \widetilde{\mathcal{U}}$, $f(V) \subseteq \widetilde{V}$, $f(e) = \widetilde{e}$
(ii) $f(x \circ y) = f(x) \circ f(y)$ for $(x, y) \in \mathcal{U}$
(iii) $f(i(x)) = \widetilde{i}(f(x))$ for $x \in V$.

□

Before giving the definition of local group-group, we state local morphism of groupoids (see [13, Definition 6.1.6] for the notion of local morphism of Lie groupoids).

Definition 3.6. Let G and H be groupoids. A local morphism from G to H consists of a map $f\colon W \to H$ defined on a subset of G including all the identities 1_x for $x \in \mathrm{Ob}(G)$ such that

(i) $f(ab) = f(a)f(b)$ for $a, b \in W$ with $t(a) = s(a)$
(ii) $f(a^{-1}) = f(a)^{-1}$.

The notion of local group-groupoid is given in [17, Definition 4.1] as follows.

Definition 3.7. A *local group-groupoid* G is a groupoid in which $\mathrm{Ob}(G)$ and G both have local group structures such that the following maps are the local morphisms of groupoids (i.e., they are both groupoid morphsims and local group morphisms):

(i) $\mu\colon \mathcal{U} \to G, (a,b) \mapsto a \circ b$
(ii) $i\colon V \to G, a \mapsto \overline{a}$
(iii) $e\colon \star \to G$, where is $\star$ is singleton.

□

In a local group-groupoid we write ab for the composition in groupoid while $a \circ b$ for the multiplication in local group; and write a^{-1} for the inverse of a in groupoid while $\overline{a}$ for the one in local groupoid. We obtain that in a local group-groupoid G,

$$(ac) \circ (bd) = (a \circ b)(c \circ d)$$

for $a, b, c, d \in G$ such that the necessary composition and multiplications are defined.

The category of local group-groupoids is denoted by LGpGpd.

Definition 3.8. [17, Definition 5.1] Let G and H be two local group-groupoids. A morphism of local group-groupoids $f\colon H \to G$ is a morphism of underlying groupoids preserving local group structure, i.e., $f(a \circ b) = f(a) \circ f(b)$ for $a, b \in \mathcal{U} \subseteq H \times H$. □

Definition 3.9. [17, Definition 5.1] A morphism $f\colon H \to G$ of local group-groupoids is called a covering morphism (resp. universal covering morphism)if it is a covering (resp. universal covering) on underlying groupoids. □

As topological version of Definition 3.7, a local morphism of topological groupoids can be stated as follows:

Definition 3.10. Let G and H be topological groupoids. A local morphism from G to H consists of a continuous local morphism $f\colon W \to H$ of groupoids defined on an open subset of G including all the identities.

Now we study on local topological group-groupoids as the following definition:

Definition 3.11. Let G be a topological groupoid. If the set G of morphisms and the set $\mathrm{Ob}(G)$ of objects have local topological group structures such that the maps

(i) $\mu\colon \mathcal{U} \to G, (a,b) \mapsto a \circ b$
(ii) $i\colon V \to G, a \mapsto \overline{a}$
(iii) $e\colon \star \to G$, where is $\star$ is singleton

are local morphisms of topological groupoids, then G is called a *local topological group-groupoid*.

□

Let us denote the category of local topological group-groupoids as LTGpGpd.

EXAMPLE 3.12. A local topological group is just a local topological group-groupoid with one object and arrows the elements of the local topological group.

EXAMPLE 3.13. Given any collection of local topological groups $L_1, L_2, ...$ their disjoint union $G = L_1 \bigsqcup L_2 \bigsqcup ...$ is a local topological group-groupoid; here a pair of morphisms of G can only be composed if they come from the same L_n in which case their composition is the product they have there.

