Korean J. Math. **27** (2019), No. 2, pp. 297–327
https://doi.org/10.11568/kjm.2019.27.2.297

A NEW MAPPING FOR FINDING A COMMON SOLUTION OF SPLIT GENERALIZED EQUILIBRIUM PROBLEM, VARIATIONAL INEQUALITY PROBLEM AND FIXED POINT PROBLEM

Mohammad Farid* and Kaleem Raza Kazmi

Abstract. In this paper, we introduce and study a general iterative algorithm to approximate a common solution of split generalized equilibrium problem, variational inequality problem and fixed point problem for a finite family of nonexpansive mappings in real Hilbert spaces. Further, we prove a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area.

1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively.

A mapping $T : C \to C$ is called *nonexpansive*, if

$$\|Tx - Ty\| \leq \|x - y\|, \; x, y \; \in C.$$

Received September 3, 2018. Revised April 10, 2019. Accepted April 23, 2019.
2010 Mathematics Subject Classification: 49J30, 47H10, 47H17, 90C99.
Key words and phrases: Split generalized equilibrium problem, Variational inequality problem, Fixed-point problem, Nonexpansive mappings.
∗ Corresponding author.

The *fixed point problem* (in short, FPP) for a mapping $T : C \to C$ is to find $x \in C$ such that

$$Tx = x. \tag{1.1}$$

The solution set of FPP(1.1) is denoted by Fix(T).

The *classical scalar nonlinear variational inequality problem* (in short, VIP) is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0,\ \forall y \in C, \tag{1.2}$$

where $B : C \to H_1$ is a nonlinear mapping. The solution set of VIP(1.2) is denoted by Ω. It is introduced by Hartman and Stampacchia [9].

In 1994, Blum and Oettli [2] introduced and studied the following *equilibrium problem* (in short, EP): Find $x \in C$ such that

$$F_1(x, y) \geq 0,\ \forall y \in C, \tag{1.3}$$

where $F_1 : C \times C \to \mathbb{R}$ is a bifunction. We denote the solution set of EP(1.3) by Sol(EP(1.3)).

In the last two decades, EP(1.3) has been generalized and extensively studied in many directions due to its importance; see for example [4, 6, 7, 11, 12, 16, 21] for the literature on the existence and iterative approximation of solution of the various generalizations of EP(1.3).

Recently, Kazmi and Rizvi [13] considered the following pair of equilibrium problems in different spaces, which is called *split equilibrium problem* (in short, SEP): Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0,\ \forall x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0,\ \forall y \in Q. \tag{1.5}$$

They introduced and studied some iterative methods for finding the common solution of SEP(1.4)-(1.5), VIP(1.2) and FPP(1.1). For related work, see [14].

In this paper, we consider the following split generalized equilibrium problem (in short, SGEP):

Let $\phi_1 : C \times C \to \mathbb{R}$, and $\phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear mappings, then SGEP is to find $x^* \in C$ such that

$$F_1(x^*, x) + \phi(x, x^*) - \phi(x^*, x^*) \geq 0, \ \forall x \in C, \tag{1.6}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \phi(y, y^*) - \phi(y^*, y^*) \geq 0, \ \forall y \in Q. \tag{1.7}$$

When looked separately, (1.6) is the generalized equilibrium problem (GEP) and we denote its solution set by Sol(GEP(1.6)). The SGEP(1.6)-(1.7) constitutes a pair of generalized equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the GEP(1.6) in H_1 is the solution of another GEP(1.6) in another space H_2. We denote the solution set of GEP(1.7) by Sol(GEP(1.7)). The solution set of SGEP(1.6)-(1.7) is denoted by $\Gamma = \{p \in \text{Sol(GEP(1.6))} : Ap \in \text{Sol(GEP(1.7))}\}$.

SGEP(1.6)-(1.7) generalize multiple-sets split feasibility problem. It also includes as special case, the split variational inequality problem [6] which is the generalization of split zero problems and split feasibility problems, see for detail [3, 5, 6, 16, 17].

Recently, Kangtunyakarn and Suantai [10] defined the new mappings

$$\begin{aligned}
U_{n,0} &= I \\
U_{n,1} &= \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1}) I \\
U_{n,2} &= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) U_{n,1} \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) U_{n,N-2} \\
K_n = U_{n,N} &= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) U_{n,N-1},
\end{aligned}$$

where $T_i : C \to C$, $i = 1, 2, ..., N$ is a finite family of nonexpansive mappings and $\{\lambda_{n,i}\}_{i=1}^N \subset (0, 1]$. Such a mapping K_n is called the K-mapping generated by $T_1, T_2, ..., T_N$ and $\{\lambda_{n,1}\}$,$\{\lambda_{n,2}\}$,...,$\{\lambda_{n,N}\}$.

Moreover, Kangtunyakarn and Suantai [10] introduced the following iterative methods to obtained a strong convergence theorem for EP(1.3) and FPP(1.1): $x_1 \in H$

$$\begin{cases} F(y_n, y) + \dfrac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta_n)I - \alpha_n A)K_n y_n. \end{cases}$$

Motivated by the work of Censor *et al.* [6], Moudafi [16, 17], Kazmi *et al.* [13, 14], Kangtunyakarn and Suantai [10] and by the ongoing research in this direction, we suggest and analyze a general iterative method for approximating the common solution to the split generalized equilibrium problem, variational inequality problem and fixed point problem for a finite family of nonexpansive mappings in Hilbert space. Furthermore, we prove that the sequence generated by the proposed iterative scheme converges strongly to the common solution of split generalized equilibrium problem, variational inequality problem and fixed point problem. The results and methods presented in this paper generalize, improve and unify many previously known results in this research area.

2. Preliminaries

We recall some concepts and results that are needed in the sequel.

Definition 2.1. A mapping $T : H_1 \to H_1$ is said to be

(i) *monotone*, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H_1;$$

(ii) *α-inverse strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ \geq \alpha\|Tx - Ty\|^2, \ \forall x, y \in H_1;$$

(iii) *β-Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta\|x - y\|, \ \forall x, y \in H_1.$$

We note that if T is α-inverse strongly monotone mapping, then T is monotone and $\frac{1}{\alpha}$-Lipschitz continuous but converse need not be true. For $\alpha = 1$, α-inverse strongly monotone mapping T is called firmly nonexpansive mapping.

DEFINITION 2.2. [1]. A mapping $T : H_1 \to H_1$ is said to be *averaged* if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H_1 \to H_1$ is nonexpansive and I is the identity operator on H_1.

The following are some key properties of averaged mappings.

LEMMA 2.3. [16].

(i) *If $T = (1-\alpha)S+\alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0, 1)$, then T is averaged;*
(ii) *The composite of finitely many averaged mappings is averaged;*
(iii) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1T_2...T_N);$$

(iv) *If T is τ-ism, then for $\gamma > 0$, γT is $\frac{\tau}{\gamma}$-ism;*
(v) *T is averaged if and only if, its complement $I - T$ is τ-ism for some $\tau > \frac{1}{2}$.*

DEFINITION 2.4. [1]. A multi-valued mapping $M : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

DEFINITION 2.5. [1]. A multi-valued monotone mapping $M : H_1 \to 2^{H_1}$ is *maximal* if the Graph(M), the graph of M, is not properly contained in the graph of any other monotone mapping.

REMARK 2.6. . It is known that a multi-valued monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$, for every $(y, v) \in \mathrm{Graph}(M)$ implies that $u \in Mx$.

