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# $L$-FUZZY BI-CLOSURE SYSTEMS AND $L$-FUZZY BI-CLOSURE OPERATORS

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Abstract. In this paper, we introduced the notions of right and left closure systems on generalized residuated lattices. In particular, we study the relations between right (left) closure (interior) operators and right (left) closure (interior) systems. We give their examples.

## 1. Introduction

The notion of closure systems and closure operators facilitated to study topological structures ,logic and lattices. Gerla [5-7] introduced closure systems and closure operators in the unit interval [0,1].

Ward et al.[16] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1-4, 11-14]. Recently, Bělohlávek [1-4] investigate the properties of fuzzy relations and fuzzy closure systems on a residuated lattice which supports part of foundation of theoretic computer science. As an Bělohlávek's extension, Fang and Yue [8] introduced strong fuzzy closure systems and strong fuzzy

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closure operators. On the other hand, Georgescu and Popescue [9,10] introduced non-commutative fuzzy Galois connection in a generalized residuated lattice which is induced by two implications.

In this paper, we introduced the notions of right and left closure (resp. interior) systems in a sense as the right and left least upper (resp. greatest lower) bound on a generalized residuated lattice. In particular, we investigated the relations between right (left) closure (interior) operators and and right (left) closure (interior) systems. We give their examples.

## 2. Preliminaries

DEFINITION 2.1. [9,10,17] A structure $(L, \vee, \wedge, \odot, \to, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded lattice where $\top$ is the upper bound and $\top$ denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$(a \odot b) \leq c \text{ iff } a \leq (b \to c) \text{ iff } b \leq (a \Rightarrow c).$$

REMARK 2.2. [9,10,15,17]

(1) A generalized residuated lattice is a residuated lattice ($\to=\Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) Let $(L, \leq, \odot)$ be a quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee\{z \in L \mid z \odot x \leq y\},$$

$$x \Rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$(x \odot y) \leq z$ iff $x \leq (y \to z)$ iff $y \leq (x \Rightarrow z)$. Hence $(L, \vee, \wedge, \odot, \to, \Rightarrow, \bot, \top)$ is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \top, \bot)$ is a complete generalized residuated lattice with the law of double negation defined as $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \bot$ and $a^* = a \Rightarrow \bot$.

LEMMA 2.3 [9,10,17] *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $(x \to y) \leq (x \to z)$ and $(z \to x) \leq (y \to x)$ for $\to \in \{\rightarrow, \Rightarrow\}$.*

(2) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$ *for $\to \in \{\rightarrow, \Rightarrow\}$.*

(3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ *and* $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(4) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ *and* $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

(5) $x \odot (x \Rightarrow y) \leq y$ *and* $(x \rightarrow y) \odot x \leq y$.

(6) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ *and* $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$.

(7) $(x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z)$ *and* $(x \rightarrow z) \leq (x \odot y) \rightarrow (z \odot y)$.

(8) $x \rightarrow y \leq (y \rightarrow z) \Rightarrow (x \rightarrow z)$ *and* $(x \Rightarrow y) \leq (y \Rightarrow z) \rightarrow (x \Rightarrow z)$.

(9) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$ *and* $(y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$.

(10) $x \rightarrow y = \top$ *iff* $x \leq y$.

(11) $x \rightarrow y = y^0 \Rightarrow x^0$ *and* $x \Rightarrow y = y^* \rightarrow x^*$.

(12) $(x \rightarrow y)^* = x \odot y^*$ *and* $(x \Rightarrow y)^0 = y^0 \odot x$.

(13) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(14) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ *and* $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

(15) $\bigwedge_{i\in\Gamma} x_i \to (\bigwedge_{i\in\Gamma} y_i) \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigvee_{i\in\Gamma} x_i \to (\bigvee_{i\in\Gamma} y_i) \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *for $\to \in \{\rightarrow, \Rightarrow\}$.*

DEFINITION 2.4.Let $X$ be a set. A function $e^r_X : X \times X \to L$ is called a *right partial order* if it satisfies the following conditions :

(O1) $e^r_X(x, x) = \top$ for all $x \in X$,

(O2) If $e^r_X(x, y) = e^r_X(y, x) = \top$, then $x = y$,

(R) $e^r_X(x, y) \odot e^r_X(y, z) \leq e^r_X(x, z)$, for all $x, y, z \in X$.