EXAMPLE 3.14. If L is a local topological group, then $G = L \times L$ is a local topological group-groupoid.
We know that $L \times L$ is a topological groupoid. Since $\mathrm{Ob}(G) = L$ is a local topological group, we prove that $L \times L$ has a local topological group structure. On the other hand $L \times L$ is a local group-groupoid [17, Example 4.1]. Here a pair (x, y) is a morphism from x to y and the groupoid composite is defined by $(x, y)(z, u) = (x, u)$ whenever $y = z$. The local group multiplication is defined by $(x, y) \circ (z, u) = (x \circ z, y \circ u)$. The maps

$$\mu' = (\mu \times \mu) \colon \mathcal{U} \times \mathcal{U} \to L \times L, ((x, y), (z, u)) \mapsto (x \circ z, y \circ u)$$

and

$$i' = (i \times i) \colon V \times V \mapsto L \times L, (x, y) \mapsto \overline{(x, y)} = (\overline{x}, \overline{y})$$

exist and since μ and i are continuous respectively, then μ' and i' are continuous such that $\mathcal{U}' = \mathcal{U} \times \mathcal{U}$ is open in $G \times G$ and $V' = V \times V$ is open in G. Then $G = L \times L$ becomes a local topological group-groupoid.
□

EXAMPLE 3.15. If L is a local topological group such that the underlying space is locally path connected and semi-locally simply connected, then the fundamental groupoid $\pi_1 L$ is a local topological group-groupoid. Let L be a local topological group such that $\mathcal{U} \subseteq L \times L$ and $e \in V \subseteq L$

are open. We know from [7, Proposition 4.2] that $\pi_1 L$ is a topological groupoid. Since the maps on local topological group structure

$$\mu\colon \mathcal{U} \to L, (x, y) \mapsto x \circ y$$

and

$$i\colon V \mapsto L, x \mapsto \overline{x}$$

are continuous, then the induced maps

$$\pi_1(\mu)\colon \pi_1\mathcal{U} \to \pi_1 L, [(a, b)] \mapsto [a \circ b]$$

and

$$\pi_1(i)\colon \pi_1 V \mapsto \pi_1 L, [a] \mapsto \overline{[a]} = [\overline{a}]$$

are well defined. Note that since (a, b) is defined in $\mathcal{U}$, then $a \circ b$ is defined. So $\pi_1 L$ is a local group-groupoid [17, Proposition 4.2]. Also the induced maps $\pi_1(\mu)$ and $\pi_1(i)$ are continuous such that $\pi_1\mathcal{U}$ and $\pi_1 V$ are open. Therefore $\pi_1 L$ becomes a local topological group-groupoid. □

THEOREM 3.16. *Let X and Y be local topological groups such that the underlying spaces are locally path connected and semi-locally simply connected. Then $\pi_1(X \times Y)$ and $\pi_1 X \times \pi_1 Y$ are isomorphic as local topological groupoids.*

Proof. We know from [1] that the topological groupoids $\pi_1(X \times Y)$ and $\pi_1 X \times \pi_1 Y$ are isomorphic as groupoids. By Theorem 1.3 and the reference [1, 6.4.4] it is possible to see that $\pi_1(X \times Y)$ and $\pi_1 X \times \pi_1 Y$ are homeomorphic. □

In [14, Theorem 3.10], it is proved that if G is a topological group-groupoid such that each star G_x has a universal cover, then the monodromy groupoid $\mathrm{Mon}(G)$ is a group-groupoid. Also Mucuk and Akız, in [19, Theorem 3.13], proved a more general result and developed the monodromy groupoid for an internal groupoid in the category of topological groups with operations, which is defined in [18]. We now give the local group-groupoid structure of monodromy groupoids.

THEOREM 3.17. *Let G be a local topological group-groupoid such that each star G_x has a universal cover. Then the monodromy groupoid* $\mathrm{Mon}(G)$ *is a local group-groupoid.*

Proof. Let G be a local topological group-groupoid as assumed. Considering the local group structures of $\mathrm{Ob}(G)$ and G, we define local group structures of $\mathrm{Ob}(\mathrm{Mon}(G))$ and $\mathrm{Mon}(G)$. Since $\mathrm{Ob}(G) = \mathrm{Ob}(\mathrm{Mon}(G))$, then it is sufficient to prove that Mon(G) satisfies the conditions of local group.

(1) Let e be the identity element of Ob(G) and so 1_e be the identity element of local group G. Then $[a_e]$ is the identity element of Mon(G), which is also identity morphism in Mon(G) from e to e, where a_e is the constant path at 1_e in G_e.

