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IDEAL RIGHT-ANGLED PENTAGONS IN HYPERBOLIC 4-SPACE

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Abstract. An ideal right-angled pentagon in hyperbolic 4-space $\mathbb{H}^4$ is a sequence of oriented geodesics $(L_1, \ldots, L_5)$ such that L_i intersects L_{i+1}, $i = 1, \ldots, 4$, perpendicularly in $\mathbb{H}^4$ and the initial point of L_1 coincides with the endpoint of L_5 in the boundary at infinity $\partial\mathbb{H}^4$. We study the geometry of such pentagons and the various possible augmentations and prove identities for the associated quaternion half side lengths as well as other geometrically defined invariants of the configurations. As applications we look at two-generator groups $\langle A, B\rangle$ of isometries acting on hyperbolic 4-space such that A is parabolic, while B and AB are loxodromic.

1. Introduction

The main purpose of this paper is to investigate configurations of ideal right-angled pentagons and their applications to two-generator groups of isometries acting on hyperbolic 4-space. We define *an ideal right-angled pentagon* in hyperbolic 4-space $\mathbb{H}^4$ to be a sequence of oriented geodesics $(L_1, \ldots, L_5)$ such that L_i intersects L_{i+1}, $i = 1, \ldots, 4$, perpendicularly in $\mathbb{H}^4$ and the initial point of L_1 coincides with the endpoint of L_5 in the boundary at infinity $\partial\mathbb{H}^4$, see Figure 1. It can be viewed as a degeneration of a right-angled hexagon, namely, when one of the sides has (real) length approaching zero, while the adjacent sides have (real) lengths approaching infinity.

The geometry of right-angled hexagons is one of the fundamental elements in the study of hyperbolic geometry. For example, in dimension 2, it is used to study the discreteness of two-generator groups of isometries [4], the collar lemma [5, 12] and the Fenchel-Nielsen coordinates of Teichmüller space [10].

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We can also see the applications and generalizations of the geometry of right-angled hexagons in hyperbolic 3-space [19, 20] and complex hyperbolic space [11, 15, 17]. An ideal right-angled pentagon naturally appears in the process of deforming a two-generator group or a geodesic on a surface.

As a generalization of such configurations in hyperbolic 2 and 3-space, right-angled hexagons in hyperbolic 4-space were studied by Tan, Wong and Zhang in [21]. In particular, they showed how to augment the alternate sides of the hexagon with geodesic planes which were perpendicular to the incident sides, and in this way, defined associated quaternion half lengths for these augmented hexagons. They then proved generalizations of the Delambre-Gauss formulas for these half lengths. In this paper, we augment an ideal right-angled pentagon $(L_1, \ldots, L_5)$ in two different ways and prove various formulas of the quaternion half lengths associated to the augmented ideal right-angled pentagons.

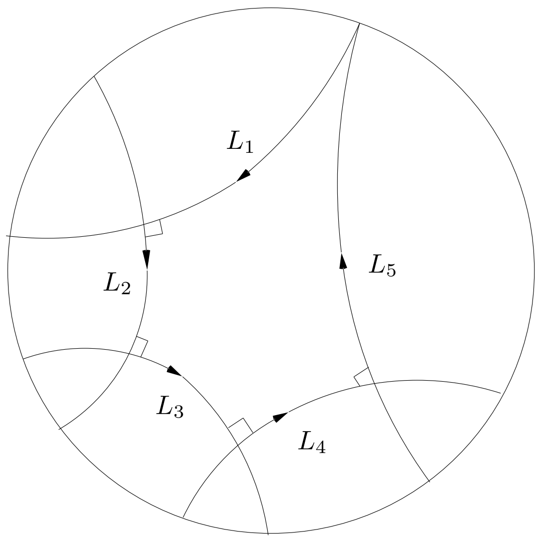

FIGURE 1. An ideal right-angled pentagon

Throughout the paper, we assume that all maps and isometries are orientation-preserving unless otherwise noted. We shall adopt the convention that all lines and all planes in hyperbolic 4-space are oriented geodesic lines and oriented geodesic 2-dimensional planes and they are denoted by L and Π respectively.

Firstly, we augment the pentagon $(L_1, \ldots, L_5)$ with two planes $\Pi_2 \supset L_2$ and $\Pi_4 \supset L_4$ such that Π_2 is perpendicular to L_1 and L_3, and Π_4 is perpendicular to L_3 and L_5 (see Figure 3). In this way, we can define complex half lengths δ_2 and δ_4 for L_2 and L_4 respectively, and a quaternion half length δ_3 for L_3. Lemma 3.1 gives the relation between δ_2, δ_3 and δ_4.

The lines L_1 and L_5 have infinite length, but we can still extract useful geometric invariants by introducing an oriented horocycle C_0, intersecting with and perpendicular to L_1 and L_5, and with orientation from L_5 to L_1. We can make C_0 unique by requiring it to have Euclidean length 1 between L_1 and L_5 (see Figure 2). We can also augment C_0 with an oriented 2-dimensional horosphere $S_0 \supset C_0$ which is perpendicular to both L_1 and L_5. The choice for S_0 is not canonical, but there is a one-dimensional family of choices.

Let p_{01} (resp. p_{50}) be the point of intersection between C_0 and L_1 (resp. C_0 and L_5). For the chosen S_0, we take a geodesic L_0 (resp. L_6) which is tangent to C_0 at p_{01} (resp. p_{50}) and a plane Π_0 (resp. Π_6) which is tangent to S_0 and contains L_0 (resp. L_6). In this way, we can choose canonically augmented pairs (L_0, Π_0) and (L_6, Π_6) although the choice of S_0 is not canonical so that we may define quaternion half lengths δ_1 and δ_5 for the sides L_1 and L_5 (See Figure 4).

This is similar to the lambda length construction of Penner in the two dimensional case. Theorem 3.2 gives the formulas of the half lengths δ_i, $i = 1, \ldots, 5$. We will see in Remark 3.3 that Equation (3) gives us the same pentagon formula (43) in hyperbolic 2-space if the initial ideal right-angled pentagon $(L_1, \ldots, L_5)$ is embedded in a 2-dimensional geodesic plane of $\mathbb{H}^4$. Thus, this is a direct generalization into $\mathbb{H}^4$ of the pentagon formula in hyperbolic 2-space.

Theorem 3.2. *For an augmented ideal right-angled pentagon* $(L_1, \Pi_2, L_3, \Pi_4, L_5)$*, the horocyle/horosphere pair* $C_0 \subset S_0$ *assigned to the ideal vertex, and canonically chosen pairs* (L_0, Π_0) *and* (L_6, Π_6)*, let* δ_i*,* $i = 1, \ldots, 5$ *be the associated quaternion or complex half lengths of the augmented pentagon. Then we have the following formulas*:

$$\text{(1)} \qquad \exp\delta_1 \Big(\cosh\delta_2 \exp\delta_3 \cosh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \sinh\delta_4\Big) \exp\delta_5 = \pm e_2;$$

$$\text{(2)} \qquad \Big(\cosh\delta_2 \exp\delta_3 \cosh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \sinh\delta_4\Big) \exp\delta_5 \exp\delta_5^*$$
$$= \cosh\delta_2 \exp\delta_3 \sinh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \cosh\delta_4;$$

$$\text{(3)} \qquad \Big(\cosh\delta_2 \exp\delta_3 \sinh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \cosh\delta_4\Big)$$
$$\times \Big(\cosh\delta_4^* \exp\delta_3^* \sinh\delta_2^* + \sinh\delta_4^* \exp(-\delta_3) \cosh\delta_2^*\Big) = -1;$$

$$\text{(4)} \qquad \sinh\delta_2 \exp\delta_3 \sinh\delta_4 + \cosh\delta_2 \exp(-\delta_3^*) \cosh\delta_4 = 0.$$

On the other hand, we may also augment the sides L_1, L_3 and L_5 by three planes $\Pi_1 \supset L_1$, $\Pi_3 \supset L_3$ and $\Pi_5 \supset L_5$ (see Figure 7). The plane Π_3 is chosen to be perpendicular to L_2 and L_4, and is generically unique. A priori, Π_1 and Π_5 are only required to be perpendicular to L_2 and L_4 respectively, so generically, there is a one-dimensional family of choices for Π_1 and Π_5 respectively. This augmentation now gives quaternion half lengths δ_2 and δ_4 for L_2 and L_4

respectively and a complex half length δ_3 for L_3. Lemma 3.4 is about the relation between these lengths.