For every point $x \in H_1$, there exists a unique nearest point to x in C denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \ \forall y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H_1 onto C. It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \ \forall x, y \in H_1. \tag{2.2}$$

Moreover, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \ \forall y \in C. \tag{2.3}$$

This implies that

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H_1, \ \forall y \in C. \tag{2.4}$$

In a real Hilbert space H_1, it is well known that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.5}$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

It is also known that every Hilbert space H_1 satisfies:

1. Opial's condition [18], i.e., for any sequence $\{x^n\}$ with $x^n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x^n - x\| < \liminf_{n\to\infty} \|x^n - y\| \tag{2.6}$$

holds for every $y \in H_1$ with $y \neq x$;

2.

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in H_1. \tag{2.7}$$

LEMMA 2.7. *[20]. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then*

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

LEMMA 2.8. *[15]. Assume that B is a strongly positive self-adjoint bounded linear operator on a Hilbert space H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|B\|^{-1}$. Then $\|I - \rho B\| \le 1 - \rho\overline{\gamma}$.*

Lemma 2.9. [22]. *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. [10]. *Let* C *be a nonempty closed convex set of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, ..., \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, ..., N-1$ *and* $0 < \lambda_N \leq 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$. *Then* $\mathrm{Fix}(K) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

Lemma 2.11. [10]. *Let* C *be a nonempty convex subset of a Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into itself and* $\{\lambda_{n,i}\}_{i=1}^N$ *be sequences in* $[0,1]$ *such that* $\lambda_{n,i} \to \lambda_i (i = 1, 2, ..., N)$. *Moreover, for every* $n \in N$, *let* K *and* K_n *be the* K*-mappings generated by* $T_1, T_2, ..., T_N$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$ *and* $T_1, T_2, ..., T_N$ *and* $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, ..., \{\lambda_{n,N}\}$, *respectively. Then for every* $x \in C$,

$$\lim_{n\to\infty} \|K_n x - Kx\| = 0.$$

Assumption 2.12. *Let* $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ *be bimappings satisfy the following conditions:*

(1) $F_1(x,x) = 0,\ \forall x \in C$;

(2) F_1 *is monotone, i.e.,*

$$F_1(x,y) + F_1(y,x) \leq 0,\ \forall x, y \in C;$$

(3) *For each* $y \in C$, $x \to F_1(x,y)$ *is weakly upper semicontinuous;*

(4) *For each* $x \in C$, $y \to F_1(x,y)$ *is convex and lower semicontinuous;*

(5) $\phi_1(.,.)$ *is weakly continuous and* $\phi_1(.,y)$ *is convex;*

(6) ϕ_1 *is skew-symmetric, i.e.,*

$$\phi_1(x,x) - \phi_1(x,y) + \phi_1(y,y) - \phi_1(y,x) \geq 0,\ \forall x, y \in C.$$

Now, we define $T_r^{(F_1,\phi_1)} : H_1 \to C$ as follows:

$$\begin{aligned} T_r^{(F_1,\phi_1)}(z) &= \{x \in C : F_1(x,y) + \phi_1(y,x) - \phi_1(x,x) \\ &\quad + \frac{1}{r}\langle y - x, x - z\rangle \geq 0,\ \forall y \in C\}, \end{aligned} \tag{2.8}$$

where r is a positive real number.

Lemma 2.13. [8]. *Let H_1 be a real Hilbert space and let C be a nonempty, closed and convex subset of H_1. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ be nonlinear mappings satisfying the Assumption 2.12. Assume that for each $z \in H_1$ and for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$F_1(y, z_x) + \phi_1(z_x, y) - \phi_1(y,y) + \frac{1}{r}\langle z_x - y, y - z\rangle < 0.$$

Let the mapping $T_r^{(F_1,\phi_1)}$ be defined by (2.8). Then the following conclusions hold:

(i) $T_r^{(F_1,\phi_1)}(z)$ *is nonempty for each $z \in H_1$;*
(ii) $T_r^{(F_1,\phi_1)}$ *is single-valued;*
(iii) $T_r^{(F_1,\phi_1)}$ *is a firmly nonexpansive mapping, i.e., for all $z_1, z_2 \in H_1$,*

$$\|T_r^{(F_1,\phi_1)}(z_1) - T_r^{(F_1,\phi_1)}(z_2)\|^2 \leq \langle T_r^{(F_1,\phi_1)}(z_1) - T_r^{(F_1,\phi_1)}(z_2), z_1 - z_2\rangle;$$

(iv) Fix($T_r^{(F_1,\phi_1)}$)= *Sol(GEP*(1.6)*);*
(v) *Sol(GEP*(1.6)*) is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.12. For $s > 0$ and for all $u \in H_2$, define a mapping $T_s^{(F_2,\phi_2)} : H_2 \to Q$ as follows:

$$\begin{aligned} T_s^{(F_2,\phi_2)}(u) &= \{v \in Q : F_2(v,w) + \phi_2(w,v) - \phi_2(v,v) \\ &\quad + \frac{1}{s}\langle w - v, v - u\rangle \geq 0,\ \forall w \in Q\}. \end{aligned} \tag{2.9}$$

Then, we easily observe that $T_s^{(F_2,\phi_2)}$ is nonempty, single-valued, firmly nonexpansive, Fix($T_s^{(F_2,\phi_2)}$)= Sol(GEP(1.7)) and Sol(GEP(1.7)) is closed and convex.

LEMMA 2.14. [8]. *Let F_1 and ϕ_1 satisfy Assumption 2.12 and let the mapping $T_r^{(F_1,\phi_1)}$ be defined by (2.9). Let $x_1, x_2 \in H_1$ and $r_1, r_2 > 0$, then*

$$\|T_{r_2}^{(F_1,\phi_1)}(x_2) - T_{r_1}^{(F_1,\phi_1)}(x_1)\| \leq \|x_2 - x_1\| + \frac{|r_2 - r_1|}{r_2}\|T_{r_2}^{(F_1,\phi_1)}(x_2) - x_2\|.$$

3. Main Results

In this section, we prove a strong convergence theorem based on the proposed iterative scheme for computing the approximate common solution of SGEP(1.6)-(1.7), VIP(1.2) and FPP(1.1) for a finite family of nonexpansive mappings in real Hilbert spaces.

THEOREM 3.1. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$, $\phi_1 : C \times C \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$ are nonlinear mappings satisfying Assumption 2.12 and F_2 is upper semicontinuous in first argument. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, ..., N$ such that $\Theta = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \bigcap \Gamma \bigcap \Omega \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and D be a strongly positive bounded linear self adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $B : C \to H_1$ be a τ-inverse strongly monotone mapping. For a given $x_0 \in C$ arbitrarily, let the sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by the following iterative schemes:*

$$\begin{aligned} u_n &= T_{r_n}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n); \\ y_n &= P_C(u_n - \mu_n B u_n) \\ x_{n+1} &= \alpha_n \gamma f(K_n x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n D)K_n y_n, \end{aligned}$$

where $\{\mu_n\} \subset (0, 2\tau)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ are sequences in $(0, 1)$; $\delta \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, and $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, ..., N - 1$ and $0 < \lambda_N \leq 1$, $\lambda_{n,i} \to \lambda_i (i = 1, 2, ..., N)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 2\tau$ and $\lim_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$;

(v) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty,\ \forall i = 1, 2, ..., N.$

Then the sequence $\{x_n\}$ *converges strongly to some* $z_* \in \Theta$, *where* $z_* = P_\Theta(\gamma f + (I - D))z_*$ *which solves the following variational inequality:*

$$\langle (D - \gamma f)z_*, z - z_*\rangle \geq 0, \text{ for any } z \in \Theta. \tag{3.1}$$

Proof. We divide the proof into four claims.

Claim 1. $\{x_n\}$ is a bounded sequence.

Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, without loss of generality, that $\alpha_n < \|D\|^{-1}$, $\forall n \geq 1$. Then, $\alpha_n < \frac{1}{\overline{\gamma}}$, $\forall n \geq 1$. By Lemma 2.8, $\|I - \alpha_n D\| \leq 1 - \alpha_n\overline{\gamma}$.

Since D is a strongly positive bounded linear operator therefore $\langle Dx, x\rangle \geq \overline{\gamma}\|x\|^2$ and $\|D\| = \sup\{|\langle Dx, x\rangle| : x \in H_1, \|x\| = 1\}$.

Now, we observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n D)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Dx, x\rangle\\ &\geq 1 - \beta_n - \alpha_n\|D\|\\ &\geq 0,\ \forall x \in H_1.\end{aligned}$$

This shows that $(1-\beta_n)I - \alpha_n D$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n D\| &= \sup\{|\langle((1-\beta_n)I - \alpha_n D)x, x\rangle| :\\ &\qquad x \in H_1, \|x\| = 1\}\\ &= \sup\{1 - \beta_n - \alpha_n\langle Dx, x\rangle : x \in H_1, \|x\| = 1\}\\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}. \qquad (3.2)\end{aligned}$$

For any $x, y \in C$, we have

$$\begin{aligned}\|(I - \mu_n B)x - (I - \mu_n B)y\|^2 &= \|(x-y) - \mu_n(Bx - By)\|^2\\ &\leq \|x-y\|^2 - 2\mu_n\langle x - y, Bx - By\rangle\\ &\quad + \mu_n^2\|Bx - By\|^2\\ &\leq \|x - y\|^2. \qquad (3.3)\end{aligned}$$

This shows that the mapping $I - \mu_n B$ is nonexpansive.

Let for each $i = 1, 2, ..., N$, $q := P_\Theta(\gamma f + (I - D))$. Since f is a contraction mapping with constant $\alpha \in (0, 1)$, it follows that for all $x, y \in H_1$

$$\begin{aligned}\|q(I-D+\gamma f)(x)-q(I-D+\gamma f)(y)\| &\leq \|(I-D+\gamma f)(x) \\ &\quad -(I-D+\gamma f)(y)\| \\ &\leq \|I-D\|\|x-y\| \\ &\quad +\gamma\|f(x)-f(y)\| \\ &\leq (1-\bar{\gamma})\|x-y\| \\ &\quad +\gamma\alpha\|x-y\| \\ &\leq (1-(\bar{\gamma}-\gamma\alpha))\|x-y\|.\end{aligned}$$

Therefore the mapping $q(I-D+\gamma f)$ is a contraction mapping from H_1 into itself. It follows from the Banach contraction principle that there exists an element $z \in H_1$ such that $z = q(I-D+\gamma f)z = P_{\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)\bigcap\Gamma\bigcap\Omega}(I-D+\gamma f)z$.

Let $p \in \Theta := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \bigcap \Gamma \bigcap \Omega$, i.e., $p \in \Gamma$, we have $p = T_{r_n}^{(F_1,\phi_1)} p$ and $Ap = T_{r_n}^{(F_2,\phi_2)}(Ap)$.

We estimate

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n)-p\|^2 \\ &\leq \|T_{r_n}^{(F_1,\phi_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n - T_{r_n}^{(F_1,\phi_1)}p\|^2 \\ &\leq \|x_n+\delta A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n-p\|^2 \\ &\leq \|x_n-p\|^2+\delta^2\|A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2 \\ &\quad +2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\rangle. \end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}\|u_n-p\|^2 &\leq \|x_n-p\|^2+\delta^2\langle (T_{r_n}^{(F_2,\phi_2)}-I)Ax_n, AA^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\rangle \\ &\quad +2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\rangle. \end{aligned} \tag{3.5}$$

Now, we have

$$\begin{aligned}&\delta^2\langle (T_{r_n}^{(F_2,\phi_2)}-I)Ax_n, AA^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\rangle \\ &\qquad \leq L\delta^2\langle (T_{r_n}^{(F_2,\phi_2)}-I)Ax_n, (T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\rangle \\ &\qquad = L\delta^2\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2. \end{aligned} \tag{3.6}$$

Denoting $\Lambda := 2\delta\langle x_n - p, A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle$ and using (2.8), we have

$$\begin{aligned}
\Lambda &= 2\delta\langle x_n - p, A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle \\
&= 2\delta\langle A(x_n - p), (T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle \\
&= 2\delta\langle A(x_n - p) + (T_{r_n}^{(F_2,\phi_2)} - I)Ax_n \\
&\quad -(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n, (T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle \\
&= 2\delta\Big\{\langle T_{r_n}^{(F_2,\phi_2)}Ax_n - Ap, (T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle \\
&\quad -\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2\Big\} \\
&\leq 2\delta\Big\{\frac{1}{2}\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2 - \|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2\Big\} \\
&\leq -\delta\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2. \qquad (3.7)
\end{aligned}$$

Using (3.5), (3.6) and (3.7), we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \delta(L\delta - 1)\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2. \qquad (3.8)$$

Since, $\delta \in (0, \frac{1}{L})$, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \qquad (3.9)$$

By using(3.3), we have

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I - \mu_n B)u_n - p\| \\
&\leq \|(I - \mu_n B)u_n - (I - \mu_n B)p\| \\
&\leq \|u_n - p\|. \qquad (3.10)
\end{aligned}$$

Further, we estimate

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\gamma f(K_nx_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nD)K_ny_n-p\| \\
&= \|\alpha_n(\gamma f(K_nx_n)-Dp)+\beta_n(x_n-p) \\
&\quad +((1-\beta_n)I-\alpha_nD)(K_ny_n-p)\| \\
&\leq \alpha_n\|\gamma f(K_nx_n)-Dp\|+\beta_n\|x_n-p\| \\
&\quad +((1-\beta_n)I-\alpha_n\overline{\gamma})\|K_ny_n-p\| \\
&\leq \alpha_n\|\gamma f(K_nx_n)-\gamma f(p)+\gamma f(p)-Dp\|+\beta_n\|x_n-p\| \\
&\quad +((1-\beta_n)I-\alpha_n\overline{\gamma})\|y_n-p\| \\
&\quad \text{(using nonexpansivity of } K_n) \\
&\leq \alpha_n\gamma\|f(K_nx_n)-f(p)\|+\alpha_n\|\gamma f(p)-Dp\| \\
&\quad +\beta_n\|x_n-p\|+((1-\beta_n)I-\alpha_n\overline{\gamma})\|x_n-p\| \\
&\quad \text{(using (3.9) and (3.10))} \\
&\leq \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Dp\| \\
&\quad +(1-\alpha_n\overline{\gamma})\|x_n-p\| \\
&\leq (1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-p\|+\alpha_n\|\gamma f(p)-Dp\| \\
&\leq \max\{\|x_n-p\|,\tfrac{\|\gamma f(p)-Dp\|}{\overline{\gamma}-\gamma\alpha}\},\ n\geq 0.
\end{aligned}$$

By induction, we have

$$\|x_{n+1}-p\|\leq \max\{\|x_0-p\|,\frac{\|\gamma f(p)-Dp\|}{\overline{\gamma}-\gamma\alpha}\}.$$

Hence $\{x_n\}$ is bounded, so $\{y_n\}$, $\{K_ny_n\}$ and $\{f(K_nx_n)\}$ are bounded.

Claim 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\|x_n-K_ny_n\|=0$, $\lim_{n\to\infty}\|u_n-x_n\|=0$ and $\lim_{n\to\infty}\|u_n-y_n\|=0$.