A function $e^l_X : X \times X \to L$ is called a *left partial order* if it satisfies (O1), (O2) and

(L) $e^l_X(y, z) \odot e^l_X(x, y) \leq e^l_X(x, z)$, for all $x, y, z \in X$.

The triple $(X, e^r_X, e^l_X)$ is a bi-partial ordered set.

EXAMPLE 2.5.

(1) We define a function $e^r_L, e^l_L : L \times L \to L$ as

$$e^r_L(x, y) = (x \Rightarrow y),\ e^l_L(x, y) = (x \rightarrow y).$$

By Lemma 2.3 (6), $(L, e^r_L, e^l_L)$ is a bi-partial ordered set.

(2) We define a function $e^r_{L^X}, e^l_{L^X} : L^X \times L^X \to L$ as

$$e^r_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \Rightarrow B(x)),$$

$$e^l_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \to B(x)).$$

By Lemma 2.3 (6), $(L^X, e^r_{L^X}, e^l_{L^X})$ is a bi-partial ordered set.

## 3. $L$-fuzzy bi-closure systems and $L$-fuzzy bi-closure operators

DEFINITION 3.1. A map $S^r : L^X \to L$ is called an *$L$-fuzzy right closure system* if

(S1) $S^r(\top_X) = \top$,

(S2) $S^r(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} S^r_X(A_i)$, for all $A_i \in L^X$,

(RS) $S^r(\alpha \Rightarrow A) \geq S^r_X(A)$, for all $A \in L^X$ and $\alpha \in L$.

A map $S^l : L^X \to L$ is called an *$L$-fuzzy left closure system* if it satisfies (S1), (S2) and

(LS) $S^l(\alpha \to A) \geq S^l_X(A)$, for all $A \in L^X$ and $\alpha \in L$.

The triple $(X, S^r, S^l)$ is called an $L$-fuzzy bi-closure system. A map $f : (X, S^r_X, S^l_X) \to (Y, S^r_Y, S^l_Y)$ is called bi-continuous if $S^r_X(f^{\leftarrow}(B)) \geq S^r_Y(B)$ and $S^l_X(f^{\leftarrow}(B)) \geq S^l_Y(B)$ for each $B \in L^Y$.

DEFINITION 3.2. Let $(X, e^r_{L^X}, e^l_{L^X})$ be a bi-partial ordered set. An operator $C^r : L^X \to L^X$ is called an *$L$-fuzzy right closure operator* on $X$ if it satisfies the following conditions:

(CR1) $e^r_{L^X}(A, C^r(A)) = \top$, for all $A \in L^X$.

(CR2) $e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r(A), C^r(B))$ for all $A, B \in L^X$.

(CR3) $\alpha \odot e^r_{L^X}(C^r(A)) \leq C^r(\alpha \odot A)$ for all $A \in L^X$ and $\alpha \in L$..

An operator $C^l : L^X \to L^X$ is called an *$L$-fuzzy left closure operator* on $X$ if it satisfies the conditions

(CL1) $e^l_{L^X}(A, C^l(A)) = \top$ for all $A \in L^X$,

(CL2) $e^l_{L^X}(A,B) \leq e^l_{L^X}(C^l(A), C^l(B))$ for all $A, B \in L^X$.

(CR3) $e^l_{L^X}(C^l(A)) \odot \alpha \leq C^l(A \odot \alpha)$ for all $A \in L^X$ and $\alpha \in L$.