For the purpose of applications to two-generator groups of isometries acting on hyperbolic 4-space, we want to consider all possible choices for Π_1 and Π_5. On the other hand, from a geometric point of view, there are more canonical choices which would pin down Π_1 and Π_5 precisely. We may use the horocycle C_0 defined in the earlier augmentation to do this. Namely, we now require Π_1 and Π_5 to be perpendicular to C_0 as well as L_2 and L_4 respectively (see Figure 8). Generically, this extra condition determines Π_1 and Π_5 and we now have complex half lengths δ_1 and δ_5 for L_1 and L_5 respectively. We also have an angle invariant θ_0 associated to C_0 as generically, the isometry leaving C_0 invariant and mapping the pair (L_5, Π_5) to (L_1, Π_1) is a composition of a parabolic isometry with the half rotation angle θ_0 and an involution. Here, the involution is needed only because of the orientation of L_1 and L_5. Theorem 3.5 gives the formulas of the half lengths δ_i, $i = 1, \dots, 5$ and θ_0.

Theorem 3.5. *Let $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ be an augmented ideal right-angled pentagon in $\mathbb{H}^4$ by the horocycle C_0 assigned to the ideal vertex and planes $\Pi_1 \supset L_1$, $\Pi_3 \supset L_3$ and $\Pi_5 \supset L_5$ as above. Let δ_i, $i = 1, \dots, 5$ be the associated quaternion half lengths of the augmented pentagon and θ_0 the half rotation angle associated to C_0. Then we have the following formulas*:

$$\exp\delta_1 \Big(\cosh\delta_2 \exp\delta_3 \cosh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \sinh\delta_4\Big) \exp\delta_5 = \pm\alpha e_2; \tag{5}$$

$$\Big(\cosh\delta_2 \exp\delta_3 \cosh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \sinh\delta_4\Big) \exp\delta_5 \exp\delta_5^* \tag{6}$$
$$= \cosh\delta_2 \exp\delta_3 \sinh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \cosh\delta_4;$$

$$\Big(\cosh\delta_2 \exp\delta_3 \sinh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \cosh\delta_4\Big) \tag{7}$$
$$\times \Big(\cosh\delta_4^* \exp\delta_3^* \sinh\delta_2^* + \sinh\delta_4^* \exp(-\delta_3) \cosh\delta_2^*\Big) = -1;$$

$$\sinh\delta_2 \exp\delta_3 \sinh\delta_4 + \cosh\delta_2 \exp(-\delta_3^*) \cosh\delta_4 = 0, \tag{8}$$

where $\alpha = \cos\theta_0 + \sin\theta_0 e_1 e_2$,

If the half rotation angle θ_0 is 0, Equation (5) is the same as Equation (1). We also note that Equations (6), (7) and (8) are the same as (2), (3) and (4), respectively. This means that the formulas of side lengths of an ideal right-angled pentagon we induced do not depend on how we augment the given pentagon. Therefore, we call Equation (3) (or Equation (7)) *a generalized pentagon formula in hyperbolic 4-space*.

As mentioned earlier, for the purposes of applications to linked two-generator groups of isometries acting on hyperbolic 4-space, we want generic planes $\Pi_1 \supset L_1$ and $\Pi_5 \supset L_5$. We can see this as follows. Let $\mathrm{Isom}(\mathbb{H}^n)$ be the group of orientation-preserving isometries acting on hyperbolic n-space. Let $\langle A, B, C \mid ABC = I \rangle$ be a linked two-generator subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ such that A is parabolic and B, C are loxodromic. By the definition of *linked*, this means that $A = PQ$, $B = QR$ and $C = RP$ where P, Q, R are half turns around planes, say, Π_5, Π_1 and Π_3 respectively. We note that for a plane $\Pi \subset \mathbb{H}^4$, a half turn around Π is a π-rotation around Π in this paper. Since $A = PQ$ is parabolic, $\overline{\Pi}_5 \cap \overline{\Pi}_1$ is a point $p_\infty \in \partial\mathbb{H}^4$ (where $\overline{\Pi}$ denotes $\Pi \cup \partial\Pi \subset \mathbb{H}^4 \cup \partial\mathbb{H}^4$). Since B and C are loxodromic, $\overline{\Pi}_1 \cap \overline{\Pi}_3 = \overline{\Pi}_3 \cap \overline{\Pi}_5 = \emptyset$. Denote by L_2 (respectively L_4) the unique line perpendicular to Π_1 and Π_3 (respectively Π_3 and Π_5) and denote by p_{ij} the point of intersection of L_i with Π_j, $|j - i| = 1$. By construction, the line L_3 passing through p_{23} and p_{34} lies in Π_3 and is perpendicular to L_2 and L_4. The line L_1 from p_∞ through p_{12} lies in Π_1 and is perpendicular to L_2, similarly, the line L_5 through p_{45} to p_∞ lies in Π_5 and is perpendicular to L_4. We hence recover the ideal right-angled pentagon $(L_1, \dots, L_5)$ and the second augmentation by Π_1, Π_3, Π_5. Conversely, half turns around Π_1, Π_3 and Π_5 respectively coming from the augmented ideal right-angled pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ generates a linked two-generator subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ with a parabolic isometry A. Note that while Π_3 is generically well defined, Π_1 and Π_5 are not. Lemma 4.1 gives us the relations for this general configuration.

If the associated ideal right-angled pentagon $(L_1, \dots, L_5)$ is embedded in a 2-dimensional plane of $\mathbb{H}^4$, the group generated by the three half turns around Π_1, Π_3 and Π_5 respectively is conjugate to a Fuchsian group by an isometry of $\mathbb{H}^4$. A deformation of the group in $\mathrm{Isom}(\mathbb{H}^4)$ is said to be *type-preserving* if A remains parabolic and B, C loxodromic. We then have the following result.

Theorem 4.3. *Let $\Gamma = \langle A, B, C \mid ABC = 1 \rangle$ be a discrete two-generator subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ where A is parabolic and B, C are loxodromic. Then there exists a 6-dimensional parameter space $\mathcal{P}$ containing the identity representation in the deformation space $\mathcal{D}(\Gamma)$ which is the set of discrete, faithful and type-preserving representations of Γ into $\mathrm{Isom}(\mathbb{H}^4)$ up to the conjugation action of $\mathrm{Isom}(\mathbb{H}^4)$.*

2. Preliminaries and notations

In this section we will give definitions and basic facts of hyperbolic 4-space $\mathbb{H}^4$, Vahlen matrices and geometric configurations. For the basics on hyperbolic geometry, the reader is referred to [7, 8, 16, 18]; for Vahlen matrices, to [1–3, 9, 13, 14, 21–23]. We will follow the notation and terminology of [21] closely as many of the basic technical details for our results are worked out there.