(2) There is a local topological group structure on the set G of morphisms with the local group multiplication

$$\mu\colon \mathcal{U} \to G, (a,b) \mapsto a \circ b$$

defined on a subset $\mathcal{U}$ of $G \times G$. Considering the functor

$$\mathrm{Mon}\colon \mathsf{STGd} \to \mathsf{Gpd},$$

and taking $\mathrm{Mon}(U) = \widetilde{\mathcal{U}}$, we have the following multiplication

$$\widetilde{\mu}\colon \widetilde{\mathcal{U}} \to \mathrm{Mon}(G), ([a],[b]) \mapsto [a] \circ [b] = [a \circ b]$$

defined on the subset $\widetilde{\mathcal{U}} \subseteq \mathrm{Mon}(G) \times \mathrm{Mon}(G)$ such that $(\{[a_e]\} \times \mathrm{Mon}(G)) \cup (\mathrm{Mon}(G) \times \{[a_e]\}) \subseteq \widetilde{\mathcal{U}}$ is well defined.

(3) There is an inversion map

$$i\colon V \to G, a \mapsto \overline{a}$$

defined on a subset $1_e \in V \subseteq G$ are continuous morphisms of local groupoids. By taking $\mathrm{Mon}(V) = \widetilde{V}$, we have the following map

$$i\colon \widetilde{V} \to \mathrm{Mon}(G), [a] \mapsto [\overline{a}]$$

defined on the subset $[a_e] \in \widetilde{V} \subseteq \mathrm{Mon}(G)$ such that $\widetilde{V} \times i(\widetilde{V}) \subseteq \widetilde{\mathcal{U}}$ and $i(\widetilde{V}) \times \widetilde{V} \subseteq \widetilde{\mathcal{U}}$.

In addition to these properties, we have to shove that the following conditions are satisfied:

(i) Identity: $\widetilde{\mu}([a_e],[a]) = [a_e] \circ [a] = \widetilde{\mu}([a],[a_e])$, for all $[a] \in \mathrm{Mon}(G)$.

(ii) Inverse: $\widetilde{\mu}([i(a)],[a]) = \widetilde{\mu}([\overline{a}],[a]) = [\overline{a} \circ a] = [a_e]$ and on the other hand we have $\widetilde{\mu}([a],[i(a)]) = \widetilde{\mu}([a],[\overline{a}]) = [a \circ \overline{a}] = [a_e]$, for all $[a] \in \widetilde{V}$.

(iii) Associativity: If $([a],[b]), ([b],[c]), (\widetilde{\mu}([a],[b]),[c])$ and $([a],\widetilde{\mu}([b],[c]))$ all belong to $\widetilde{\mathcal{U}}$, then by the local group structure of G we have,

$$\begin{aligned}\widetilde{\mu}([a],\widetilde{\mu}([b],[c])) &= \widetilde{\mu}([a],[b\circ c]) \\ &= [a]\circ([b]\circ[c]).\end{aligned}$$

On the other hand

$$\begin{aligned}\widetilde{\mu}(\widetilde{\mu}([a],[b]),[c]) &= \widetilde{\mu}([a\circ b],[c]) \\ &= ([a\circ b])\circ[c].\end{aligned}$$

Then $[a]\circ([b]\circ[c]) = ([a\circ b])\circ[c]$

Now we have to prove that the morphisms $\widetilde{\mu}$ and i are morphisms of local groups. For the morphism $\widetilde{\mu}\colon \widetilde{\mathcal{U}} \to \mathrm{Mon}(G), ([a],[b]) \mapsto [a\circ b]$, since

$$\widetilde{\mu}(([a],[b])([c],[d])) = \widetilde{\mu}([ac],[bd]) = [ac\circ bd]$$

and

$$\widetilde{\mu}([a],[b])\widetilde{\mu}([c],[d]) = [a\circ b][c\circ d] = [ac\circ bd],$$

then we have

$$\widetilde{\mu}(([a],[b])([c],[d])) = \widetilde{\mu}([a],[b])\widetilde{\mu}([c],[d]).$$

So $\widetilde{\mu}$ is a local morphism.

For the morphism $i\colon \widetilde{V} \to G, a \mapsto \overline{a}$, since

$$i([a][b]) = i([ab]) = \overline{[ab]} = [\overline{a}\overline{b}] = [\overline{a}][\overline{b}]$$

and

$$i([a])i([b]) = [\overline{a}][\overline{b}],$$

we have

$$i([a][b]) = i([a])i([b]).$$

So i is a local morphism.