The triple $(X, C^r, C^l)$ is called an *$L$-fuzzy bi-closure space.* A map $f : (X, C^r_X, C^l_X) \to (Y, C^r_Y, C^l_Y)$ is called an $L$-fuzzy bi-closed map if

$C^r_X(f^{\leftarrow}(B)) \leq f^{\leftarrow}(C^r_Y(B))$ and $C^l_X(f^{\leftarrow}(B)) \leq f^{\leftarrow}(C^l_Y(B))$ for each $B \in L^Y$.

THEOREM 3.3. *Let $(L^X, e^r_{L^X}, e^l_{L^X})$ be a bi-partial ordered set and $(X, S^r, S^l)$ be an $L$-fuzzy bi-closure system. Define two maps $C^r_{S^r}, C^l_{S^l} : L^X \to L^X$ as follows:*

$$C^r_{S^r}(A) = \bigwedge_{B\in L^X} (e^r_{L^X}(A,B) \odot S^r(B) \to B),$$

$$C^l_{S^l}(A) = \bigwedge_{B\in L^X} (S^l(B) \odot e^l_{L^X}(A,B) \Rightarrow B).$$

*Then $(X, C^r_{S^r}, C^l_{S^l})$ is an $L$-fuzzy bi-closure space.*

*Proof.* (CR1) For each $A \in L^X$, by Lemma 2.3(2,4),

$$\begin{aligned}
&e^r_{L^X}(A, C^r_{S^r}(A))\\
&= \textstyle\bigwedge_{x\in X}(A(x) \Rightarrow (\bigwedge_{B\in L^X}(e^r_{L^X}(A,B) \odot S^r(B) \to B))\\
&= \textstyle\bigwedge_{x\in X}\bigwedge_{B\in L^X}(A(x) \Rightarrow ((e^r_{L^X}(A,B) \odot S^r(B) \to B)))\\
&= \textstyle\bigwedge_{x\in X}\bigwedge_{B\in L^X}((e^r_{L^X}(A,B) \odot S^r(B) \to (A(x) \Rightarrow B)))\\
&= \textstyle\bigwedge_{B\in L^X}((e^r_{L^X}(A,B) \odot S^r(B) \to \bigwedge_{x\in X}(A(x) \Rightarrow B(x)))) = \top.
\end{aligned}$$

(CR2) We will show that $e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r_{S^r}(A), C^r_{S^r}(B))$.

$$\begin{aligned}
&C^r_{S^r}(A) \odot e^r_{L^X}(A,B) \odot e^r_{L^X}(B,D) \odot S^r(D)\\
&\leq \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(A,B) \odot S^r(B) \to B) \odot e^r_{L^X}(A,D) \odot S^r(D)\\
&\leq D \text{ (by Lemma 2.3(5))}.
\end{aligned}$$

Then $C^r_{S^r}(A) \odot e^r_{L^X}(A,B) \leq C^r_{S^r}(B)$. Hence $e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r_{S^r}(A), C^r_{S^r}(B))$.

(CR3) For each $A, B \in L^X$, since $S^r(\alpha \Rightarrow B) \geq S^r(B)$,

$$\begin{aligned}
&C^r_{S^r}(A) = \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(A,B) \odot S^r(B) \to B)\\
&= \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(A, \alpha \Rightarrow B) \odot S^r(\alpha \Rightarrow B) \to (\alpha \Rightarrow B))\\
&= \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(\alpha \odot A, B) \odot S^r(\alpha \Rightarrow B) \to (\alpha \Rightarrow B))\\
&= \alpha \Rightarrow \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(\alpha \odot A, B) \odot S^r(\alpha \Rightarrow B) \to B)\\
&\leq \alpha \Rightarrow \textstyle\bigwedge_{B\in L^X}(e^r_{L^X}(\alpha \odot A, B) \odot S^r(B) \to B)\\
&= \alpha \Rightarrow C^r_{S^r}(\alpha \odot A)
\end{aligned}$$

Hence $\alpha \odot C^r_{S^r}(A) \leq C^r_{S^r}(\alpha \odot A)$. Thus $C^r_{S^r}$ is an $L$-fuzzy right closure operator. Similarly, $C^l_{S^l}$ is an $L$-fuzzy left closure operator.

