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CHARACTERIZING FUNCTIONS FIXED BY A WEIGHTED BEREZIN TRANSFORM IN THE BIDISC

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Abstract. For $c > -1$, let ν_c denote a weighted radial measure on $\mathbb{C}$ normalized so that $\nu_c(D) = 1$. For $c_1, c_2 > -1$ and $f \in L^1(D^2,\ \nu_{c_1} \times \nu_{c_2})$, we define the weighted Berezin transform $B_{c_1,c_2} f$ on D^2 by

$$\big(B_{c_1,c_2}\big)f(z,w) = \int_D \int_D f\big(\varphi_z(x), \varphi_w(y)\big)\ d\nu_{c_1}(x) d\nu_{c_2}(y).$$

This paper is about the space $M^p_{c_1,c_2}$ of function $f \in L^p(D^2,\ \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1,c_2} f = f$ for $1 \le p < \infty$. We find the identity operator on $M^p_{c_1,c_2}$ by using invariant Laplacians and we characterize some special type of functions in $M^p_{c_1,c_2}$.

1. Introduction

Let D be the unit dics of $\mathbb{C}$ and ν be the Lebesgue measure on $\mathbb{C}$ normalized to $\nu(D) = 1$. For $c > -1$, we define a measure ν_c by $d\nu_c(z) = (\alpha + 1)(1 - |z|^2)^\alpha\ d\nu(z)$ so that $\nu_c(D) = 1$. If $u \in L^1(D, \nu_c)$ and $z \in D$, we define $T_c u$ the weighted Berezin transform of u by

$$\big(T_c u\big)(z) = \int_D (u \circ \varphi_z)\ d\nu_c,$$

where $\varphi_a \in \mathrm{Aut}(D)$ is defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

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For $c_1, c_2 > -1$ and $f \in L^1(D^2,\ \nu_{c_1} \times \nu_{c_2})$, we define the weighted Berezin transform $B_{c_1,c_2}f$ on D^2 by

$$\big(B_{c_1,c_2}\big)f(z,w) = \int_D \int_D f\big(\varphi_z(x), \varphi_w(y)\big)\ d\nu_{c_1}(x) d\nu_{c_2}(y).$$

A function $f \in C^2(D)$ with $\Delta_1 f = \Delta_2 f = 0$ (i, e, harmonic in each variable) is called $2-$harmonic. If $f \in L^1(D^2,\ \nu_{c_1} \times \nu_{c_2})$ is $2-$harmonic, then we can easily see that $B_{c_1,c_2}f = f$ for every $c_1, c_2 > -1$. Conversely, Furstenberg ([3]) proved that a function $f \in L^\infty(D^2)$ satisfying $B_{c_1,c_2}f = f$ for some $c_1, c_2 > -1$ has to be $2-$harmonic, whose complete analytic proof is given in [5]. The author([4]) proved that for every $1 \le p < \infty$ and $c_1, c_2 > -1$, a function $f \in L^p(D^2,\ \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1,c_2}f = f$ needs not be $2-$harmonic. Indeed, for every $1 \le p < \infty$ and $c_1, c_2 > -1$, there exist uncountably many joint eigenfunctions $f \in L^p(D^2,\ \nu_{c_1} \times \nu_{c_2})$ of invariant Laplacians satisfying $B_{c_1,c_2}f = f$ (theorem 1.1 of [4]).

This paper is about the space $M^p_{c_1,c_2}$ of function $f \in L^p(D^2,\ \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1,c_2}f = f$ for $1 \le p < \infty$ and $c_1, c_2 > -1$. We express the identity operator on $M^p_{c_1,c_2}$ as an entire function of invariant Laplacians. Then we find the joint spectrum of invariant Laplacians in an attempt to express $M^p_{c_1,c_2}$ by using spectral decompositions. Our original aim is to prove that the space $M^p_{c_1,c_2}$ is generated by the joint eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$. However, we are unable to provide the entire proof so that we leave it as a conjecture. In this paper, instead, we prove the conjecture for $f \in M^p_{c_1,c_2}$ of the form $f(z,w) = u(z)v(w)$.

In Section 2, we mention some preliminaries on eigenspaces and eigenvalues of invariant Laplacians, most of which have appeared in [4] and [7]. In Section 3, we mention some important properties of the operator the operator $B_{k+c_1,\ell+c_2}$ where k, ℓ are non-negative integers. In Section 4, we suggest a conjecture on $M^p_{c_1,c_2}$ and provide related propositions.