2.1. Hyperbolic 4-space and Vahlen matrices

We first describe the setting for hyperbolic 4-space $\mathbb{H}^4$ and its boundary at infinity $\partial\mathbb{H}^4$. An isometry of hyperbolic space can be represented as a 2×2 matrix whose entries are the Clifford numbers satisfying some conditions. The action of the 2×2 matrix is the usual action of Möbius transformations. This is a natural generalization of the classical settings, $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, via identifying the real numbers $\mathbb{R}$ with the Clifford algebra $\mathbb{A}_0$ and the complex numbers $\mathbb{C}$ with the Clifford algebra $\mathbb{A}_1$.

The Clifford algebra $\mathbb{A}_n$ is the associative algebra over the real numbers generated by the elements $e_1, e_2, \ldots, e_n$ subject to the relations $e_i^2 = -1$ for all $i = 1, \ldots, n$ and $e_i e_j = -e_i e_j$ for $i \neq j$. The Clifford algebra $\mathbb{A}_2$ is isomorphic to the quaternions as an algebra, where we identify e_1, e_2 and $e_1 e_2$ with i, j and k respectively. This identification however does not take into account the grading of $\mathbb{A}_2$.

As a graded real algebra with $e_1, \ldots, e_n$ all having degree 1, $\mathbb{A}_n$ decomposes as the direct sum of its degree p vector subspaces $\mathbb{A}_n^{(p)}$, $p = 0, 1, \ldots, n$, with $\mathbb{A}_n^{(0)} = \mathbb{R}$, and, every element $a \in \mathbb{A}_n$ decomposes as $a = a^{(0)} + a^{(1)} + \cdots + a^{(n)}$, where, for $p = 0, 1, \ldots, n$, $a^{(p)} \in \mathbb{A}_n^{(p)}$ is the degree p part of a. We identify the Euclidean space $\mathbb{R}^{n+1}$ with $\mathbb{A}_n^{(0)} + \mathbb{A}_n^{(1)}$, denoted by $\mathbb{A}_n^{(0,1)}$, which is a vector space with basis $\{1, e_1, \ldots, e_n\}$.

We set the upper half-space model of the hyperbolic 4-space as

$$\mathbb{H}^4 = \{x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R},\ x_3 > 0\} \subset \mathbb{A}_3^{(0,1)}.$$

The boundary at infinity is

$$\partial\mathbb{H}^4 = \hat{\mathbb{R}^3} = \mathbb{A}_2^{(0,1)} \cup \{\infty\} = \{x_0 + x_1 e_1 + x_2 e_2 \mid x_0, x_1, x_2 \in \mathbb{R}\} \cup \{\infty\}.$$

There are three involutions in the Clifford algebra $\mathbb{A}_n$:

(1) The main involution $a \mapsto a'$ is obtained by replacing each e_i with $-e_i$. Thus, $(ab)' = a'b'$ and $(a+b)' = a' + b'$.
(2) The reversion $a \mapsto a^*$ is obtained by replacing each $e_{v_1} e_{v_2} \cdots e_{v_p}$ with $e_{v_p} e_{v_{p-1}} \cdots e_{v_1}$. Therefore, $(ab)^* = b^* a^*$ and $(a+b)^* = a^* + b^*$.
(3) The conjugation $a \mapsto \overline{a}$ is the composition of the main involution and the reversion, i.e., $\overline{a} = (a')^* = (a^*)'$.

Every non-zero $x \in \mathbb{A}_n^{(0,1)}$ is invertible with its multiplicative inverse $x^{-1} = \frac{\overline{x}}{|x|^2}$ where $|x|$ is the Euclidean norm of $\mathbb{A}_n^{(0,1)} \simeq \mathbb{R}^{n+1}$. A Clifford group Γ_n is a multiplicative group generated by all non-zero elements of $\mathbb{A}_n^{(0,1)}$. Note that $\Gamma_n = \mathbb{A}_n \setminus \{0\}$ is true for only $n = 0, 1, 2$.

Definition 2.1. A matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is said to be a *Vahlen matrix* if the following conditions are satisfied:

(1) $a, b, c, d \in \Gamma_n \cup \{0\}$.
(2) $ad^* - bc^* = 1$.
(3) $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^{n+1}$.

A Vahlen matrix A has a multiplicative inverse $A^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ which is also a Vahlen matrix. Hence, all Vahlen matrices form a group, denoted by $\mathrm{SL}(\Gamma_n)$.

A Vahlen matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_n)$ induces a Möbius transformation of $\hat{\mathbb{R}}^{n+1}$ by $Ax = (ax+b)(cx+d)^{-1}$ for any vector $x = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^{n+1}$, and $\infty \mapsto \infty$ if $c = 0$ and $\infty \mapsto ac^{-1}$, $-c^{-1}d \mapsto \infty$ if $c \neq 0$. Moreover, any orientation-preserving Möbius transformation of $\hat{\mathbb{R}}^{n+1}$ can be presented as a Vahlen matrix. Replacing x with $x + x_{n+1}e_{n+1} \in \mathbb{R}^{n+2}$, we can automatically extend the action of A to a Möbius transformation of $\hat{\mathbb{R}}^{n+2}$: $x+x_{n+1}e_{n+1} \mapsto (a(x+x_{n+1}e_{n+1})+b)(c(x+x_{n+1}e_{n+1})+d)^{-1}$. The coefficient of the last generator e_{n+1} of the image is $\frac{x_{n+1}}{|cx+d|^2}$. This shows that the extension keeps hyperbolic space $\mathbb{H}^{n+2}$ invariant. In fact, the group of Vahlen matrices modulo $\pm$I is isomorphic to the group of orientation-preserving isometries of $\mathbb{H}^{n+2}$, denoted by $\mathrm{Isom}(\mathbb{H}^{n+2})$.

From now on, we will only consider the case where $n = 2$ which corresponds to hyperbolic 4-space $\mathbb{H}^4$.

Theorem 2.2 ([9]). (1) *An isometry* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ *is loxodromic if and only if* $|\lambda| \neq 1$.

(2) *An isometry* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ *with* $|\lambda| = 1$ *is*

$$\begin{cases} \textit{strictly parabolic if } \lambda \in \mathbb{R}, \\ \textit{screw parabolic if } \mu \notin \mathbb{R}^3, \\ \textit{elliptic otherwise.} \end{cases}$$

In particular, a parabolic isometry P_θ which fixes ∞, rotates around $\langle 1 \rangle$-axis by 2θ and sends 0 to 1 is of the form $P_\theta = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}$ where $\alpha = \cos\theta + \sin\theta e_1e_2 \in \mathbb{A}_2$, $0 \leq \theta < \pi$. We call θ *the half rotation angle of* P_θ. Any parabolic isometry can be conjugate to P_θ by an isometry.

For a loxodromic isometry $f \in \mathrm{Isom}(\mathbb{H}^4)$, we denote the axis of f by $\mathrm{Axis}(f)$, which is the geodesic line connecting the two fixed points.