□

THEOREM 3.4. *Let $(L^X, e^r_{L^X}, e^l_{L^X})$ be a bi-partial ordered set and $(X, C^r, C^l)$ be an L-fuzzy bi-closure operator. Define two maps $S^r_{C^r}, S^l_{C^l} : L^X \to L$ as follows:*

$$S^r_{C^r}(A) = e^r_{L^X}(C^r(A), A),$$
$$S^l_{C^l}(A) = e^l_{L^X}(C^l(A), A).$$

*(1) $(X, S^r_{C^r}, S^l_{C^l})$ is an L-fuzzy bi-closure system such that $C^r \leq C^r_{S^r_{C^r}}$ and $C^l \leq C^l_{S^l_{C^l}}$.*

*(2) If $(X, S^r, S^l)$ is an L-fuzzy bi-closure system, then $S^r \leq S^r_{C^r_{S^r}}$ and $S^l \leq S^l_{C^l_{S^l}}$.*

*Proof.* (1) (S1) $S^r_{C^r}(\top_X) = e^r_{L^X}(C^r(\top_X), \top_X) = \top$.

(S2) For all $A_i \in L^X$, by Lemma 2.3(15),

$$\begin{array}{l} S^r_{C^r}(\bigwedge_{i\in\Gamma} A_i) = e^r_{L^X}(C^r(\bigwedge_{i\in\Gamma} A_i), \bigwedge_{i\in\Gamma} A_i) \\ \geq e^r_{L^X}(\bigwedge_{i\in\Gamma} C^r(A_i), \bigwedge_{i\in\Gamma} A_i) \\ \geq \bigwedge_{i\in\Gamma} e^r_{L^X}(C^r(A_i), A_i) \\ = \bigwedge_{i\in\Gamma} S^r_{C^r}(A_i) \end{array}$$

(RS) For all $A \in L^X$ and $\alpha \in L$, by Lemma 2.3(3),

$$\begin{array}{l} S^r_{C^r}(\alpha \Rightarrow A) = e^r_{L^X}(C^r(\alpha \Rightarrow A), \alpha \Rightarrow A) \\ = e^r_{L^X}(\alpha \odot C^r(\alpha \Rightarrow A), A) \\ \geq e^r_{L^X}(C^r(\alpha \odot (\alpha \Rightarrow A)), A) \\ \geq e^r_{L^X}(C^r(A), A) = S^r_{C^r}(A). \end{array}$$

Hence $S^r_{C^r}$ is an $L$-fuzzy right closure system. Moreover, for all $A \in L^X$,

$$\begin{array}{l} C^r_{S^r_{C^r}}(A) = \bigwedge_{B\in L^X}(e^r_{L^X}(A, B) \odot S^r_{C^r}(B) \to B) \\ = \bigwedge_{B\in L^X}(e^r_{L^X}(A, B) \odot e^r_{L^X}(C^r(B), B) \to B) \\ \geq \bigwedge_{B\in L^X}(e^r_{L^X}(C^r(A), C^r(B)) \odot e^r_{L^X}(C^r(B), B) \to B) \\ \geq \bigwedge_{B\in L^X}(e^r_{L^X}(C^r(A), B) \to B) \\ \geq C^r(A). \end{array}$$

Similarly, $S^l_{C^l}$ is an $L$-fuzzy left closure system.