Since $T_{r_{n+1}}^{(F_1,\phi_1)}$ and $T_{r_{n+1}}^{(F_2,\phi_2)}$ both are firmly nonexpansive, they are averaged. For $\delta\in(0,\frac{1}{L})$, the mapping $(I+\delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)}-I)A)$ is averaged. It follows from Lemma 2.3(ii) that the mapping $T_{r_{n+1}}^{(F_1,\phi_1)}(I+\delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)}-I)A)$ is averaged and hence nonexpansive. Further, since $u_n=T_{r_n}^{(F_1,\phi_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n)$ and $u_{n+1}=T_{r_{n+1}}^{(F_1,\phi_1)}(x_{n+1}+$

$\delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)} - I)Ax_{n+1})$, it follows from Lemma 2.14 that

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|T_{r_{n+1}}^{(F_1,\phi_1)}(x_{n+1} + \delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)} - I)Ax_{n+1}) \\
&\quad -T_{r_{n+1}}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)} - I)Ax_n)\| \\
&\quad +\|T_{r_{n+1}}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)} - I)Ax_n) \\
&\quad -T_{r_n}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n)\| \\
&\leq \|x_{n+1} - x_n\| + \|(x_n + \delta A^*(T_{r_{n+1}}^{(F_2,\phi_2)} - I)Ax_n) \\
&\quad -(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n)\| \\
&\quad + \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n) \\
&\quad -(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n)\| \\
&\leq \|x_{n+1} - x_n\| \\
&\quad +\delta\|A\|\|T_{r_{n+1}}^{(F_2,\phi_2)}Ax_n - T_{r_n}^{(F_2,\phi_2)}Ax_n\| + \delta_n \\
&\leq \|x_{n+1} - x_n\| \\
&\quad +\delta\|A\|\left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}}^{(F_2,\phi_2)}Ax_n - Ax_n\| + \delta_n \\
&\leq \|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n, \qquad (3.11)
\end{aligned}$$

where

$$\begin{aligned}
\sigma_n &= \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_n}^{(F_2,\phi_2)}Ax_n - Ax_n\| \\
\delta_n &= \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_n}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n) \\
&\quad -(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n)\|.
\end{aligned}$$

We estimate

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \|P_C(I - \mu_{n+1}B)u_{n+1} - P_C(I - \mu_n B)u_n\| \\
&\le \|(I - \mu_{n+1}B)u_{n+1} - (I - \mu_n B)u_n\| \\
&= \|(I - \mu_{n+1}B)u_{n+1} - (I - \mu_{n+1}B)u_n \\
&\quad +(\mu_n - \mu_{n+1})Bu_n\| \\
&\le \|u_{n+1} - u_n\| + |\mu_n - \mu_{n+1}|\|Bu_n\| \\
&\le \|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n + |\mu_n - \mu_{n+1}|\|Bu_n\| \\
&\quad \text{(using (3.11))} \\
&\le \|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n + M_1|\mu_n - \mu_{n+1}|, \qquad (3.12)
\end{aligned}
$$

where $M_1 = \sup_{n\ge 1}\|Bu_n\|$.

Setting $x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n$, then we have $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ and

$$
\begin{aligned}
l_{n+1} - l_n &= \tfrac{\alpha_{n+1}\gamma f(K_{n+1}x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}D)K_{n+1}y_{n+1}}{1-\beta_{n+1}} \\
&\quad -\tfrac{\alpha_n\gamma f(K_n x_n)+((1-\beta_n)I-\alpha_n D)K_n y_n}{1-\beta_n} \\
&= (\tfrac{\alpha_{n+1}}{1-\beta_{n+1}})(\gamma f(K_{n+1}x_{n+1}) - DK_{n+1}y_{n+1}) \\
&\quad +(\tfrac{\alpha_n}{1-\beta_n})(DK_n y_n - \gamma f(K_n x_n)) \\
&\quad +K_{n+1}y_{n+1} - K_n y_n \\
&= (\tfrac{\alpha_{n+1}}{1-\beta_{n+1}})(\gamma f(K_{n+1}x_{n+1}) - DK_{n+1}y_{n+1}) \\
&\quad +(\tfrac{\alpha_n}{1-\beta_n})(DK_n y_n - \gamma f(K_n x_n)) \\
&\quad +K_{n+1}y_{n+1} - K_{n+1}y_n + K_{n+1}y_n - K_n y_n.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(K_{n+1}x_{n+1})\| + \|DK_{n+1}y_{n+1}\|) \\
&\quad +\frac{\alpha_n}{1-\beta_n}(\|DK_n y_n\| + \|\gamma f(K_n x_n)\|) \\
&\quad +\|K_{n+1}y_{n+1} - K_{n+1}y_n\| + \|K_{n+1}y_n - K_n y_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(K_{n+1}x_{n+1})\| + \|DK_{n+1}y_{n+1}\|) \\
&\quad +\frac{\alpha_n}{1-\beta_n}(\|DK_n y_n\| + \|\gamma f(K_n x_n)\|) \\
&\quad +\|y_{n+1} - y_n\| + \|K_{n+1}y_n - K_n y_n\|. \qquad (3.13)
\end{aligned}
$$

Next, we estimate $\|K_{n+1}y_n - K_n y_n\|$.
For $i \in \{2,3,...,N-2\}$, we have

$$
\begin{aligned}
\|U_{n+1,N-i}y_n &- U_{n,N-i}y_n\| \\
&= \|\lambda_{n+1,N-i}T_{N-i}U_{n+1,N-i-1}y_n + (1-\lambda_{n+1,N-i})U_{n+1,N-i-1}y_n \\
&\quad -\lambda_{n,N-i}T_{N-i}U_{n,N-i-1}y_n - (1-\lambda_{n,N-i})U_{n,N-i-1}y_n\| \\
&= \|\lambda_{n+1,N-i}T_{N-i}U_{n+1,N-i-1}y_n - \lambda_{n+1,N-i}T_{N-i}U_{n,N-i-1}y_n \\
&\quad +\lambda_{n+1,N-i}T_{N-i}U_{n,N-i-1}y_n - \lambda_{n+1,N-i})U_{n,N-i-1}y_n \\
&\quad +\lambda_{n+1,N-i})U_{n,N-i-1}y_n + (1-\lambda_{n+1,N-i})U_{n+1,N-i-1}y_n \\
&\quad -\lambda_{n,N-i}T_{N-i}U_{n,N-i-1}y_n - (1-\lambda_{n,N-i})U_{n,N-i-1}y_n\| \\
&\leq \lambda_{n+1,N-i}\|T_{N-i}U_{n+1,N-i-1}y_n - T_{N-i}U_{n,N-i-1}y_n\| \\
&\quad +(1-\lambda_{n+1,N-i})\|U_{n+1,N-i-1}y_n - U_{n,N-i-1}y_n\| \\
&\quad +|\lambda_{n+1,N-i} - \lambda_{n,N-i}|\|T_{N-i}U_{n,N-i-1}y_n\| \\
&\quad +|\lambda_{n+1,N-i} - \lambda_{n,N-i}|\|U_{n,N-i-1}y_n\| \\
&\leq \|U_{n+1,N-i-1}y_n - U_{n,N-i-1}y_n\| + M_3|\lambda_{n+1,N-i} - \lambda_{n,N-i}| \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\|U_{n+1,1}y_n - U_{n,1}y_n\| &= \|\lambda_{n+1,1}T_1y_n + (1-\lambda_{n+1,1})y_n \\
&\quad -\lambda_{n,1}T_1y_n - (1-\lambda_{n,1})y_n\| \\
&\leq |\lambda_{n+1,1} - \lambda_{n,1}|\|T_1y_n\| + |\lambda_{n+1,1} - \lambda_{n,1}|\|y_n\| \\
&\leq |\lambda_{n+1,1} - \lambda_{n,1}|M_2, \qquad (3.15)
\end{aligned}
$$

where

$$
M_2 = \sup\{\textstyle\sum_{i=2}^{N}(\|T_iU_{n,i-1}y_n\| + \|U_{n,i-1}y_n\|) + \|T_1y_n\| + \|y_n\|\} \leq \infty.
$$