2.2. The quaternion exponential and hyperbolic functions

Following [21], we define the quaternion exponential function $\exp : \mathbb{A}_2 \to \mathbb{A}_2$ by

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{A}_2.$$

In general,

$$\exp x \exp(-x) = \exp(-x) \exp x = 1, \tag{9}$$

$$\exp(x^*) = (\exp x)^*, \tag{10}$$

$$\exp x \exp y \neq \exp y \exp x \neq \exp(x+y). \tag{11}$$

The quaternion hyperbolic functions $\cosh, \sinh : \mathbb{A}_2 \to \mathbb{A}_2$ are defined by

$$\cosh x = \frac{\exp x + \exp(-x^*)}{2}, \tag{12}$$

$$\sinh x = \frac{\exp x - \exp(-x^*)}{2}. \tag{13}$$

We note that

$$\cosh(-x) = \cosh x^*, \tag{14}$$

$$\sinh(-x) = -\sinh x^*, \tag{15}$$

$$\cosh x + \sinh x = \exp x, \tag{16}$$

$$\cosh x - \sinh x = \exp(-x^*), \tag{17}$$

$$\cosh x \cosh x^* - \sinh x \sinh x^* = 1. \tag{18}$$

Since the quaternions do not commute in general, we also note that $\sinh x \neq \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ and $\cosh x \neq \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

2.3. Geometric configurations

Here we consider geometric configurations of lines and planes that we will investigate in what follows.

For distinct points $u, v \in \partial\mathbb{H}^4$, let $L_{[u,v]}$ denote the oriented geodesic from u to v. If u, v, x, y are four distinct points in $\partial\mathbb{H}^4$ such that $L_1 := L_{[u,v]}$ and $L_2 := L_{[x,y]}$ intersect in $\mathbb{H}^4$, denote the oriented plane spanned by L_1 and L_2 with orientation determined by the oriented frame (L_1, L_2) by $L_1 \vee L_2$ or $\Pi_{[u,v]\vee[x,y]}$. In particular, $L_1 \vee L_2$ and $L_2 \vee L_1$ are the same planes but with opposite orientations. A point-line-plane flag configuration (a *PLP-configuration* for short) is a triple (p, L, Π) where $p \in L \subset \Pi \subset \mathbb{H}^4$.

Definition 2.3. An ideal right-angled pentagon in $\mathbb{H}^4$ is a sequence of lines $(L_1, \ldots, L_5)$ such that L_i intersects L_{i+1}, $i = 1, \ldots, 4$, perpendicularly in $\mathbb{H}^4$ and the initial point of L_1 coincides with the endpoint of L_5 in $\partial\mathbb{H}^4$ (see Figure 1).

To the ideal vertex of an ideal right-angled pentagon $(L_1, \ldots, L_5)$ we assign the *horocycle/horosphere pair* (C_0, S_0) as follows. Let u_i and v_i $(i = 1, \ldots, 5)$ be the starting and ending points of L_i in $\partial\mathbb{H}^4$, so $L_i = L_{[u_i,v_i]}$ and $u_1 = v_5$. Denote by p_{ij} the point of intersection in $\mathbb{H}^4$ between L_i and L_{i+1}, $1 \leq i \leq 4$. Let C_0 be the horocycle centered at $u_1 = v_5$ of height 1 which intersects with L_1 and L_5 respectively. We define p_{01} (respectively p_{50}) to be the point of intersection between C_0 and L_1 (respectively L_5 and C_0). Let S_0 be any horosphere centred at $u_1 = v_5$ containing C_0. If we normalize $(L_1, \ldots, L_5)$ so that $L_1 = L_{[\infty,0]}$ and $L_5 = L_{[1,\infty]}$ (i.e., $u_1 = v_5 = \infty$, $v_1 = 0$ and $u_5 = 1$), then

$$\begin{aligned} &C_0 = \{-x + e_3 \mid x \in \mathbb{R}\}, \\ &p_{01} = e_3, \qquad p_{50} = 1 + e_3, \\ &S_0 = \{x + y(ae_1 + be_2) + e_3 \mid x, y \in \mathbb{R}\}, \end{aligned} \tag{19}$$

where a, b are fixed real constants. In particular, we often normalize them further and take $a = 1$ and $b = 0$ to define S_0 (see Figure 2).

The horocycle/horosphere pair (C_0, S_0) assigned to the ideal vertex of $(L_1, \ldots, L_5)$ determines PLP-configurations at p_{01} and p_{50} as follows: The PLP-configuration at p_{01} is (p_{01}, L, Π) where L is tangent to C_0 at p_{01} and Π is tangent to S_0 at p_{01} and similarly for p_{50}. Again if we normalize as before, and choose $a = 1$ and $b = 0$ for S_0, then the PLP-configurations at p_{01} and p_{50} are

$$(20) \qquad \left(e_3, L_{[1,-1]}, \Pi_{[1,-1]\vee[-e_1,e_1]}\right) \text{ and } \left(1+e_3, L_{[2,0]}, \Pi_{[2,0]\vee[1-e_1,1+e_1]}\right)$$

respectively (see Figure 5). We will measure quaternion half lengths for the infinite lines L_1 and L_5 using these PLP-configurations later.

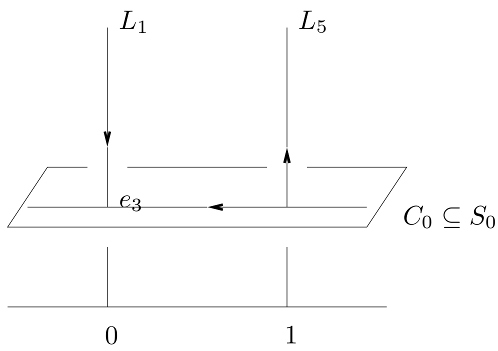

FIGURE 2. The horocycle/horosphere pair (C_0, S_0) assigned to the ideal vertex

2.4. Half side lengths of ideal right-angled pentagons

For a pair of lines L_1 and L_2 which share a common orthogonal line L in hyperbolic space, we may define a distance from L_1 to L_2 along L as a translation length of a loxodromic isometry which keeps L invariant and maps L_1 to L_2. Since we consider oriented lines, such a loxodromic isometry might not exist in the isometries of hyperbolic 2-space. For example, let L_1 and L_2 be $L_{[-1,1]}$ and $L_{[4,-4]}$ respectively. Then the isometry which maps L_1 to L_2 and keeps the common orthogonal geodesic invariant cannot be a loxodromic isometry, but it is an elliptic isometry $\begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R})$. However, in contrast to hyperbolic 2-space, a loxodromic isometry $A = \begin{pmatrix} 2e_1 & 0 \\ 0 & -\frac{1}{2}e_1 \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ maps L_1 to L_2 and keeps the common orthogonal line $L_{[0,\infty]}$ invariant in hyperbolic 4-space. Hence, we may define a quaternion distance from L_1 to L_2 along L as a translation length of A. Yet, such an isometry might not be unique. In fact, a loxodromic isometry $B = \begin{pmatrix} 2e_2 & 0 \\ 0 & -\frac{1}{2}e_2 \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ also maps L_1 to L_2 and keeps $L_{[0,\infty]}$ invariant. The quaternion translation length of B is different from A. It is because a quaternion translation length reflects the rotational action of a loxodromic isometry. Note that A rotates around the plane $\langle 1, e_2 \rangle$ and

B around $\langle 1, e_1 \rangle$. Thus, we need to use flags to define a quaternion distance between lines.