(2) Since $(a \to b) \odot a \leq b$ iff $a \leq (a \to b) \Rightarrow$,

$$\begin{array}{l} S^r_{C^r_{S^r}}(A) = e^r_{L^X}(C^r_{S^r}(A), A) \\ = e^r_{L^X}(\bigwedge_{B\in L^X}(e^r_{L^X}(A, B) \odot S^r(B) \to B), A) \\ \geq e^r_{L^X}((e^r_{L^X}(A, A) \odot S^r(A) \to A), A) \\ \geq S^r(A). \end{array}$$

Similarly, $S^l \leq S^l_{C^l_{S^l}}$. □

THEOREM 3.5. (1) *If* $f : (X, C^r_X, C^l_X) \to (Y, C^r_Y, C^l_Y)$ *is an* $L$*-fuzzy bi-closure map, then* $f : (X, S^r_{C^r_X}, S^l_{C^l_X}) \to (Y, S^r_{C^r_Y}, S^l_{C^l_Y})$ *is a bi-continuous map.*

(2) *If* $f : (X, S^r_X, S^l_X) \to (Y, S^r_Y, S^l_Y)$ *is a bi-continuous map, then* $f : (X, C^r_{S^r_X}, C^l_{S^l_X}) \to (Y, C^r_{S^r_Y}, C^l_{S^l_Y})$ *is an* $L$*-fuzzy bi-closure map.*

*Proof.* (1) Since $C^r_X(f^\leftarrow(B)) \leq f^\leftarrow(C^r_Y(B))$,

$$\begin{aligned}
&S^r_{C^r_X}(f^\leftarrow(B)) = e^r_{L^X}(C^r(f^\leftarrow(B)), f^\leftarrow(B))\\
&\geq e^r_{L^X}(f^\leftarrow(C^r_Y(B)), f^\leftarrow(B))\\
&\geq \textstyle\bigwedge_{x\in X}(C^r_Y(B)(f(x)) \Rightarrow B(f(x)))\\
&\geq \textstyle\bigwedge_{y\in Y}(C^r_Y(B)(y) \Rightarrow B(y))\\
&= S^r_{C^r_Y}(B).
\end{aligned}$$

Similarly, $S^l_{C^l_X}(f^\leftarrow(B)) \geq S^l_{C^l_Y}(B)$ for all $B \in L^Y$.

(2) Since $S^r_X(f^\leftarrow(D)) \geq S^r_Y(D)$ for all $D \in L^Y$,

$$\begin{aligned}
&f^\leftarrow(C^r_{S^r_Y}(B)) = f^\leftarrow(\textstyle\bigwedge_{D\in L^Y}(e^r_{L^Y}(B, D) \odot S^r_Y(D) \to D))\\
&= \textstyle\bigwedge_{D\in L^Y}(e^r_{L^Y}(B, D) \odot S^r_Y(D) \to f^\leftarrow(D))\\
&\geq \textstyle\bigwedge_{D\in L^Y}(e^r_{L^X}(f^\leftarrow(B), f^\leftarrow(D)) \odot S^r_X(f^\leftarrow(D)) \to f^\leftarrow(D))\\
&\geq \textstyle\bigwedge_{E\in L^X}(e^r_{L^X}(f^\leftarrow(B), E) \odot S^r_X(E) \to E)\\
&= C^r_{S^r_X}(f^\leftarrow(B)).
\end{aligned}$$

Similarly, $f^\leftarrow(C^l_{S^l_Y}(B)) \geq C^l_{S^l_X}(f^\leftarrow(B))$ for all $B \in L^Y$. □

DEFINITION 3.6. [1] Suppose that $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ are concrete functors. The pair $(F, G)$ is called a *Galois correspondence* between $\mathcal{C}$ and $\mathcal{D}$ if for each $Y \in \mathcal{C}$, $id_Y : F \circ G(Y) \to Y$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}$, $id_X : X \to G \circ F(X)$ is a $\mathcal{D}$-morphism.

If $(F, G)$ is a Galois correspondence, then it is easy to check that $F$ is a left adjoint of $G$, or equivalently that $G$ is a right adjoint of $F$.

Let **BFC** be denote the category of $L$-fuzzy bi-closure spaces and bi-closure mappings for morphisms.

Let **BCS** be denote the category of $L$-fuzzy bi-closure systems and continuous mappings for morphisms.