We now prove that the interchange law

$$[a\circ c]\bullet[b\circ d] = [a\bullet b]\circ[c\bullet d]$$

in $\mathrm{Mon}(G)$ is satisfied when $\bullet$ denotes the groupoid composition in $\mathrm{Mon}(G)$ and $a\bullet b$, $c\bullet d$ are defined and (a,c) and (c,d) are in $\mathcal{U}$. If these $a \bullet b$ and $b \bullet c$ are defined, then we have the following:

$$(a \bullet b)(t) = \begin{cases} a(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ a(1)b(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$(c \bullet d)(t) = \begin{cases} c(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ c(1)d(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$(a \bullet b) \circ (c \bullet d)(t) = \begin{cases} (a \circ c)(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ (a(1)b(2t-1)) \circ (c(1)d(2t-1)), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Hence

$$(a \bullet b) \circ (c \bullet d) = (a \circ c) \star ((a(1)b) \circ (c(1)d))).$$

On the other hand

$$(a \circ c) \bullet (b \circ d)(t) = \begin{cases} (a \circ c)(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ (a \circ c)(1)(b \circ d)(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and hence

$$(a \circ c) \bullet (b \circ d) = (a \circ c) \star (a \circ c)(1)(b \circ d).$$

By the interchange law in G, $(a \circ c)(1)(b \circ d) = (a(1)b) \circ (c(1)d))$ and so we have that

$$(a \circ c) \bullet (b \circ d) = (a \bullet b) \circ (c \bullet d).$$

which insures the interchange law in $\mathrm{Mon}(G)$.

For $a, b \in \mathrm{Mon}(G)$, where $a \bullet b$ is defined, we have the followings:

$$\overline{(a \bullet b)(t)} = \begin{cases} \overline{a(2t)}, & 0 \leqslant t \leqslant \frac{1}{2} \\ \overline{(a(1)b(2t-1))}, & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and

$$\overline{a(t)} \bullet \overline{b(t)} = \begin{cases} \overline{a(2t)}, & 0 \leqslant t \leqslant \frac{1}{2} \\ \overline{a(1)} \quad \overline{b(2t-1)}, & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

Since $\overline{(a(1)b(2t-1))} = \overline{a(1)} \quad \overline{b(2t-1)}$ in G we have that $\overline{a \bullet b} = \overline{a} \bullet \overline{b}$.

All these details complete the proof that $\mathrm{Mon}(G)$ is a local group-groupoid. □

We can now restate [14, Theorem 2.1] for local topological group-groupoids as follows:

THEOREM 3.18. *For local topological group-groupoids G and H whose stars have universal covers, the monodromy groupoids* $\mathrm{Mon}(G \times H)$ *and* $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ *are isomorphic as local group-groupoids.*

Proof. We know from the proof of [14, Theorem 2.1] that

$$f\colon\ \mathrm{Mon}(G \times H) \longrightarrow \mathrm{Mon}(G) \times \mathrm{Mon}(H), f([a]) = ([p_1 a], [p_2 a])$$

is an isomorphism of groupoids. So it is sufficient to prove that f is a local morphism.

For $[a], [b] \in \mathrm{Mon}(G)$,

$$\begin{aligned} f([a] \circ [b]) &= ([p_1(a \circ b)], [p_2(a \circ b)]) \\ &= ([p_1 a \circ p_1 b)], [p_2 a \circ p_2 b)]) \\ &= ([p_1 a], [p_2 a)]) \circ ([p_1 b], [p_2 b]) \\ &= f([a]) \circ f([b]). \end{aligned}$$

Also for $[a] \in \mathrm{Mon}(G)$,

$$\begin{aligned} f(i[a]) &= f[i(a)] \\ &= ([p_1(i(a))], [p_2(i(a))]) \\ &= ([i(p_1 a)], [i(p_2 a)]) \\ &= i([p_1 a], [p_2 a]) \\ &= i(f[a]). \end{aligned}$$

Then f becomes a local morphism. □

As a result of Theorem 3.17 we can state that we have a functor $\mathrm{Mon}\colon \mathsf{LTGpGpd} \to \mathsf{LGpGpd}$.

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