We define a *flag* $F = (L, \Pi)$ to be an ordered pair of a line L and a plane Π such that L is contained in Π. In particular, we define *the horizontal flag* $F_h = (L_h, \Pi_h)$ and *the vertical flag* $F_v = (L_v, \Pi_v)$ where

$$\begin{aligned} L_h &= L_{[-1,1]}, \\ L_v &= L_{[0,\infty]}, \\ \Pi_h &= \Pi_{[-1,1]\vee[-e_1,e_1]}, \\ \Pi_v &= \Pi_{[-e_2,e_2]\vee[0,\infty]}. \end{aligned} \tag{21}$$

The group of isometries $\mathrm{Isom}(\mathbb{H}^4)$ acts transitively on the set of all flags in $\mathbb{H}^4$. That is to say, for a given flag $F = (L, \Pi)$ there exists an isometry which sends L to L_h and Π to Π_h. A line L' is said to be orthogonal to a flag $F = (L, \Pi)$ if the line L' intersects with L orthogonally and is perpendicular to Π.

Let L_1 and L_2 be a pair of lines which share a common orthogonal line L in $\mathbb{H}^4$. For a pair of flags $F_1 = (L_1, \Pi_1)$ and $F_2 = (L_2, \Pi_2)$ which are orthogonal to L, we define *a quaternion distance from F_1 to F_2 along L* by a translation length of a loxodromic isometry which maps F_1 to F_2 and keeps L invariant. For a flag $F = (L, \Pi)$ which is orthogonal to L_1 and L_2, we define *an e_2-complex distance from L_1 to L_2 along F* by a translation length of a loxodromic isometry which maps L_1 to L_2 and keeps F invariant. We refer the precise definitions to [21] and provide here two propositions which we use in what follows.

Proposition 5.12 ([21])**.** *Suppose $F_1 = (L_1, \Pi_1)$ and $F_2 = (L_2, \Pi_2)$ are two flags both orthogonal to a line L in $\mathbb{H}^4$. Let $\tau \in \mathrm{Isom}(\mathbb{H}^4)$ be such that $\tau(L) = L$ and $\tau(F_1) = F_2$, and $\iota \in \mathrm{Isom}(\mathbb{H}^4)$ be such that $\iota(L) = L_v$ and $\iota(F_1) = F_h$. Then the isometry $\iota\tau\iota^{-1}$ has Vahlen matrices $\pm\begin{pmatrix} \exp\delta & 0 \\ 0 & \exp(-\delta^*) \end{pmatrix}$ where $\delta = d_L(F_1, F_2) \in \mathbb{A}_2$ is the quaternion half distance from F_1 to F_2 along L.*

Proposition 5.17 ([21])**.** *Suppose L_1 and L_2 are two lines both orthogonal to a flag $F = (L, \Pi)$ in $\mathbb{H}^4$. Let $\tau \in \mathrm{Isom}(\mathbb{H}^4)$ be such that $\tau(F) = F$ and $\tau(L_1) = L_2$, and $\iota \in \mathrm{Isom}(\mathbb{H}^4)$ be such that $\iota(F) = F_h$ and $\iota(L_1) = L_v$. Then the isometry $\iota\tau\iota^{-1}$ has Vahlen matrices $\pm\begin{pmatrix} \cosh\delta & \sinh\delta \\ \sinh\delta & \cosh\delta \end{pmatrix}$ where $\delta = d_F(L_1, L_2) \in \mathbb{R} + \mathbb{R}e_2$ is the e_2-complex half distance from L_1 to L_2 along F.*

We will close this section with a couple of remarks on isometries we use in the rest of the paper.

Let $i = \frac{e_1+e_2}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ be the isometry which maps the PLP-configuration (e_3, L_h, F_h) to (e_3, L_v, F_v). Note that i is an involution, i.e., $i^2 = id$.

$$i : L_{[-1,1]} \leftrightarrow L_{[0,\infty]},\ L_{[-e_2,e_2]} \leftrightarrow L_{[-e_1,e_1]}. \tag{22}$$

For a plane $\Pi \subset \mathbb{H}^4$, we call a π-rotation around Π a *half turn* around Π. In particular, let I_v (respectively I_h) be the half turn around Π_v (respectively Π_h):

$$I_v = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix}, \quad I_h = iI_v i^{-1} = \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix}. \tag{23}$$

3. Augmented ideal right-angled pentagons

3.1. Augmented pentagons with two planes

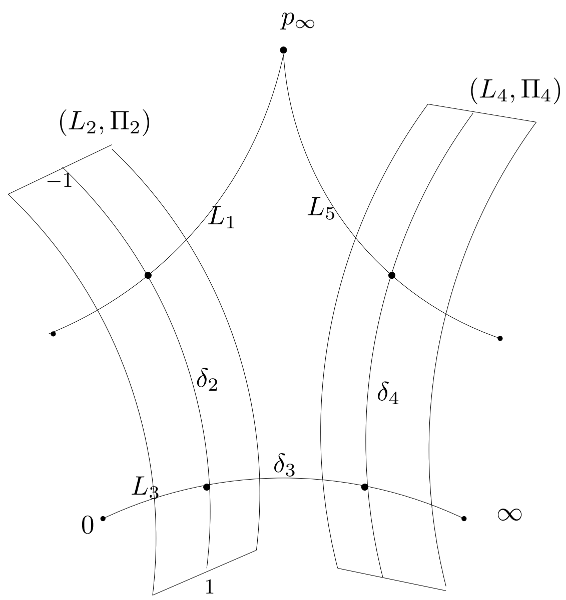

FIGURE 3. A normalized augmented ideal right-angled pentagon $(L_1, \Pi_2, L_3, \Pi_4, L_5)$

Here, we augment an ideal right-angled pentagon $(L_1, \dots, L_5)$ with two planes $\Pi_2 \supset L_2$ and $\Pi_4 \supset L_4$ such that Π_2 is perpendicular to L_1 and L_3, and Π_4 is perpendicular to L_3 and L_5. In this way, we can associate a quaternion half length δ_3 to L_3 and e_2-complex half lengths δ_2 and δ_4 to L_2 and L_4 respectively.

Lemma 3.1. *For an augmented ideal right-angled pentagon $(L_1, \Pi_2, L_3, \Pi_4, L_5)$ with a quaternion half length $\delta_3 \in \mathbb{A}_2$ and e_2-complex half lengths $\delta_2, \delta_4 \in \mathbb{R} + \mathbb{R}e_2$, we have the following formula*

$$\exp \delta_3 (\sinh \delta_4{}^{-1} \cosh \delta_4) \exp \delta_3{}^* = \cosh \delta_2{}^{-1} \sinh \delta_2. \tag{24}$$

Proof. Without loss of generality, we may normalize the configuration so that $(L_2, \Pi_2) = (L_h, \Pi_h)$ and $L_3 = L_v$ (see Figure 3). Let η_i $(i = 2, 3, 4)$ be a loxodromic isometry such that $\mathrm{Axis}(\eta_i) = L_i$ and

$$\begin{aligned} &\eta_2(\Pi_2) = \Pi_2,\ \eta_2(L_1) = L_3, \\ &\eta_3(\Pi_2) = \Pi_4,\ \eta_3(L_2) = L_4, \\ &\eta_4(\Pi_4) = \Pi_4,\ \eta_4(L_3) = L_5. \end{aligned} \tag{25}$$