2. Preliminaries

Here we mention some preliminaries on function theories in the bidisc, related with eigenspaces and eigenvalues of invariant Laplacians. For $u \in C^2(D)$, $\tilde{\Delta}u$ the invariant Laplacian of u is defined by $\tilde{\Delta}u(z) = (1-|z|^2)^2 \Delta u(z)$. Acting on $f \in C^2(D^2)$, $\tilde{\Delta}_1, \tilde{\Delta}_2$ are the invariant Laplacians with respect to the first and second variable respectively, such as

$(\tilde{\Delta}_1 f)(z,w) = (1-|z|^2)^2(\Delta_1 f)(z,w)$. For $\lambda, \mu \in \mathbb{C}$, we define the joint eigenspace $X_{\lambda,\mu}$ by

$$X_{\lambda,\mu} = \{f \in C^2(D^2) \mid \tilde{\Delta}_1 f = \lambda f \text{ and } \tilde{\Delta}_2 f = \mu f\}.$$

It is known that (section 3 of [4]) for $c_1, c_2 > -1$ and $1 \le p < \infty$,

$$L^p(D^2, \nu_{c_1} \times \nu_{c_2}) \cap X_{\lambda,\mu} \neq \{0\} \quad \text{if and only if} \quad \alpha \in \Sigma_{c_1,p} \text{ and } \beta \in \Sigma_{c_2,p}$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $\lambda = -4\alpha(1-\alpha)$, $\mu = -4\beta(1-\beta)$ and

$$\Sigma_{c,p} = \left\{ \ \alpha \in \mathbb{C} \mid \ -\frac{c+1}{p} < \operatorname{Re} \alpha < 1 + \frac{c+1}{p} \ \right\}.$$

The key idea of theorem 1.1 of [4] is that $L^p(D^2, \nu_{c_1} \times \nu_{c_2}) \cap X_{\lambda,\mu}$ is also an eigenspace of B_{c_1,c_2} with eigenvalue

$$\frac{\Gamma(c_1+1+\alpha)\Gamma(c_1+2-\alpha)}{\Gamma(c_1+1)\Gamma(c_1+2)} \frac{\Gamma(c_2+1+\beta)\Gamma(c_2+2-\beta)}{\Gamma(c_2+1)\Gamma(c_2+2)},$$

and there exist uncountably many pairs of $(\alpha,\beta) \in \Sigma_{c_1,p} \times \Sigma_{c_2,p}$ satisfying

$$\frac{\Gamma(c_1+1+\alpha)\Gamma(c_1+2-\alpha)}{\Gamma(c_1+1)\Gamma(c_1+2)} \frac{\Gamma(c_2+1+\beta)\Gamma(c_2+2-\beta)}{\Gamma(c_2+1)\Gamma(c_2+2)} = 1.$$

Moreover, if $\lambda = -4\alpha(1-\alpha)$ then we get

$$\frac{\Gamma(c_1+1+\alpha)\Gamma(c_1+2-\alpha)}{\Gamma(c_1+1)\Gamma(c_1+2)} = \frac{1}{G_{c_1}(\lambda)},$$

where

$$G_c(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{(j+c)(j+c+1)}\right)$$

is an entire function.

3. the operator $B_{k+c_1,\ell+c_2}$

Definition 3.1. For $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ and $k, \ell = 0, 1, 2, \cdots$, we define the operator $B_{k+c_1,\ell+c_2}$ on $L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ by the obvious way such as

$$\big(B_{k+c_1,\ell+c_2} f\big)(z,w) = (k+1)(\ell+1) \cdot \int\int_{D^2} (1-|x|^2)^k \ (1-|y|^2)^\ell \ f(\varphi_z(x), \varphi_w(y)) \ d\nu_{c_1}(x) d\nu_{c_2}(y).$$

Just the same way as the proof of Proposition 3.2 of [4], it is easy to see that

$$\big(B_{k+c_1,\ell+c_2} f\ \big)\circ\psi \ =\ B_{k+c_1,\ell+c_2}\big(\ f\circ\psi\ \big)$$

for $k,\ell \ge 0$, $\psi \in Aut(D^2)$ and $f \in L^1(D^2, \nu_{c_1}\times\nu_{c_2})$. Also, the operators B_{c_1,c_2} and $B_{k+c_1,\ell+c_2}$ commute on $L^1(D^2,\nu_{c_1}\times\nu_{c_2})$.