By (3.14) and (3.15), we have

$$
\begin{aligned}
\|K_{n+1}y_n - K_ny_n\| &= \|U_{n+1,N}y_n - U_{n,N}y_n\| \\
&\leq \|U_{n+1,N_1}y_n - U_{n,N_1}y_n\| + M_2|\lambda_{n+1,N} - \lambda_{n,N}| \\
&\leq \|U_{n+1,N_2}y_n - U_{n,N_2}y_n\| \\
&\quad +M_2|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + M_2|\lambda_{n+1,N} - \lambda_{n,N}| \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \|U_{n+1,1}y_n - U_{n,1}y_n\| + M_2\sum_{i=2}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\leq M_2\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \qquad (3.16)
\end{aligned}
$$

Using (3.12) and (3.16) in (3.13), we have

$$
\begin{aligned}
\|l_{n+1} - l_n\| \leq\; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(K_{n+1}x_{n+1})\| + \|DK_{n+1}y_{n+1}\|) \\
& +\tfrac{\alpha_n}{1-\beta_n}(\|DK_ny_n\| + \|\gamma f(K_nx_n)\|) \\
& +\|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n + M_1|\mu_n - \mu_{n+1}| \\
& +M_2\textstyle\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq\; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(K_{n+1}x_{n+1})\| \\
& +\|DK_{n+1}y_{n+1}\|) + \tfrac{\alpha_n}{1-\beta_n}(\|DK_ny_n\| \\
& +\|\gamma f(K_nx_n)\|) \\
& +\delta\|A\|\sigma_n + \delta_n + M_1|\mu_n - \mu_{n+1}| \\
& +M_2\textstyle\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned}
$$

Taking lim sup and using the conditions (i)-(v), in above inequality, we have

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.7, we have

$$\lim_{n\to\infty}\|l_n - x_n\| = 0. \qquad (3.17)$$

Since $x_{n+1} = (1-\beta_n)l_n + \beta_nx_n$ therefore

$$\|x_{n+1} - x_n\| = \|(1-\beta_n)(l_n - x_n)\|.$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.18}$$

Now,

$$\begin{aligned} \|x_n - K_n y_n\| &= \|x_n - x_{n+1} + x_{n+1} - K_n y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n \gamma f(K_n x_n) + \beta_n x_n \\ &\quad +((1-\beta_n)I - \alpha_n D)K_n y_n - K_n y_n\| \\ &= \|x_{n+1} - x_n\| + \|\alpha_n(\gamma f(K_n x_n) - DK_n y_n)\| \\ &\quad +((1-\beta_n)I - \alpha_n D)(K_n y_n - K_n y_n) + \beta_n(x_n - K_n y_n) \\ &\leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(K_n x_n) - DK_n y_n\| \\ &\quad +\beta_n\|x_n - K_n y_n\|. \end{aligned}$$

Thus,

$$(1-\beta_n)\|x_n - K_n y_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(K_n x_n) - DK_n y_n\|.$$

Taking limit and using the conditions (i)-(ii) and (3.18) in above inequality, we have

$$\lim_{n\to\infty} \|x_n - K_n y_n\| = 0. \tag{3.19}$$

As $\{x_n\}$ is bounded, we may assume a nonnegative real number K such that $\|x_n - p\| \leq K$. It follows from (3.8) and (2.8) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(K_n x_n) - Dp) + \beta_n(x_n - K_n y_n) \\ &\quad +(1-\alpha_n D)(K_n y_n - p)\|^2 \\ &\leq \|(1-\alpha_n D)(K_n y_n - p) + \beta_n(x_n - K_n y_n)\|^2 \\ &\quad +2\langle \alpha_n \gamma f(K_n x_n) - Bp, x_{n+1} - p\rangle \\ &\leq [\|(1-\alpha_n D)(K_n y_n - p)\| + \beta_n\|x_n - K_n u_n\|]^2 \\ &\quad +2\alpha_n\langle \gamma f(K_n x_n) - Dp, x_{n+1} - p\rangle \\ &\leq [(1-\alpha_n\bar{\gamma})\|y_n - p\| + \beta_n\|x_n - K_n y_n\|]^2 \\ &\quad +2\alpha_n\langle \gamma f(K_n x_n) - Dp, x_{n+1} - p\rangle \\ &= (1-\alpha_n\bar{\gamma})^2\|y_n - p\|^2 + \beta_n^2\|x_n - K_n y_n\|^2 \\ &\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n - p\| \\ &\quad \times\|x_n - K_n y_n\| + 2\alpha_n\langle \gamma f(K_n x_n) - Dp, x_{n+1} - p\rangle \end{aligned} \tag{3.20}$$

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\bar{\gamma})^2[\|x_n-p\|^2+\delta(L\delta-1)\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2] \\
&\quad +\beta_n^2\|x_n-K_ny_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n)-Dp,x_{n+1}-p\rangle \\
&= (1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})^2\delta(L\delta-1)\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2 \\
&\quad +\beta_n^2\|x_n-K_ny_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n)-Bp,x_{n+1}-p\rangle \\
&\leq \|x_n-p\|^2+(\alpha_n\bar{\gamma})^2)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})^2\delta(L\delta-1)\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2 \\
&\quad +\beta_n^2\|x_n-K_nu_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n)-Dp,x_{n+1}-p\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha_n\bar{\gamma})^2\delta(1-L\delta)\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2 \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\beta_n^2\|x_n-K_ny_n\|^2+\alpha_n\bar{\gamma}^2\|x_n-p\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n)-Dp,x_{n+1}-p\rangle \\
&\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\
&\quad +\beta_n^2\|x_n-K_ny_n\|^2+\alpha_n\bar{\gamma}^2\|x_n-p\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
&\quad +2\alpha_n(\gamma\|f(K_nx_n)\|+\|Dp\|)K.
\end{aligned}$$

Since $\delta(1-L\delta)>0$, $\|x_{n+1}-x_n\|\to 0$ and $\|x_n-K_ny_n\|\to 0$ as $n\to\infty$ and from conditions (i) and (ii), we obtain

$$\lim_{n\to\infty}\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|^2=0. \tag{3.21}$$

We estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n) - p\|^2 \\
&\leq \|T_{r_n}^{(F_1,\phi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n) - T_{r_n}^{(F_1,\phi_1)}p\|^2 \\
&\leq \langle u_n - p, x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n - p\rangle \\
&= \frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n - p\|^2 \\
&\quad -\|(u_n - p) - [x_n + \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n - p]\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 \\
&\quad -\|u_n - x_n - \delta A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 \\
&\quad -[\|u_n - x_n\|^2 + \delta^2\|A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|^2 \\
&\quad -2\delta\langle u_n - x_n, A^*(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\rangle]\Big\}.
\end{aligned}$$

Hence, we obtain

(3.22)

$$\|u_n-p\|^2 \leq \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\|.$$

From (3.20), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\alpha_n\bar{\gamma})^2\|y_n - p\|^2 + \beta_n^2\|x_n - K_ny_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n - p\|\|x_n - K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n) - Dp, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2[\|x_n - p\|^2 - \|u_n - x_n\|^2 \\
&\quad +2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\|] \\
&\quad +\beta_n^2\|x_n - K_ny_n\|^2 + 2(1-\alpha_n\bar{\gamma})\beta_n\|y_n - p\|\|x_n - K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n) - Dp, x_{n+1} - p\rangle \\
&\leq \|x_n - p\|^2 + (\alpha_n\bar{\gamma})^2\|x_n - p\|^2 - (1-\alpha_n\bar{\gamma})^2\|u_n - x_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,\phi_2)} - I)Ax_n\| \\
&\quad +\beta_n^2\|x_n - K_ny_n\|^2 + 2(1-\alpha_n\bar{\gamma})\beta_n\|y_n - p\|\|x_n - K_ny_n\| \\
&\quad +2\alpha_n\langle\gamma f(K_nx_n) - Dp, x_{n+1} - p\rangle. \qquad (3.23)
\end{aligned}$$