Then η_2, η_3, η_4 can be written as

$$\begin{aligned} \eta_2 &= \begin{pmatrix} \cosh\delta_2 & \sinh\delta_2 \\ \sinh\delta_2 & \cosh\delta_2 \end{pmatrix}, \\ \eta_3 &= \begin{pmatrix} \exp\delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix}, \\ \eta_4 &= \eta_3 \begin{pmatrix} \cosh\delta_4 & \sinh\delta_4 \\ \sinh\delta_4 & \cosh\delta_4 \end{pmatrix} \eta_3^{-1}. \end{aligned} \tag{26}$$

Then

$$\begin{aligned} &\eta_2^{-1}(0) = -\cosh{\delta_2}^{-1}\sinh\delta_2, \\ &\eta_4(\infty) = \exp\delta_3(-\sinh{\delta_4}^{-1}\cosh\delta_4)\exp(-\delta_3^*)^{-1}. \end{aligned} \tag{27}$$

Equation (24) comes from the fact that $\eta_2^{-1}(0) = \eta_4(\infty)$. □

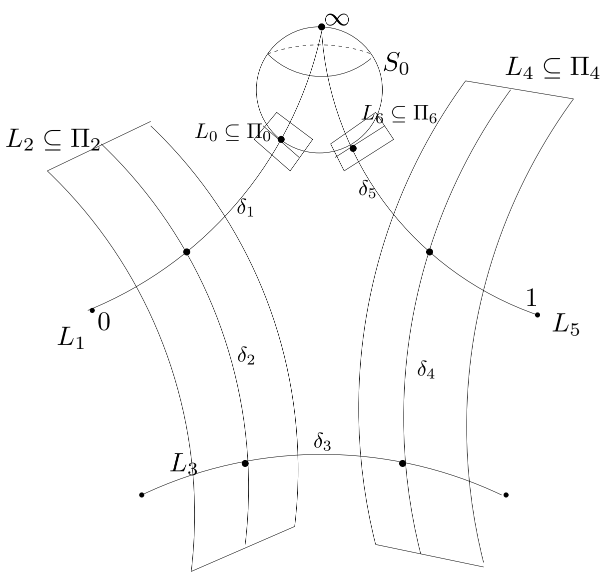

FIGURE 4. An augmented ideal right-angled pentagon $(L_1, \Pi_2, L_3, \Pi_4, L_5)$ with (C_0, S_0)

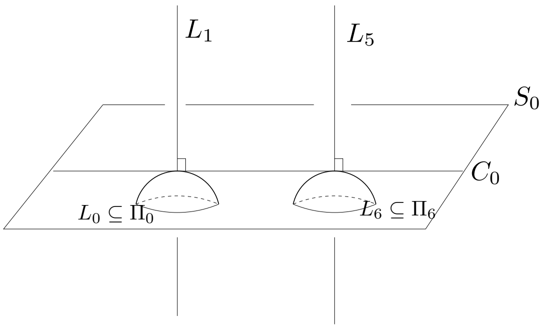

FIGURE 5. PLP-configurations at p_{01} and p_{50}

For the augmented ideal right-angled pentagon $(L_1, \Pi_2, L_3, \Pi_4, L_5)$, consider the horocyle/horosphere pair $C_0 \subset S_0$ assigned to the ideal vertex. Let p_{01} (resp. p_{50}) be the point of intersection between C_0 and L_1 (resp. C_0 and L_5). For the chosen S_0, we take a geodesic L_0 (resp. L_6) which is tangent to C_0 at p_{01} (resp. p_{50}) and a plane Π_0 (resp. Π_6) which is tangent to S_0 and contains L_0 (resp. L_6) (see Figure 4). In this way, we can choose canonically augmented pairs (L_0, Π_0) and (L_6, Π_6) although the choice of S_0 is not canonical.

Theorem 3.2. *For an augmented ideal right-angled pentagon $(L_1, \Pi_2, L_3, \Pi_4, L_5)$, the horocyle/horosphere pair $C_0 \subset S_0$ assigned to the ideal vertex, and canonically chosen pairs (L_0, Π_0) and (L_6, Π_6), let δ_i, $i = 1, \dots, 5$ be the associated quaternion or complex half lengths of the augmented pentagon. Then we have the following formulas*:

$$\exp \delta_1 \left(\cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm e_2; \tag{28}$$

$$\begin{aligned} &\left(\cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \exp \delta_5^* \\ &= \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4; \end{aligned} \tag{29}$$

$$\begin{aligned} &\left(\cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \\ &\times \left(\cosh \delta_4^* \exp \delta_3^* \sinh \delta_2^* + \sinh \delta_4^* \exp(-\delta_3) \cosh \delta_2^* \right) = -1; \end{aligned} \tag{30}$$

$$\sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 = 0. \tag{31}$$

Proof. Without loss of generality, we may assume that the common ideal vertex of L_1 and L_5 is the point $\infty \in \partial \mathbb{H}^4$, $L_0 = L_{[1,-1]}$, $\Pi_0 = \Pi_{[1,-1] \vee [-e_1, e_1]}$, $L_1 = L_{[\infty, 0]}$ and $L_5 = L_{[1, \infty]}$ such that the intersection point of L_1 and Π_0 is $e_3 \in \mathbb{H}^4$. Let η_i $(i = 1, \dots, 5)$ be loxodromic isometries such that $\mathrm{Axis}(\eta_i) = L_i$

and

$$\begin{aligned} \eta_i(L_{i-1}) &= L_{i+1},\ i=1,\dots,5, \\ \eta_i(\Pi_{i-1}) &= \Pi_{i+1},\ i=1,3,5, \\ \eta_i(\Pi_i) &= \Pi_i,\ i=2,4. \end{aligned} \tag{32}$$

In particular, let $\eta_6 = I_h \circ P$ where P is a strictly parabolic isometry which fixes ∞ and translates by -1 and I_h is the half turn around Π_h. So

$$\eta_6(L_5) = L_1,\ \eta_6(L_6) = L_0,\ \eta_6(\Pi_6) = \Pi_0.$$

Then η_6 has a Vahlen matrix of the form $\pm \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Since the isometry $\eta_6 \cdots \eta_2\eta_1$ maps not only the PLP-configuration (e_3, L_0, Π_0) onto itself, but also L_1 onto itself, it is the identity map, id.

Define ι_0 to be the identity map and for $i = 1, \dots, 6$,

$$\begin{aligned} \iota_i &= (\eta_i \cdots \eta_2\eta_1)^{-1}, \\ \psi_i &= \iota_{i-1}\eta_i\iota_{i-1}^{-1}. \end{aligned} \tag{33}$$

Then

$$\begin{aligned} &\iota_i(L_i) = L_0,\ i=1,3,5, \\ &\iota_i : (L_i, \Pi_i) \mapsto (L_0, \Pi_0),\ i=2,4,5. \end{aligned} \tag{34}$$

In particular, for $i = 1, 3, 5$ (respectively $i = 2, 4$) ψ_i keeps 0 and ∞ (respectively -1 and 1). Thus, ψ_i has a Vahlen matrix of the following form:

$$\psi_i = \begin{cases} \begin{pmatrix} \exp\delta_i & 0 \\ 0 & \exp(-\delta_i^*) \end{pmatrix} & \text{if } i = 1,3,5, \\ \begin{pmatrix} \cosh\delta_i & \sinh\delta_i \\ \sinh\delta_i & \cosh\delta_i \end{pmatrix} & \text{if } i = 2,4. \end{cases} \tag{35}$$