The following lemma comes directly from Proposion 2.1 and Proposion 2.2 of [6].

LEMMA 3.2. *For $k,\ell \ge 0, B_{k+c_1,\ell+c_2}$ is a bounded operator on $L^p(D^2,\nu_{c_1}\times \nu_{c_2})$ when $p > 1$. And for $k,\ell > 0, B_{k+c_1,\ell+c_2}$ is a bounded linear operator on $L^1(D^2,\nu_{c_1}\times\nu_{c_2})$.*

Using Lemma 3.2, we get the following proposition.

PROPOSITION 3.3. *If $1 \le p < \infty$, then for every $f \in L^p(D^2,\nu_{c_1}\times\nu_{c_2})$ we have*

$$\lim_{n\to\infty}\ \|f - B_{n+c_1,n+c_2} f\|_p \ =\ 0.$$

Proof. By Proposition 2.2 of [6], we get $\lim_{n\to\infty}\|B_{n+c_1,n+c_2}\| = 1$ on $L^1(D^2,\nu_{c_1}\times\nu_{c_2})$.
Since $B_{n+c_1,n+c_2}$ is a contraction which fixes $2-$harmonic functions on $L^\infty(D^2)$, an interpolation theorem gives $\lim_{n\to\infty}\|B_{n+c_1,n+c_2}\| = 1$ on $L^p(D^2,\nu_{c_1}\times\nu_{c_2})$ for $1 \le p < \infty$.

If $g \in C(\overline{D^2})$, then Definition 2.1 shows that $(B_{n+c_1,n+c_2}g)(z,w) \to g(z,w)$ for every $z,w \in D$ as $n \to \infty$. Hence by dominated convergence theorem, $\lim_{n\to\infty}\|g - B_{n+c_1,n+c_2}g\|_p = 0$.

If $1 \le p < \infty$ and $f \in L^p(D^2,\nu_{c_1}\times\nu_{c_2})$ then there is a sequence $\{g_k\}$ in $C(\overline{D^2})$ such that $\|f - g_k\|_p \to 0$ as $k \to \infty$. Hence, we get the proof from the inequality

$$\|f - B_{n+c_1,n+c_2} f\|_p \le \|f - g_k\|_p + \|g_k - B_{n+c_1,n+c_2} g_k\|_p + \|B_{n+c_1,n+c_2}(g_k - f)\|_p.$$

□

The following lemma directly comes from Proposition 2.4 of [1].

LEMMA 3.4. *For $k,\ell \ge 0$, $\psi \in Aut(D^2)$, and $f \in L^1(D^2,\nu_{c_1}\times\nu_{c_2})$ we get*

$$\tilde{\Delta}_1 B_{k+c_1,\ell+c_2} f \ =\ 4(k+1+c_1)(k+2+c_1)\big(B_{k+c_1,\ell+c_2} f \ -\ B_{k+1+c_1,\ell+c_2} f\big)$$

$$\tilde{\Delta}_2 B_{k+c_1,\ell+c_2} f \ =\ 4(\ell+1+c_2)(\ell+2+c_2)\big(B_{k+c_1,\ell+c_2} f \ -\ B_{k+c_1,\ell+1+c_2} f\big)$$

and

$$B_{k+c_1,\ell+c_2}f \;=\; G_{k,c_1}(\tilde\Delta_1)G_{\ell,c_2}(\tilde\Delta_2)B_{c_1,c_2}f$$

where

$$G_{m,c}(z) \;=\; \prod_{i=1}^{m}\Big(1-\frac{z}{4(c+i)(c+i+1)}\Big)$$

is an entire function.

4. The space $M^p_{c_1,c_2}$

In this section, for $c_1, c_2 > -1$ and $1 \le p < \infty$, we make an attempt to characterize

$$M^p_{c_1,c_2} = \{f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2}) : B_{c_1,c_2}f = f\}$$

which is a Banach space of real analytic functions. The entire function

$$G_c(z) = \prod_{j=1}^{\infty}\Big(1+\frac{z}{(j+c)(j+c+1)}\Big)$$

mentioned at the end of Section 2 plays an important role.

