

SOLVABILITY OF SYLVESTER OPERATOR EQUATION WITH BOUNDED SUBNORMAL OPERATORS IN HILBERT SPACES

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ABSTRACT. The aim of this paper is to present some necessary and sufficient conditions for existence of solution of Sylvester operator equation involving bounded subnormal operators in a Hilbert space. Our results improve and generalize some results in the literature involving normal operators.

1. Introduction

In recent years, many problems in control theory, optimization, dynamical systems and quantum mechanics require the use of Sylvester matrix equation $AX - XB = C$ or its generalization $AX - YB = C$. In fact, Sylvester equation has the improved technique to give necessary and sufficient conditions for the existence of solution.

Roth [13] proved that, if A and B are finite matrices over a field, then Sylvester equation has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and

Received February 17, 2019. Revised May 27, 2019. Accepted June 13, 2019.

2010 Mathematics Subject Classification: 47A62, 47A52, 15A24.

Key words and phrases: Sylvester equation, subnormal operators, similar operators, Fuglede-Putnam property.

This work was supported by Laboratory of operator theory, Algeria.

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$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar. Later Rosenblum [11] showed that the result remains true even A and B are bounded self adjoint operators. On other hand Bhatia [1] gave the solution based on chain convergence. Lancaster et al. [8] proved that the Lyapunov equation $AX + XB = C$ has a unique solution if and only if A and B have no common eigenvalues. In [14] the author generalized the results of Roth for normal operators for the infinite case. Recently, Mecheri and Mansour [10] replaced the condition of normality of the operators A and B by the normality of one operator A , assuming the pair (B, A) to satisfy Fuglede-Putnam property.

In this paper, we resolve the Sylvester equation $AX - XB = C$ assuming A and B to be subnormal operators. Further, we give some consequences for the case of rank one operators, which have applications in physics and dynamical systems.

2. Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H .

DEFINITION 2.1. [3] Let S be an operator in $B(H)$. S is called normal if and only if it commutes with its adjoint, i.e., $SS^* = S^*S$.

DEFINITION 2.2. [3] Let S be an operator in $B(H)$. S is called subnormal if there exists a Hilbert space K , on which S admits an extension N_S such that:

1. $H \subset K$.
2. N_S is normal on K .
3. $N_S|_H = S$.

In general, if S is subnormal, one can take $K = H \oplus H^\perp$, so N_S is given by $N_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix}$, where $Q : H^\perp \rightarrow H$ and $T : H^\perp \rightarrow H^\perp$.

EXAMPLE 2.1. Let S be an operator in $B(H)$ and let $S = M(S^*S)^{\frac{1}{2}}$ be its polar decomposition, where M is an isometry, i.e., $M^*M = I$. If M commutes with $(S^*S)^{\frac{1}{2}}$, then S is subnormal. In fact, if $P = MM^*$, then P is the projection onto the final space of

M , thus $(I - P)M = M^*(I - P) = 0$.

We define L and T by

$$L = \begin{pmatrix} M & I - P \\ 0 & M^* \end{pmatrix}, \quad T = \begin{pmatrix} (S^*S)^{\frac{1}{2}} & 0 \\ 0 & (S^*S)^{\frac{1}{2}} \end{pmatrix}.$$

Since $S = M(S^*S)^{\frac{1}{2}} = (S^*S)^{\frac{1}{2}}M$ we get

$$MS = MM(S^*S)^{\frac{1}{2}} = M(S^*S)^{\frac{1}{2}}M = SM$$

and

$$M^*(S^*S)^{\frac{1}{2}} = (S^*S)^{\frac{1}{2}}M^*.$$

Then L is unitary, $LT = TL$ and T positive.

Let $N = LT = \begin{pmatrix} S & (I - P)(S^*S)^{\frac{1}{2}} \\ 0 & M^*(S^*S)^{\frac{1}{2}} \end{pmatrix}$. Clearly N is normal and it is an extension of S . Consequently, S is a subnormal operator.

LEMMA 2.1. [3] *Every normal operator is subnormal, but the converse does not hold in general.*

EXAMPLE 2.2. Let $H = \ell^2$. The shift operator $T \in B(H)$ is given by

$$T(x_1, x_2, \dots, x_n, \dots) = (0, x_1, x_2, \dots, x_n, \dots).$$

Then T is a subnormal operator, but it is not normal since $T^*T \neq TT^*$.