Therefore,

$$\begin{aligned} \psi_1\psi_2\cdots\psi_6 &= \eta_1(\iota_1\eta_2\iota_1^{-1})\cdots(\iota_5\eta_6\iota_5^{-1}) \\ &= \eta_1\eta_1^{-1}\eta_2\eta_1(\eta_2\eta_1)^{-1}\eta_2(\eta_1\eta_1)\cdots(\eta_5\cdots\eta_1)^{-1}\eta_6(\eta_5\cdots\eta_1) \\ &= \eta_6\eta_5\cdots\eta_1 = id. \end{aligned} \tag{36}$$

From the fact that $\psi_6 = \eta_6$ and $\psi_1 \cdots \psi_5 = \psi_6^{-1}$, we have

$$\begin{pmatrix} \exp\delta_1 & 0 \\ 0 & \exp(-\delta_1^*) \end{pmatrix} \begin{pmatrix} \cosh\delta_2 & \sinh\delta_2 \\ \sinh\delta_2 & \cosh\delta_2 \end{pmatrix} \begin{pmatrix} \exp\delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix}$$
$$\begin{pmatrix} \cosh\delta_4 & \sinh\delta_4 \\ \sinh\delta_4 & \cosh\delta_4 \end{pmatrix} \begin{pmatrix} \exp\delta_5 & 0 \\ 0 & \exp(-\delta_5^*) \end{pmatrix} = \pm \begin{pmatrix} e_2 & e_2 \\ e_2 & 0 \end{pmatrix}.$$

Therefore, we have

$$\exp\delta_1 \Big(\cosh\delta_2 \exp\delta_3 \cosh\delta_4 + \sinh\delta_2 \exp(-\delta_3^*) \sinh\delta_4 \Big) \exp\delta_5 = \pm e_2, \tag{37}$$

(38)
$$\exp\delta_1\left(\cosh\delta_2\exp\delta_3\sinh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4\right)\exp(-\delta_5^*)=\pm e_2,$$

(39)
$$\exp(-\delta_1^*)\left(\sinh\delta_2\exp\delta_3\cosh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\sinh\delta_4\right)\exp\delta_5=\pm e_2,$$

(40)
$$\exp(-\delta_1^*)\left(\sinh\delta_2\exp\delta_3\sinh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\cosh\delta_4\right)\exp(-\delta_5^*)=0.$$

Equation (37) and (40) give us (28) and (31) respectively. Equation (29) comes from the fact that the left hand sides of (37) and (38) are the same. Using (38) and (39), we have Equation (30):

$$\begin{aligned}&\exp\delta_1\left(\cosh\delta_2\exp\delta_3\sinh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4\right)\exp(-\delta_5^*)\\&\times\left(\exp(-\delta_1^*)\left(\sinh\delta_2\exp\delta_3\cosh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\sinh\delta_4\right)\exp\delta_5\right)^*\\&=e_2e_2^*=-1.\end{aligned}$$
□

Remark 3.3. In the above theorem, if the ideal right-angled pentagon $(L_1,\ldots,L_5)$ embeds in a plane in $\mathbb{H}^4$, we recover the pentagon formula (43) in hyperbolic 2-space (see Theorem 7.18.1 of [7]). We may assume that the ideal right-angled pentagon $(L_1,\ldots,L_5)$ embeds in the plane $\langle 1,e_3\rangle$ under our normalization during the proof. Then δ_i $(i=2,3,4)$ becomes an e_2-complex number $d_i+\frac{\pi}{2}e_2$ for a real number $d_i\in\mathbb{R}$ and they commute each other. We recall that for an e_2-complex number $d+\frac{\pi}{2}e_2\in\mathbb{R}+\mathbb{R}e_2$,

(41)
$$\begin{aligned}\exp(d+\frac{\pi}{2}e_2)&=e_2\exp d,\\\exp(-d-\frac{\pi}{2}e_2)&=-e_2\exp(-d),\\\cosh(d+\frac{\pi}{2}e_2)&=e_2\sinh d,\\\sinh(d+\frac{\pi}{2}e_2)&=e_2\cosh d,\end{aligned}$$

(see Propositions 5.18 and 5.19 of [21]). Thus, Equation (30) becomes

(42)
$$\begin{aligned}&\left(-e_2\sinh d_2\exp d_3\cosh d_4+e_2\cosh d_2\exp(-d_3)\sinh d_4\right)\\&\times\left(-e_2\sinh d_4\exp d_3\cosh d_2+e_2\cosh d_4\exp(-d_3)\sinh d_2\right)=-1.\end{aligned}$$

Therefore, we have

$$2\cosh(2d_3)\cosh d_2\sinh d_2\cosh d_4\sinh d_4=\sinh^2 d_2\sinh^2 d_4+\cosh^2 d_2\cosh^2 d_4$$

which induces the pentagon formula associated to the ideal right-angled pentagon in hyperbolic 2-space (see Figure 6).

$$\sinh 2d_2 \cosh 2d_3 \sinh 2d_4 = \cosh 2d_2 \cosh 2d_4 + 1. \tag{43}$$

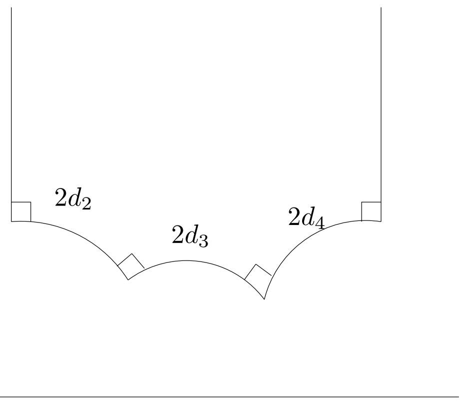

FIGURE 6. An ideal right-angled pentagon in hyperbolic 2-space

3.2. Augmented pentagons with three planes

Now, we augment an ideal right-angled pentagon $(L_1, \dots, L_5)$ with three planes $\Pi_1 \supset L_1$, $\Pi_3 \supset L_3$ and $\Pi_5 \supset L_5$ (See Figure 7). The plane Π_3 is chosen to be perpendicular to L_2 and L_4, and is generically unique. A priori, Π_1 and Π_5 are only required to be perpendicular to L_2 and L_4 respectively, so generically, there is a one dimensional family of choices for Π_1 and Π_5 respectively. This augmentation gives us quaternion half lengths δ_2 and δ_4 for L_2 and L_4 respectively, and an e_2-complex half length δ_3 for L_3. Lemma 3.4 is about the relation between these three lengths.