Proposition 4.1. *$G_{c_1}(\tilde\Delta_1)G_{c_2}(\tilde\Delta_2)$ is the identity operator on $M^p_{c_1,c_2}$.*

Proof. By Lemma 3.4, for $f \in M^p_{c_1,c_2}$ we get

$$\tilde\Delta_1 f \;=\; \tilde\Delta_1 B_{c_1,c_2}f = 4(1+c_1)(2+c_1)\big(f - B_{1+c_1,c_2}f\big).$$

Since B_{1+c_1,c_2} is bounded on $L^p(D^2)$ and commutes with B_{c_1,c_2},

$$\begin{aligned} B_{c_1,c_2}\big(\,\tilde\Delta_1 f\,\big) &= 4(1+c_1)(2+c_1)\big(B_{c_1,c_2}f - B_{c_1,c_2}B_{1+c_1,c_2}f\big)\\ &= 4(1+c_1)(2+c_1)\big(f - B_{1+c_1,c_2}B_{c_1,c_2}f\big)\\ &= 4(1+c_1)(2+c_1)\big(f - B_{1+c_1,c_2}f\big) \;=\; \tilde\Delta_1 f \end{aligned}$$

Likewise we get $B_{c_1,c_2}\big(\,\tilde\Delta_2 f\,\big) = \tilde\Delta_2 f$ for $f \in M^p_{c_1,c_2}$. Hence $\tilde\Delta_1, \tilde\Delta_2$ are bounded operators on $M^p_{c_1,c_2}$. From Lemma 3.4, for $f \in M^p_{c_1,c_2}$ and $n \in \mathbb{N}$,

$$B_{n+c_1,n+c_2}f \;=\; G_{n,c_1}(\tilde\Delta_1)G_{n,c_2}(\tilde\Delta_2)B_{c_1,c_2}f.$$

For $c > -1$, the function

$$G_c(z) = \prod_{j=1}^{\infty}\Big(1+\frac{z}{(j+c)(j+c+1)}\Big)$$

is entire, so that we have

$$G_{n,c_1}(\tilde{\Delta}_1) \;\to\; G_{c_1}(\tilde{\Delta}_1) \;\text{ and }\; G_{n,c_2}(\tilde{\Delta}_2) \;\to\; G_{c_2}(\tilde{\Delta}_2)$$

in the operator norm since $G_{n,c_1} \;\to\; G_{c_1}$ and $G_{n,c_2} \;\to\; G_{c_2}$ uniformly on compact set of $\mathbb{C}$.
Now take $n \to \infty$, by Proposition 3.3 we get

$$f \;=\; G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2)f.$$

Therefore, $G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2) = I$ on $M^p_{c_1,c_2}$. □

On the other hand, from Section 2 we get

$$G_c(\lambda) = \frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma(c+1+\alpha)\Gamma(c+2-\alpha)}$$

if $c > -1$ and $\lambda = -4\alpha(1-\alpha)$. Hence, if we define

$$\Omega_{c,p} \;= \big\{\lambda = -4\alpha(1-\alpha) \mid -\frac{c+1}{p} < \text{Re } \alpha < 1 + \frac{c+1}{p}\big\},$$

then we have

$$L^p(D^2, \nu_{c_1} \times \nu_{c_2}) \cap X_{\lambda,\mu} \;\neq \{0\} \quad \text{if and only if} \quad \lambda \in \Omega_{c_1,p} \text{ and } \mu \in \Omega_{c_2,p}$$

Therefore, the set

$$E \;=\; \{(\lambda,\mu) \in \Omega_{c_1,p} \times \Omega_{c_2,p} \mid G_{c_1}(\lambda)G_{c_2}(\mu) = 1\}$$

is the set of all joint eigenvalues of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $M^p_{c_1,c_2}$. Since $G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2) = I$ on $M^p_{c_1,c_2}$, by the holomorphic functional calculus (3.11 of [2]) ,

$$1 \;=\; \sigma\big(G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2)\big) \;=\; \{G_{c_1}(\lambda)G_{c_2}(\mu) \mid (\lambda,\mu) \in \sigma(\tilde{\Delta}_1, \tilde{\Delta}_2)\}.$$

Therefore, we get the following proposition.