Thus,

$$
\begin{aligned}
(1-\alpha_n\bar{\gamma})^2\|u_n-x_n\|^2 \leq\; & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
& +\beta_n^2\|x_n-K_ny_n\|^2+(\alpha_n\bar{\gamma})^2\|x_n-p\|^2 \\
& +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
& +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\| \\
& \times\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\| \\
& +2\alpha_n\langle\gamma f(K_nx_n)-Dp,x_{n+1}-p\rangle \\
\leq\; & (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\
& +\beta_n^2\|x_n-K_ny_n\|^2 \\
& +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\| \\
& \times\|(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n\| \\
& +(\alpha_n\bar{\gamma})^2\|x_n-p\|^2 \\
& +2(1-\alpha_n\bar{\gamma})\beta_n\|y_n-p\|\|x_n-K_ny_n\| \\
& +2\alpha_n(\gamma\|f(K_nx_n)\|+\|Dp\|)K. \qquad (3.24)
\end{aligned}
$$

Using (3.18), (3.19), (3.21) and conditions (i)-(ii), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \qquad (3.25)$$

Next, we prove $\lim_{n\to\infty}\|u_n-y_n\|=0$.

We estimate

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(K_nx_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nD)K_ny_n-p\|^2 \\
&= \|(1-\beta_n)(K_ny_n-p)+\beta_n(x_n-p) \\
&\quad +\alpha_n(\gamma f(K_nx_n)-DK_ny_n)\|^2 \\
&\leq (1-\beta_n)\|K_ny_n-p\|^2+\beta_n\|x_n-p\|^2 \\
&\quad +2\alpha_n\langle\rho_n,x_{n+1}-p\rangle \\
&\leq (1-\beta_n)\|K_ny_n-p\|^2+\beta_n\|x_n-p\|^2+2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\|y_n-p\|^2+\beta_n\|x_n-p\|^2+2\lambda^2\alpha_n. \qquad (3.26)
\end{aligned}
$$

In the above inequality we set $\rho_n = \gamma f(K_n x_n) - DK_n y_n$ and let $\lambda > 0$ be an appropriate constant such that $\lambda \geq \sup_{n,k}\{\|\rho_n\|, \|x_k - p\|\}$. Thus,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\{\|P_C(u_n - \mu_n Bu_n) - P_C(p - \mu_n Bp)\|^2\} \\
&\quad +\beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\{\|u_n - p\|^2 + \mu_n(\mu_n - 2\tau)\|Bu_n - Bp\|^2\} \\
&\quad +\beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\{\|x_n - p\|^2 + \mu_n(\mu_n - 2\tau)\|Bu_n - Bp\|^2\} \\
&\quad +\beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\mu_n(\mu_n - 2\tau)\|Bu_n - Bp\|^2 \\
&\quad +\|x_n - p\|^2 + 2\lambda^2\alpha_n
\end{aligned}$$

which yields,

$$\begin{aligned}
(1-\beta_n)\mu_n(2\tau - \mu_n)\|Bu_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad \times\|x_n - x_{n+1}\| + 2\lambda^2\alpha_n.
\end{aligned}$$

Using conditions (i)-(ii) and (3.18), we have

$$\lim_{n\to\infty}\|Bu_n - Bp\| = 0. \tag{3.27}$$

By using (2.8), we estimate

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \mu_n Bu_n) - P_C(p - \mu_n Bp)\|^2 \\
&\leq \langle y_n - p, (u_n - \mu_n Bu_n) - (p - \mu_n Bp)\rangle \\
&\leq \tfrac{1}{2}\{\|y_n - p\|^2 + \|(u_n - \mu_n Bu_n) \\
&\quad -(p - \mu_n Bp)\|^2 - \|(y_n - u_n) + \mu_n(Bu_n - Bp)\|^2\} \\
&\leq \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n - p\|^2 - \|(y_n - u_n) + \mu_n(Bu_n - Bp)\|^2\} \\
&\leq \|u_n - p\|^2 - \|y_n - u_n\|^2 - \mu_n^2\|Bu_n - Bp\|^2 \\
&\quad +2\mu_n\langle y_n - u_n, Bu_n - Bp\rangle \\
&\leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\mu_n\|y_n - u_n\|\|Bu_n - Bp\| \\
&\leq \|x_n - p\|^2 - \|y_n - u_n\|^2 + 2\mu_n\|y_n - u_n\|\|Bu_n - Bp\|.
\end{aligned}$$

From (3.26), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n \\
&\leq (1-\beta_n)\{\|x_n - p\|^2 - \|y_n - u_n\|^2 \\
&\quad +2\mu_n\|y_n - u_n\|\|Bu_n - Bp\|\} + \beta_n\|x_n - p\|^2 + 2\lambda^2\alpha_n
\end{aligned}$$

which yields

$$\begin{aligned}(1-\beta_n)\|y_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2(1-\beta_n)\mu_n\|y_n-u_n\|\|Bu_n-Bp\|+2\lambda^2\alpha_n\\ &\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad +2(1-\beta_n)\mu_n\|y_n-u_n\|\|Bu_n-Bp\|+2\lambda^2\alpha_n\end{aligned}$$

Using conditions (i)-(ii), (3.18) and (3.27), we have

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.28}$$

Claim 3. We show that $\limsup_{n\to\infty}\langle(\gamma f-D)z, x_n-z\rangle \leq 0$, where $z=P_\Theta(I-D+\gamma f)z$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-B)z, u_n-z\rangle=\lim_{i\to\infty}\langle(\gamma f-B)z, u_{n_i}-z\rangle. \tag{3.29}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to some $w\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup w$. From $\|K_ny_n-x_n\|\to 0$, we obtain $K_ny_{n_i}\rightharpoonup w$.

Now, we prove that $w\in \mathrm{Sol}(\mathrm{GEP}(1.6))$.

Since $u_n=T_{r_n}^{(F_1,\phi_1)}d_n$ where $d_n:=x_n+\delta A^*(T_{r_n}^{(F_2,\phi_2)}-I)Ax_n$, we have

$$F_1(u_n,y)+\phi_1(y,u_n)-\phi_1(u_n,u_n)+\frac{1}{r_n}\langle y-u_n, u_n-d_n\rangle\geq 0,\ \forall y\in C,$$

which implies that

$$\begin{aligned}\phi_1(y,u_n)-\phi_1(u_n,u_n) \quad + \quad &\tfrac{1}{r_n}\langle y-u_n,u_n-d_n\rangle \geq F_1(y,u_n),\ \forall y\in C,\\ &(\text{using monotonocity of } F_1).\end{aligned}$$

Hence,

$$\phi_1(y,u_{n_k})-\phi_1(u_{n_k},u_{n_k})+\langle y-u_{n_k},\frac{u_{n_k}-d_{n_k}}{r_{n_k}}\rangle\geq F_1(y,u_{n_k}),\ \forall y\in C. \tag{3.30}$$