DEFINITION 2.3. Let S and T be two operators in $B(H)$. The pair (S, T) is said to satisfy Fuglede-Putnam property if for any operator $Q \in B(H)$ such that $SQ = QT$, $S^*Q = QT^*$.

THEOREM 2.2. [11] *If S and T are two normal operators in $B(H)$ and $Q \in B(H)$ such that $SQ = QT$, then $S^*Q = QT^*$.*

LEMMA 2.3. [15] *If S is a subnormal operator on a Hilbert space H , then $\alpha S + \beta S^*$ is subnormal, where α, β are complex numbers.*

THEOREM 2.4. [5] *If S and T^* are subnormal and Q is an operator such that $SQ = QT$, then $S^*Q = QT^*$.*

By combining Lemma 2.3 and Theorem 2.4, we get the following proposition.

PROPOSITION 2.1. *If S and T are subnormal operators in $B(H)$ and Q is in $B(H)$ such that $SQ = QT$, then $S^*Q = QT^*$.*

Proof. Since T is subnormal, from Lemma 2.3 its adjoint T^* is subnormal, too. Applying Theorem 2.4 we obtain the required result. \square

DEFINITION 2.4. [1] Let A, B and C be bounded operators in a Hilbert space H . Two operators $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are said to be similar if and only if there exists an invertible operator $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ in $B(H \oplus H)$ such that

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} Q & R \\ S & T \end{pmatrix}.$$

THEOREM 2.5. [10] Let A be a normal operator in $B(H)$ and let B and C be two bounded operators, such that the pair (B, A) satisfies Fuglede-Putnam property. Then the equation $AX - XB = C$ has a solution in $B(H)$ if and only if the two operators $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

LEMMA 2.6. [14] Let Q, R, S and T be operators in $B(H)$. If $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ is invertible, then $S^*S + Q^*Q$ is invertible.

DEFINITION 2.5. An operator $S \in B(H)$ is called a rank one operator if its range is one dimensional.

LEMMA 2.7. An operator $S \in B(H)$ is a rank one operator if and only if it can be written in the form of tensorial product of two vectors in H , i.e., there exist u and v in H such that $S = u \otimes v$.

Proof. Since S is rank one operator, its range is one dimensional, i.e., its range is generated by a single element $u \in H$. Hence for all $x \in H$, there exists $\lambda_x \in \mathbb{C}$ such that;

$$Sx = \lambda_x u.$$

The mapping $x \mapsto \lambda_x$ is continuous linear form, so by Riez decomposition theorem, there exists $v \in H$ such that

$$\lambda_x = \langle x, v \rangle, \quad x \in H.$$

Thus

$$Sx = \lambda_x u = \langle x, v \rangle u = (u \otimes v)x, \quad x \in H.$$

Consequently, $S = u \otimes v$. The converse is evident. \square

3. Main results

Let A, B and C be operators in $B(H)$. Consider the equation:

$$(3.1) \quad AX - XB = C.$$

THEOREM 3.1. *Let A be a subnormal operator and B, C in $B(H)$. Assume that the pair (B, A) satisfies Fuglede-Putnam property. Then the equation (3.1) has a solution in $B(H)$ if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.*

Proof. If X is a solution of (3.1), then we have

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AX - XB \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

which implies that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

Conversely, if the two operators $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar, then there exists an invertible operator $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ such that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \begin{pmatrix} Q & R \\ S & T \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

which gives

$$\begin{pmatrix} AQ & AR \\ BS & BT \end{pmatrix} = \begin{pmatrix} QA & QC + RB \\ SA & SC + TB \end{pmatrix}.$$

Hence we get $AQ = QA$.

Since A is subnormal, by Proposition 2.1 we get

$$A^*Q = QA^*.$$

We also have

$$\begin{aligned} AR - RB &= QC, \\ BS &= SA \end{aligned}$$

and

$$BT - TB = SC.$$

Since the pair (B, A) satisfies Fuglede-Putnam property and $SA = BS$, $B^*S = SA^*$.

Since A commutes with Q and Q^* , it commutes with Q^*Q .
On other hand taking the adjoint in

$$B^*S = SA^*,$$

we get

$$S^*B = AS^*.$$

Since $BS = SA$,

$$S^*BS = S^*SA,$$

but $S^*B = AS^*$, so

$$AS^*S = S^*SA,$$

which implies that A commutes with S^*S , so it commutes with the sum $S^*S + Q^*Q$.