PROPOSITION 4.2. *The joint spectrum $\sigma(\tilde{\Delta}_1, \tilde{\Delta}_2)$ of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $M^p_{c_1,c_2}$ is*

$$\sigma(\tilde{\Delta}_1, \tilde{\Delta}_2) \;=\; \{(\lambda,\mu) \in \bar{\Omega}_{c_1,p} \times \bar{\Omega}_{c_2,p} \mid G_{c_1}(\lambda)G_{c_2}(\mu) = 1\}.$$

In view of Proposition 4.2 we may conjecture that the space $M^p_{c_1,c_2}$ is generated by the joint eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ in $M^p_{c_1,c_2}$. But this conjecture is very hard for us to prove partly because the operators $\tilde{\Delta}_1, \tilde{\Delta}_2$ are not normal so that any type of spectral decomposition of $M^p_{c_1,c_2}$ with respect to $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ is unavailable. The author hope to return to this problem in the future work.

Here instead, we will prove the conjecture for $f \in M^p_{c_1,c_2}$ of the form $f(z,w) = u(z)v(w)$.

Proposition 4.3. *Given $1 \le p < \infty$ and $c_1, c_2 > -1$, if $f(z,w) = u(z)v(w) \in M^p_{c_1,c_2}$, then then f can be written as a finite sum of joint eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$.*

Proof. If $f(z,w) = u(z)v(w)$ for some $u \in L^p(D, \nu_{c_1})$ and $v \in L^p(D, \nu_{c_2})$, then

$$\begin{aligned}(B_{c_1,c_2} f)(z,w) &= \int_D u(\varphi_z(x))\, d\nu_{c_1}(x) \int_D v(\varphi_w(y))\, d\nu_{c_2}(y) \\ &= \big(T_{c_1} u\big)(z)\ \big(T_{c_2} v\big)(w).\end{aligned}$$

Fix $1 \le p < \infty$ and $c_1, c_2 > -1$, then let H be the set of all $r \in \mathbb{C} \setminus \{0\}$ such that both $\{u \in L^p(D, \nu_{c_1}) \mid T_{c_1} u = ru\}$ and $\{v \in L^p(D, \nu_{c_2}) \mid T_{c_2} v = \frac{1}{r} v\}$ are non-empty.

For $s, t \in H$ let us define the space

$K^1_s = \{u \in L^p(D, \nu_{c_1}) \mid T_{c_1} u = su\}$ and $K^2_t = \{u \in L^p(D, \nu_{c_2}) \mid T_{c_2} u = tu\}$.

Then, for $f = uv \in M^p_{c_1,c_2}$ there exists an $r \in H$, such that $u \in K^1_r$ and $v \in K^2_{1/r}$.

Just as in Lemma 3.4 and Proposion 4.1, for $r \in H$ we have the following :

(1) $\tilde{\Delta}$ is a bounded operator on K^1_r.

(2) $rG_{c_1}(\tilde{\Delta})$ is the identity operator on K^1_r.

(3) The set $\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\}$ is the set of all eigenvalues of $\tilde{\Delta}$ on K^1_r.

Moreover, by Proposition 3.7 (d) of [1], $\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\}$ is a finite set.

Therefore, if

$$\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\} = \{\lambda_1, \cdots, \lambda_N\}$$

and

$$Q(z) = \prod_{i=1}^{N} (z - \lambda_i),$$

then $Q(\tilde{\Delta}) = 0$ on K^1_r. Hence by Lemma 4.1 of [1], every $u \in K^1_r$ is a sum

$$u = u_{\lambda_1} + \cdots + u_{\lambda_N},$$

where $u_{\lambda_i} \in L^p(D,\nu_{c_1})$ satisfies $\tilde{\Delta} u_{\lambda_i} = \lambda_i u_{\lambda_i}$ for $1 \le i \le N$. By the same way every $v \in K^2_{1/r}$ is a sum

$$v = v_{\mu_1} + \cdots + v_{\mu_m},$$

where $\{\mu_1 , \cdots , \mu_m\} = \{ \mu \in \Omega_{c_2,p} \mid G_{c_2}(\mu) = r\}$ and $v_{\mu_j} \in L^p(D,\nu_{c_2})$ satisfies

$$\tilde{\Delta} v_{\mu_j} = \mu_j v_{\mu_j} \text{ for } 1 \le j \le m.$$

Hence we can write f as

$$f(z,w) = (u_{\lambda_1}(z) + \cdots + u_{\lambda_N}(z)) \; (v_{\mu_1}(w) + \cdots + v_{\mu_m}(w))$$

which is a finite sum of joint eigenfunctions. □

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