Let $y_t=(1-t)w+ty$ for all $t\in(0,1]$. Since $y\in C$ and $w\in C$, we get $y_t\in C$ and from (3.30), we have

$$\begin{aligned}0 &\leq F_1(y_t,u_{n_k})-\phi_1(y_t,u_{n_k})+\phi_1(u_{n_k},u_{n_k})\\ &\quad -\langle y_t-u_{n_k},\tfrac{u_{n_k}-x_{n_k}}{r_{n_k}}+\delta A^*(\tfrac{(T_{r_{n_k}}^{(F_2,\phi_2)}-I)Ax_{n_k}}{r_{n_k}})\rangle.\end{aligned}$$

Since A^* is bounded linear, it follows from (3.21), (3.25) and $\liminf r_n > 0$ that $\frac{u_{n_k} - x_{n_k}}{r_{n_k}} \to 0$ and $A^*(\frac{(T_{r_{n_k}}^{(F_2,\phi_2)} - I)Ax_{n_k}}{r_{n_k}}) \to 0$ and so

$$\phi_1(y_t, w) - \phi_1(w, w) \leq F_1(y_t, w).$$

Now, for $t > 0$,

$$\begin{array}{rcl} 0 & = & F_1(y_t, y_t) \\ & = & tF_1(y_t, y) + (1-t)F_1(y_t, w) \\ & \geq & tF_1(y_t, y) + (1-t)[\phi_1(y_t, w) - \phi_1(w, w)] \\ & \geq & tF_1(y_t, y) + (1-t)t[\phi_1(y, w) - \phi_1(w, w)] \\ & \geq & F_1(y_t, y) + (1-t)[\phi_1(y, w) - \phi_1(w, w)]. \end{array}$$

Letting $t \to 0$, we have

$$F_1(w, y) + \phi_1(y, w) - \phi_1(w, w) \geq 0, \ \forall y \in C.$$

This implies that $w \in \mathrm{Sol}(\mathrm{GEP}(1.6))$.

Next, we show that $Aw \in \mathrm{Sol}(\mathrm{GEP}(1.7))$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w$ and since A is a bounded linear operator so that $Ax_{n_k} \rightharpoonup Aw$.

Now setting $v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{(F_2,\phi_2)} Ax_{n_k}$. It follows that from (3.21) that $\lim_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{(F_2,\phi_2)} Ax_{n_k}$.

Therefore, from Lemma 2.13, we have

$$\begin{array}{rcl} F_2(Ax_{n_k} & - & v_{n_k}, z) + \phi_1(z, u_{n_k}) - \phi_1(u_{n_k}, u_{n_k}) \\ & + & \frac{1}{r_{n_k}} \langle z - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k} \rangle \geq 0, \ \forall z \in Q. \end{array}$$

Since F_2 is upper semicontinuous in first argument, taking limit superior to above inequality as $k \to \infty$ and using condition (iii), we obtain

$$F_2(Aw, z) + \phi_1(z, u_{n_k}) - \phi_1(u_{n_k}, u_{n_k}) \geq 0, \ \forall z \in Q,$$

which means that $Aw \in \mathrm{Sol}(\mathrm{GEP}(1.7))$ and hence $w \in \Gamma$.

Next, we prove $w \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Then by Lemma 2.11, we have, for every $x \in C$,

$$K_{n_i} x \to Kx. \tag{3.31}$$

Moreover, from Lemma 2.10,

$$\mathrm{Fix}(K) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i).$$

Suppose for contradiction $w \notin \mathrm{Fix}(K)$. Then $w \neq Kw$. From (2.6), (3.19) and (3.31), we have

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Kw\| \\
&\leq \liminf_{i\to\infty} (\|y_{n_i} - K_{n_i} y_{n_i}\| \\
&\quad + \|K_{n_i} y_{n_i} - K_{n_i} w\| + \|K_{n_i} w - Kw\|) \\
&\leq \liminf_{i\to\infty} \|K_{n_i} y_{n_i} - K_{n_i} w\| \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - w\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $w \in \mathrm{Fix}(K)$. Thus $w \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

Next, we prove $w \in \Omega$. Since $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, there exist subsequences $\{u_{n_i}\}$ and $\{y_{n_i}\}$ of $\{u_n\}$ and $\{y_n\}$, respectively such that $u_{n_i} \rightharpoonup w$ and $y_{n_i} \rightharpoonup w$.

Define the mapping M as

$$M(z) = \begin{cases} B(z) + N_C(z), & \text{if } z \in C, \\ \emptyset, & \text{if } z \notin C, \end{cases} \tag{3.32}$$

where $N_C(z) := \{v \in H_1 : \langle z - u, v\rangle \geq 0, \ \forall u \in C\}$ is the normal cone to C at $z \in H_1$. In this case, the mapping M is maximal monotone and hence $0 \in Mz$ mapping if and only if $z \in \mathrm{Sol(VIP(1.2))}$. Let $(z, v) \in \mathrm{graph}(M)$. Then, we have $v \in Mz = Bz + N_C(z)$ and hence $v - Bz \in N_C(z)$. So, we have $\langle z - u, v - Bz\rangle \geq 0$, for all $u \in C$. On the other hand, from $y_n = P_C(u_n - \mu_n Bu_n)$ and $z \in C$, we have

$$\langle (u_n - \mu_n Bu_n) - y_n, y_n - z\rangle \geq 0.$$

This implies that

$$\langle z - y_n, \frac{y_n - u_n}{\mu_n} + Bu_n\rangle \geq 0.$$

Since $\langle z-u, v-Bz\rangle \geq 0$, for all $u \in C$ and $y_{n_i} \in C$, using monotonicity of B, we have

$$\begin{aligned}
\langle z-y_{n_i}, v\rangle &\geq \langle z-y_{n_i}, Bz\rangle \\
&\geq \langle z-y_{n_i}, Bz\rangle - \langle z-y_{n_i}, \frac{y_{n_i}-u_{n_i}}{\mu_n} + Bu_{n_i}\rangle \\
&= \langle z-y_{n_i}, Bz-By_{n_i}\rangle + \langle z-y_{n_i}, By_{n_i}-Bu_{n_i}\rangle \\
&\quad -\langle z-y_{n_i}, \frac{y_{n_i}-u_{n_i}}{\mu_n}\rangle \\
&\geq \langle z-y_{n_i}, By_{n_i}-Bu_{n_i}\rangle - \langle z-y_{n_i}, \frac{y_{n_i}-u_{n_i}}{\mu_n}\rangle.
\end{aligned}$$

Since B is continuous therefore on taking limit $i \to \infty$, we have $\langle z-w, v\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}(0)$ and hence $w \in \Omega$. Thus $w \in \Theta$.

Next, we claim that $\limsup_{n\to\infty}\langle(\gamma f - D)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(I-D+\gamma f)z$. Now from (2.3), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - D)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - D)z, K_n y_n - z\rangle \\
&\leq \limsup_{i\to\infty}\langle(\gamma f - B)z, K_n y_{n_i} - z\rangle \\
&= \langle(\gamma f - B)z, w - z\rangle \\
&\leq 0. \qquad (3.33)
\end{aligned}$$