Then we have

$$\begin{aligned} (S^*S + Q^*Q)C &= S^*SC + Q^*QC \\ &= S^*(BT - TB) + Q^*(AR - RB) \\ &= Q^*(AR - RB) + S^*(BT - TB) \\ &= Q^*AR - Q^*RB + S^*BT - S^*TB. \end{aligned}$$

Since $A^*Q = QA^*$, passing to the adjoint we get $Q^*A = AQ^*$. Further, since $S^*B = AS^*$, we get

$$\begin{aligned} (S^*S + Q^*Q)C &= AQ^*R - Q^*RB + AS^*T - S^*TB \\ &= A(Q^*R + S^*T) - (Q^*R + S^*T)B. \end{aligned}$$

Since $S^*S + Q^*Q$ is invertible (from Lemma 2.6), A commutes with $(S^*S + Q^*Q)^{-1}$, because it commutes with $S^*S + Q^*Q$. Then

$$C = A(S^*S + Q^*Q)^{-1}(Q^*R + S^*T) - (S^*S + Q^*Q)^{-1}(Q^*R + S^*T)B.$$

This yields that the solution of equation (3.1) is given by

$$X = (S^*S + Q^*Q)^{-1}(Q^*R + S^*T).$$

□

COROLLARY 3.1. *Let A and B be bounded subnormal operators on a complex Hilbert space H and $C \in B(H)$. Then the equation (3.1) has a solution in $B(H)$ if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is similar to $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.*

Proof. This corollary follows by Proposition 2.1 and Theorem 3.1. □

COROLLARY 3.2. *Let B be a bounded subnormal operator on a complex Hilbert space H , let $A, C \in B(H)$ and suppose that the pair (A^*, B^*) satisfies Fuglede-Putnam property. Then the equation $B^*X - XA^* = C$ has a solution in $B(H)$ if and only if $\begin{pmatrix} B^* & C \\ 0 & A^* \end{pmatrix}$ is similar to $\begin{pmatrix} B^* & 0 \\ 0 & A^* \end{pmatrix}$.*

Proof. It follows immediately from Lemma 2.3 and Theorem 3.1. \square

COROLLARY 3.3. *Let A be a bounded subnormal operator on a complex Hilbert space H and $B, C \in B(H)$. Assume that A and B are invertible and the pair (B, A) satisfies Fuglede-Putnam property. Then the equation $A^{-1}X - XB^{-1} = C$ has a solution in $B(H)$ if and only if $\begin{pmatrix} A^{-1} & C \\ 0 & B^{-1} \end{pmatrix}$ is similar to $\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.*

Proof. If X is a solution of the equation $A^{-1}X - XB^{-1} = C$, then

$$\begin{aligned} & \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A^{-1} & A^{-1}X - XB^{-1} \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} A^{-1} & C \\ 0 & B^{-1} \end{pmatrix}. \end{aligned}$$

Hence $\begin{pmatrix} A^{-1} & C \\ 0 & B^{-1} \end{pmatrix}$ and $\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$ are similar.

If $\begin{pmatrix} A^{-1} & C \\ 0 & B^{-1} \end{pmatrix}$ is similar to $\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$, then their inverses are similar, i.e., $\begin{pmatrix} A & -ACB \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are similar. Hence by Theorem 3.1 the equation $AX - XB = -ACB$ has a solution X . Since A and B are invertible, we obtain

$$AXB^{-1} - X = -AC.$$

Thus we get

$$XB^{-1} - A^{-1}X = -C,$$

which implies

$$A^{-1}X - XB^{-1} = C.$$

\square

If $A = B$, we get the following result.

THEOREM 3.2. *Let A be a subnormal operator in $B(H)$ and let C be an operator in $B(H)$. Then the matrices*

$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ are similar if and only if C is in the range of the derivation δ_A ($\delta_A(X) = AX - XA$).

Proof. It suffices to apply Theorem 2.4 with replacing B by A . \square

If C is a rank one operator, then from Lemma 2.7 it can be written as $C = a \otimes b$, where a and b are two vectors in H . We have the following result.

THEOREM 3.3. *Let A be a bounded subnormal operator and let B be an operator in $B(H)$ such that the pair (B, A) satisfies Fuglede-Putnam property. Then the equation $AX - XB = a \otimes b$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is similar to $\begin{pmatrix} A & a \otimes b \\ 0 & B \end{pmatrix}$.*

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