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SIMPLIFYING COEFFICIENTS IN A FAMILY OF ORDINARY DIFFERENTIAL EQUATIONS RELATED TO THE GENERATING FUNCTION OF THE MITTAG–LEFFLER POLYNOMIALS

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Abstract. In the paper, by virtue of the Faà di Bruno formula, properties of the Bell polynomials of the second kind, and the Lah inversion formula, the author simplifies coefficients in a family of ordinary differential equations related to the generating function of the Mittag–Leffler polynomials.

1. Motivation and main results

In [4, Theorem 2.2], it was established inductively and recursively that the family of differential equations

$$F^{(n)}(t)=\frac{F(t)}{(1-t)^n}\sum_{i=1}^{n}a_i(n)\frac{\langle x\rangle_i}{(1+t)^i},\quad n\in\mathbb{N} \tag{1}$$

has a solution

$$F(t)=\left(\frac{1+t}{1-t}\right)^x, \tag{2}$$

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where $a_1(n) = 2n!$,

$$a_i(n) = 2^i \sum_{k_{i-1}=0}^{n-i} \sum_{k_{i-2}=0}^{n-i-k_{i-1}} \cdots \sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2} \prod_{\ell=2}^{i} \left\langle n-i-1+2\ell-\sum_{j=\ell}^{i-1} k_j \right\rangle_{k_{\ell-1}} \left(n-i+1-\sum_{j=1}^{i-1} k_j\right)! \quad (3)$$

for $2 \le j \le n$,

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is the falling factorial, and the function $F(t)$ in (2) can be used to generate the Mittag-Leffler polynomials $M_n(x)$ by

$$F(t) = \left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}.$$

Hereafter, the expression (3) was employed in [4, Theorem 3.1].

It is not difficult to see that

1. the expression (3) is too complicated to be remembered, understood, and computed easily;
2. the original proof of [4, Theorem 2.2] is long and tedious.

In this paper, we will provide a nice and standard proof for [4, Theorem 2.2] and, more importantly, discover a simple, meaningful, and significant form for $a_i(n)$.

Our main results can be stated as the following theorem.

Theorem 1. *For $n \ge 0$, the function $F(t)$ defined by* (2) *satisfies*

$$F^{(n)}(t) = \frac{n!}{(1-t)^n}\left[\sum_{k=0}^{n} \frac{2^k}{k!}\binom{n-1}{k-1}\frac{\langle x\rangle_k}{(1+t)^k}\right]F(t) \quad (4)$$

and

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n-1}{k-1}(1-t)^k F^{(k)}(t) = \frac{2^n\langle x\rangle_n}{n!(1+t)^n}F(t) \quad (5)$$

where $\binom{-1}{-1} = 1$ *and* $\binom{k}{-1} = 0$ *if* $k \ge 0$.

2. Proof of Theorem 1

The Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ are defined [3, p. 134, Theorem A] and [3, p. 139, Theorem C] by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!} \prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The famous Faà di Bruno formula reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big) \tag{6}$$

for $n \ge 0$. The function $F(t)$ in (2) can be rearranged as

$$F(t) = \left(\frac{2}{1-t} - 1\right)^x.$$

Applying $u = h(t) = \frac{2}{1-t} - 1$ and $f(u) = u^x$ to (6) gives

$$\begin{aligned}
F^{(n)}(t) &= \sum_{k=0}^{n} \frac{\mathrm{d}^k\,u^x}{\mathrm{d}\,u^k}\,\mathrm{B}_{n,k}\left(\frac{1!2}{(1-t)^2}, \frac{2!2}{(1-t)^3}, \dots, \frac{(n-k+1)!2}{(1-t)^{n-k+2}}\right)\\
&= \sum_{k=0}^{n} \langle x\rangle_k u^{x-k} 2^k \left(\frac{1}{1-t}\right)^{n+k} \mathrm{B}_{n,k}(1!, 2!, \dots, (n-k+1)!)\\
&= \sum_{k=0}^{n} \langle x\rangle_k \left(\frac{2}{1-t} - 1\right)^{x-k} 2^k \left(\frac{1}{1-t}\right)^{n+k} \frac{n!}{k!}\binom{n-1}{k-1}\\
&= \sum_{k=0}^{n} \langle x\rangle_k \left(\frac{1+t}{1-t}\right)^{x-k} 2^k \left(\frac{1}{1-t}\right)^{n+k} \frac{n!}{k!}\binom{n-1}{k-1},
\end{aligned}$$

where we used the identities

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n\,\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

and

$$\mathrm{B}_{n,k}(1!, 2!, \dots, (n-k+1)!) = \frac{n!}{k!}\binom{n-1}{k-1}$$

in [3, p. 135] and [7, Remark 3.5]. The formula (4) is thus proved.

The Lah inversion theorem in [1, p. 96, Corollary 3.38 (iii)] and [2, pp. 60–61, Exercise 2.9] reads that

$$(-1)^n a_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} b_k$$

if and only if

$$(-1)^n b_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} a_k.$$

Combining this Lah inversion theorem with (4) arrives at

$$\frac{2^n \langle x \rangle_n}{(1+t)^n} F(t) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \binom{n-1}{k-1} (1-t)^k F^{(k)}(t)$$

which can be rewritten as (5). The proof of Theorem 1 is complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 1. Comparing (1) with (4) reveals that

$$a_k(n) = 2^k \frac{n!}{k!} \binom{n-1}{k-1}$$

for $n \ge k \ge 0$. This form for $a_k(n)$ is apparently simpler, more meaningful, and more significant than the one (3) obtained in [4, Theorem 2.2].

Remark 2. The motivations in the papers [5,6,8–13,15–29] are same as the one in this paper.

Remark 3. This paper is a modified version of the preprint [14].

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