Claim 4. Finally, we show that $x_n \to w$. Using (3.9) and (3.10), we estimate

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &= \langle \alpha_n(\gamma f(K_n x_n) - Dw) + \beta_n(x_n - w) \\
&\quad +((1-\beta_n)I - \alpha_n D)(K_n y_n - w)\rangle \\
&= \alpha_n\langle \gamma f(K_n x_n) - Dw, x_{n+1} - w\rangle + \beta_n\langle x_n - w, x_{n+1} - w\rangle \\
&\quad +\langle((1-\beta_n)I - \alpha_n D)(K_n y_n - w) - w\rangle \\
&\leq \alpha_n\left(\gamma\langle f(K_n x_n) - f(w), x_{n+1} - w\rangle + \langle \gamma f(w) - Dw, x_{n+1} - w\rangle\right) \\
&\quad +\beta_n\|x_n - w\|\|x_{n+1} - w\| \\
&\quad +\|(1-\beta_n)I - \alpha_n D\|\|K_n y_n - w\|\|x_{n+1} - w\| \\
&\leq \alpha_n\alpha\gamma\|x_n - w\|\|x_{n+1} - w\| + \alpha_n\langle \gamma f(w) - Dw, x_{n+1} - w\rangle \\
&\quad +\beta_n\|x_n - w\|\|x_{n+1} - w\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - w\|\|x_{n+1} - w\| \\
&\leq \alpha_n\alpha\gamma\|x_n - w\|\|x_{n+1} - w\| + \alpha_n\langle \gamma f(w) - Dw, x_{n+1} - w\rangle \\
&\quad +\beta_n\|x_n - w\|\|x_{n+1} - w\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - w\|\|x_{n+1} - w\| \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - w\|\|x_{n+1} - w\| \\
&\quad +\alpha_n\langle \gamma f(w) - Dw, x_{n+1} - w\rangle \\
&\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2}\left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2\right) \\
&\quad +\alpha_n\langle \gamma f(w) - Dw, x_{n+1} - w\rangle \\
&\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2}\|x_n - w\|^2 + \frac{1}{2}\|x_{n+1} - w\|^2 \\
&\quad +\alpha_n\langle \gamma f(w) - Dw, x_{n+1} - w\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(3.34) \quad \|x_{n+1} - w\|^2 &\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - w\|^2 \\
&\quad +2\alpha_n(\langle \gamma f(w) - Dw, x_{n+1} - w\rangle \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - w\|^2 + 2\alpha_n M_n.
\end{aligned}
$$

Since $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, it is easy to see that $\lim \sup\limits_{n\to\infty} M_n \leq 0$. Hence, from (3.33), (3.34) and Lemma 2.9, we deduce that $x_n \to w$, where $w = P_\Theta(I + \gamma f - D)$. This completes the proof.

□

Finally, we have the following consequence of Theorem 3.1, which is obtained by taking $A = I, H_1 = H_2, C = Q, F_1 = F_2, \phi_1 = \phi_2$ in Theorem 3.1.

Corollary 3.2. *Let H_1 be a real Hilbert space and let C be a nonempty closed convex subset of H_1. Assume that $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be nonlinear mappings satisfying Assumption 2.12. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, ..., N$ such that $\Theta = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \bigcap \mathrm{Sol}(\mathrm{GEP}(1.6)) \bigcap \Omega \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and D be a strongly positive bounded linear self adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $B : C \to H_1$ be a τ-inverse strongly monotone mapping. For a given $x_0 \in C$ arbitrarily, let the sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by the following iterative schemes:*

$$\begin{aligned} u_n &= T_{r_n}^{(F_1,\phi_1)}(x_n); \\ y_n &= P_C(u_n - \mu^n B(u_n)) \\ x_{n+1} &= \alpha_n \gamma f(K_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)K_n y_n, \end{aligned}$$

where $\{\mu_n\} \subset (0, 2\tau)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ are sequences in $(0, 1)$, and $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, ..., N-1$ and $0 < \lambda_N \leq 1$, $\lambda_{n,i} \to \lambda_i (i = 1, 2, ..., N)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(iv) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 2\tau$ and $\lim_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$;
(v) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$, $\forall i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converges strongly to some $z_* \in \Theta$, where $z_* = P_\Theta(\gamma f + (I - D))z_*$ which solves the following variational inequality:

$$\langle (D - \gamma f)z_*, z - z_* \rangle \geq 0, \text{ for any } z \in \Theta.$$

The following corollary is due to Kangtunyakarn and Suantai [10], which is obtained by taking $A = I, H_1 = H_2, C = Q, F_1 = F_2, \phi_1 = \phi_2$ and $B = 0$ in Theorem 3.1.

COROLLARY 3.3. *Let H_1 be a real Hilbert space and let C be a nonempty closed convex subset of H_1. Assume that $F_1 : C \times C \to \mathbb{R}$ be a nonlinear mapping satisfying Assumption 2.12. Let $T_i : C \to C$ be a nonexpansive mapping for each $i = 1, 2, ..., N$ such that $\Theta = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \bigcap \mathrm{Sol}(\mathrm{EP}(1.3)) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0,1)$ and D be a strongly positive bounded linear self adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. For a given $x_0 \in C$ arbitrarily, let the sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by the following iterative schemes:*

$$\begin{aligned} u_n &= T_{r_n}^{(F_1)}(x_n); \\ x_{n+1} &= \alpha_n \gamma f(K_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)K_n u_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ are sequences in $(0,1)$, and $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, ..., N-1$ and $0 < \lambda_N \leq 1$, $\lambda_{n,i} \to \lambda_i (i = 1, 2, ..., N)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(iv) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$, $\forall i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converges strongly to some $z^* \in \Theta$, where $z^* = P_\Theta(\gamma f + (I - D))z^*$.

Acknowledgements

The authors are grateful to the referees for useful suggestions which improve the contents of this paper.

References

[1] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, London, (2011).

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. **63** (1994), 123–145.

[3] C. Byrne, Y. Censor, A. Gibali and S. Reich, *Weak and strong convergence of algorithms for the split common null point problem*, J. Nonlinear Convex Anal., **13** (4) (2012), 759–775.

[4] L.C. Ceng and J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., **214** (2008), 186–201.

[5] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems in intensity modulated radiation therapy*, Physics in Medicine and Biology, **51** (2006), 2353–2365.

[6] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, **59** (2012), 301–323.

[7] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136.

[8] B. Djafari Rouhani, K. R. Kazmi and Mohd Farid, *Common solutions to some systems of variational inequalities and fixed point problems*, Fixed Point Theory, **18** (1) (2017), 167–190.

[9] P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equation*, Acta Mathematica, **115** (1966), 271–310.

[10] A. Kangtunyakarn and S. Suantai, *A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings*, Nonlinear Analysis, **71** (10) (2009), 4448–4460.

[11] K.R. Kazmi, A. Khaliq and A. Raouf, *Iterative approximation of solution of generalized mixed set-valued variational inequality problem*, Math. Inequal. Appl., **10** (2007), 677–691.

[12] K.R. Kazmi and S.H. Rizvi, *Iterative algorithms for generalized mixed equilibrium problems*, J. Egyptian Math. Soc., **21** (3) (2013), 340–345.

[13] K.R. Kazmi and S.H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egyptian Math. Soc., **21** (2013), 44–51.

[14] K.R. Kazmi and S.H. Rizvi, *An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping*, Optimization Letters, **8** (3) (2014), 1113–1124.

[15] G. Marino and H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318** (2006), 43–52.

[16] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl., **150** (2011), 275–283.

[17] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems, **26** (5) (2010), 6 Pages.

[18] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (4) (1967), 595–597.

[19] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Transactions of the American Mathematical Society, **149** (1970), 75-88.

[20] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305** (2005), 227–239.

[21] S. Takahashi and W. Takahashi, *Viscosity approximation method for equilibrium problems and fixed point problems in Hilbert space*, J. Math. Anal. Appl., **331** (2007), 506–515.

[22] H.K. Xu, *Viscosity approximation method for nonexpansive mappings*, J. Math. Anal. Appl., **298** (2004), 279–291.

Mohammad Farid
Unaizah College of Engineering
Qassim University-51911, Kingdom of Saudi Arabia
E-mail: mohdfrd55@gmail.com

Kaleem Raza Kazmi
Department of Mathematics
Faculty of Science and Arts - Rabigh
King Abdulaziz University
Jeddah, Kingdom of Saudi Arabia.

Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
E-mail: krkazmi@gmail.com