THEOREM 3.7. (1) $\mathbf{F}:\mathbf{BFC}\to\mathbf{BCS}$ *defined as* $\mathbf{F}(X,C^r_X,C^l_X)=(X,S^r_{C^r_X},S^l_{C^l_X})$ *is a functor.*

(2) $\mathbf{G}:\mathbf{BCS}\to\mathbf{BFC}$ *defined as* $\mathbf{G}(X,S^r_X,S^l_X)=(X,C^r_{S^r_X},C^l_{S^l_X})$ *is a functor.*

(3) *The pair* $(\mathbf{F},\mathbf{G})$ *is a Galois correspondence between* $\mathbf{BFC}$ *and* $\mathbf{BCS}$.

*Proof.* (1) and (2) follows from Theorem 3.5.

(3) By Theorem 3.4(2), if $(X,S^r_X,S^l_X)$ is an $L$-fuzzy bi-closure system, then $\mathbf{F}(\mathbf{G}(X,S^r_X,S^l_X))=(X,S_{C^r_{S^r_X}},S^l_{C^l_{S^l_X}})\geq(X,S^r_X,S^l_X)$. Hence, the identity map $id_X:(X,S_{C^r_{S^r_X}},S^l_{C^l_{S^l_X}})=\mathbf{F}(\mathbf{G}(X,S^r_X,S^l_X))\to(X,S^r_X,S^l_X)$ is a bi-continuous map. Moreover, if $(X,C^r_X,C^l_X)$ is an $L$-fuzzy bi-closure system, by Theorem 3.4(1), $\mathbf{G}(\mathbf{F}(X,C^r_X,C^l_X))=(X,C^r_{S^r_{C^r_X}},C^l_{S^l_{C^l_X}})\geq(X,C^r_X,C^l_X)$. Hence the identity map $id_X:(X,C^r_X,C^l_X)\to\mathbf{G}(\mathbf{F}(X,C^r_X,C^l_X))=(X,C^r_{S^r_{C^r_X}},C^l_{S^l_{C^l_X}})$ is a continuous map. Therefore $(\mathbf{F},\mathbf{G})$ is a Galois correspondence.

□

EXAMPLE 3.8. Let $M=\{(x,y)\in R^2\mid y>0\}$ be a set and we define an operation $\otimes:M\times M\to M$ as follows:

$$(x_1,y_1)\otimes(x_2,y_2)=(x_1+y_1x_2,y_1y_2).$$

Then $(M,\otimes)$ is a group with $e=(0,1)$, $(x,y)^{-1}=(-\frac{x}{y},\frac{1}{y})$.

We have a positive cone $P=\{(a,b)\in R^2\mid b=1,a\geq 0\text{ ,or } y>1\}$ because $P\cap P^{-1}=\{(0,1)\}$, $P\otimes P\subset P$, $(a,b)^{-1}\otimes P\times(a,b)=P$ and $P\cup P^{-1}=L$. For $(x_1,y_1),(x_2,y_2)\in M$, we define

$$\begin{aligned}&(x_1,y_1)\leq(x_2,y_2)\\ &\Leftrightarrow(x_1,y_1)^{-1}\otimes(x_2,y_2)\in P,\ (x_2,y_2)\otimes(x_1,y_1)^{-1}\in P\\ &\Leftrightarrow y_1<y_2\ \text{ or }\ y_1=y_2,x_1\leq x_2.\end{aligned}$$

Then $(M,\leq\otimes)$ is a lattice-group. Put $L=\{(x,y)\in M\mid(1,\frac{1}{2})\leq(x,y)\leq(0,1)\}$. Then $(L,\odot,\Rightarrow,\to,(1,\frac{1}{2}),(0,1))$ is a generalized residuated lattice where $(1,\frac{1}{2})$ is the least element and $(0,1)$ is the greatest element from the following statements:

$$\begin{aligned}(x_1,y_1)\odot(x_2,y_2) &= (x_1,y_1)\otimes(x_2,y_2)\vee(1,\tfrac{1}{2})\\ &= (x_1+y_1x_2, y_1y_2)\vee(1,\tfrac{1}{2}),\\ (x_1,y_1)\Rightarrow(x_2,y_2) &= ((x_1,y_1)^{-1}\otimes(x_2,y_2))\wedge(0,1)\\ &= (\tfrac{-x_1+x_2}{y_1},\tfrac{y_2}{y_1})\wedge(0,1),\\ (x_1,y_1)\to(x_2,y_2) &= ((x_2,y_2)\otimes(x_1,y_1)^{-1})\wedge(0,1)\\ &= (x_2-\tfrac{x_1y_2}{y_1},\tfrac{y_2}{y_1})\wedge(0,1).\end{aligned}$$

It is not commutative because

$$(\frac{2}{3},\frac{3}{4})\odot(4,\frac{1}{2}) = (3+\frac{2}{3},\frac{3}{8}) \neq (4,\frac{1}{2})\odot(\frac{2}{3},\frac{3}{4}) = (4+\frac{1}{3},\frac{3}{8}).$$

Furthermore, we have $(x,y)=(x,y)^{*\circ}=(x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y)\Rightarrow(1,\frac{1}{2}) = (\frac{-x+1}{y},\frac{1}{2y}),$$

$$(x,y)^\circ = (x,y)\to(1,\frac{1}{2}) = (1-\frac{x}{2y},\frac{1}{2y}).$$

Let $X=\{a,b,c\}$ and $A\in L^X$ as follows:

$$A(a)=(1,0.6), A(b)=(2,0.8), A(c)=(0,0.6).$$

Define two maps $S^r, S^l : L^X\to L$ as follows:

$$S^r(B) = \begin{cases} (0,1), & \text{if } B=\alpha\Rightarrow A,\\ (1,\frac{1}{2}), & \text{otherwise,}\end{cases}$$

$$S^l(B) = \begin{cases} (0,1), & \text{if } B=\alpha\to A,\\ (1,\frac{1}{2}), & \text{otherwise.}\end{cases}$$

Then $(X,S^r,S^r)$ is an $L$-fuzzy bi-closure system. For each $D\in L^X$, by Lemma 2.3(9) and Theorem 3.3,

$$\begin{aligned}&C_{S^r}(D) = \bigwedge_{B\in L^X}(e^r_{L^X}(D,B)\odot S^r(B)\to B)\\ &= \bigwedge_{\alpha\in L}(e^r_{L^X}(D,\alpha\Rightarrow A)\to(\alpha\Rightarrow A))\\ &C_{S^l}(D) = \bigwedge_{B\in L^X}(S^l(B)\odot e^l_{L^X}(D,B)\Rightarrow B)\\ &= \bigwedge_{\alpha\in L}(e^r_{L^X}(D,\alpha\to A)\Rightarrow(\alpha\to A))\end{aligned}$$

By Theorems 3.3 and 3.4,$(X,C_{S^r},C_{S^l})$ is an $L$-fuzzy bi-closure space. Moreover, by Theorem 3.4, we have

$$\begin{aligned}&S^r_{C_{S^r}}(D) = e^r_{L^X}(C_{S^r}(D),D)\\ &= e^r_{L^X}(\bigwedge_{\alpha\in L}(e^r_{L^X}(D,\alpha\Rightarrow A)\to(\alpha\Rightarrow A)),D),\\ &S^l_{C_{S^l}}(D) = e^l_{L^X}(C_{S^l}(D),D)\\ &= e^l_{L^X}(\bigwedge_{\alpha\in L}(e^l_{L^X}(D,\alpha\to A)\Rightarrow(\alpha\to A)),D).\end{aligned}$$

Since $S^r_{C_{S^r}}(\alpha \Rightarrow A) = S^l_{C_{S^l}}(D)(\alpha \Rightarrow A) = (0,1)$, we have

$$S^r_{C_{S^r}} \geq S^r, \quad S^l_{C_{S^l}} \geq S^l.$$

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