Lemma 3.4. *For an augmented ideal right-angled pentagon* $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ *with quaternion half lengths* $\delta_2, \delta_4 \in \mathbb{A}_2$ *and an* e_2*-complex half length* $\delta_3 \in \mathbb{R} + \mathbb{R}e_2$*, we have the following formula*

$$\begin{aligned} &\Big(\sinh\delta_3 \exp\delta_4 + \cosh\delta_3 \exp(-\delta_4^*)\Big) \\ &+ \exp\delta_2^* \exp\delta_2 \Big(\cosh\delta_3 \exp\delta_4 + \sinh\delta_3 \exp(-\delta_4^*)\Big) = 0. \end{aligned} \tag{44}$$

Proof. Without loss of generality, we may assume that $L_2 = L_{[\infty,0]}$, $L_3 = L_h$ and $\Pi_3 = \Pi_h$ (see Figure 7). Let η_i $(i = 2, 3, 4)$ be a loxodromic isometry such

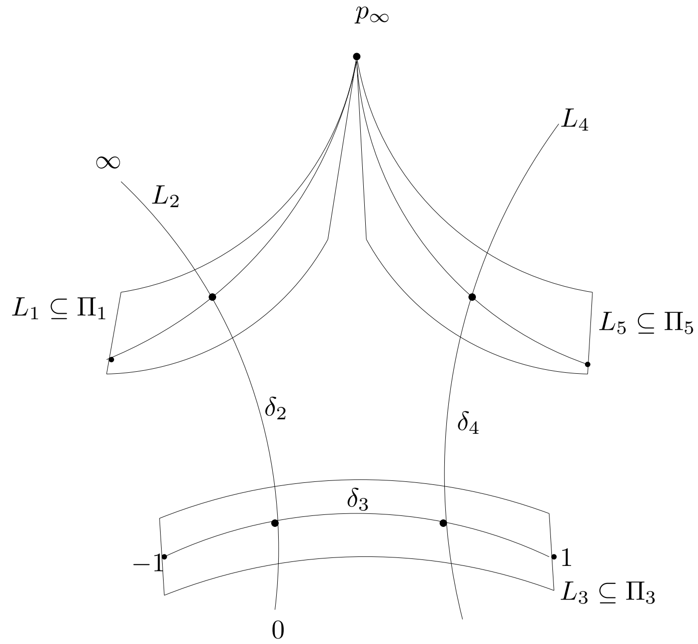

FIGURE 7. A normalized augmented pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$

that $\mathrm{Axis}(\eta_i) = L_i$ and

$$\begin{aligned} &\eta_2 : (L_1, \Pi_1) \mapsto (L_3, \Pi_3), \\ &\eta_3(L_2) = L_4, \ \eta_3(\Pi_3) = \Pi_3, \\ &\eta_4 : (L_3, \Pi_3) \mapsto (L_5, \Pi_5). \end{aligned} \tag{45}$$

Then η_2, η_3, η_4 can be written as

$$\begin{aligned} \eta_2 &= \begin{pmatrix} \exp\delta_2 & 0 \\ 0 & \exp(-\delta_2^*) \end{pmatrix}, \\ \eta_3 &= \begin{pmatrix} \cosh\delta_3 & \sinh\delta_3 \\ \sinh\delta_3 & \cosh\delta_3 \end{pmatrix}, \\ \eta_4 &= \eta_3 \begin{pmatrix} \exp\delta_4 & 0 \\ 0 & \exp(-\delta_4^*) \end{pmatrix} \eta_3^{-1} \\ &= \begin{pmatrix} \cosh\delta_3\exp\delta_4 & \sinh\delta_3\exp(-\delta_4^*) \\ \sinh\delta_3\exp\delta_4 & \cosh\delta_3\exp(-\delta_4^*) \end{pmatrix} \eta_3^{-1}. \end{aligned} \tag{46}$$

Since $\eta_2^{-1}(-1) = \eta_4(1)$,

$$-\exp(-\delta_2)\exp(-\delta_2^*) = \Big(\cosh\delta_3\exp\delta_4 + \sinh\delta_3\exp(-\delta_4^*)\Big) \tag{47}$$

$$\left(\sinh\delta_3\exp\delta_4+\cosh\delta_3\exp(-\delta_4^*)\right)^{-1}$$

which implies Equation (44). □

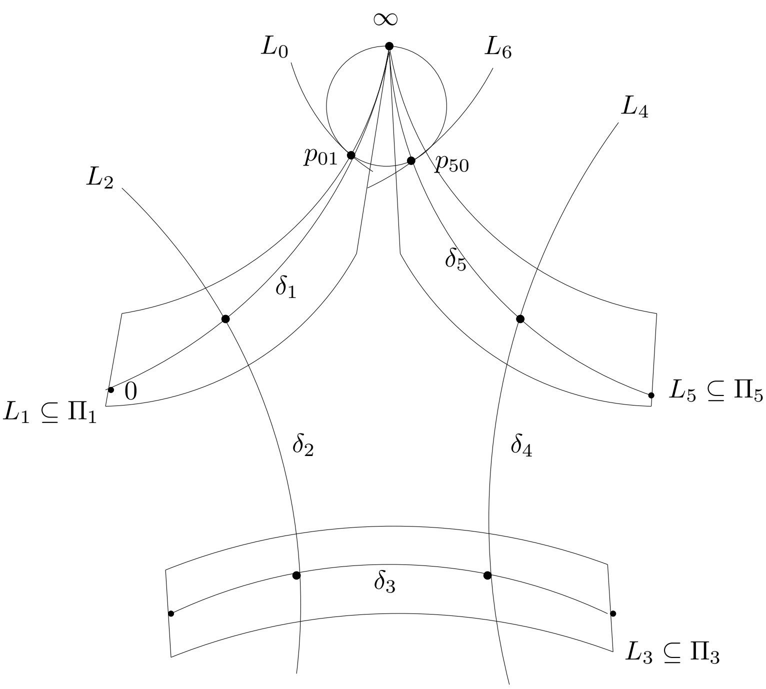

FIGURE 8. An augmented ideal right-angled pentagon with three planes and a horocycle

Now we may assume that the common ideal vertex of L_1 and L_5 is the point $\infty\in\hat{\mathbb{R}}^3$ and will consider a horocycle C_0 based at the ideal vertex ∞ so that we choose planes Π_1 and Π_5 uniquely as follows (See Figure 8). Let C_0 be the horocycle based at the ideal vertex ∞ with height 1 after normalization and intersect with L_1 and L_5. We call the intersection points $C_0\cap L_1$ and $C_0\cap L_5$ by p_{01} and p_{50} respectively. Let L_0 (respectively L_6) be the geodesic passing through p_{01} (respectively p_{50}) and be tangent to the horocycle C_0 at p_{01} (respectively p_{50}). Then, we choose uniquely a plane Π_1 (respectively Π_5) which is perpendicular L_2 and L_0 (respectively L_4 and L_6) and contains L_1 (respectively L_5). We consider an isometry which maps a PLP-configuration (p_{50},L_5,Π_5) to a PLP-configuration (p_{01},L_1,Π_1) and keeps C_0 invariant. In particular, this isometry can be seen as a composition of a parabolic isometry which fixes∞ and a half-turn around Π_1. Using this isometry, we assign an angle invariant θ_0 associated to C_0 as the rotation angle of the parabolic isometry. Theorem 3.5 gives the relations between the half lengths δ_i, $i = 1, \ldots, 5$ and θ_0.

Theorem 3.5. *Let* $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ *be an augmented ideal right-angled pentagon in* $\mathbb{H}^4$ *by the horocycle* C_0 *assigned to the ideal vertex and planes* $\Pi_1 \supset L_1$, $\Pi_3 \supset L_3$ *and* $\Pi_5 \supset L_5$ *as above. Let* δ_i, $i = 1, \ldots, 5$ *be the associated quaternion half lengths of the augmented pentagon and* θ_0 *the half rotation angle associated to* C_0. *Then we have the following formulas*:

$$\begin{aligned}(48)\qquad & \exp\delta_1\Big(\cosh\delta_2\exp\delta_3\cosh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\sinh\delta_4\Big)\exp\delta_5\\ &= \pm\,\alpha e_2;\end{aligned}$$

$$\begin{aligned}(49)\qquad & \Big(\cosh\delta_2\exp\delta_3\cosh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\sinh\delta_4\Big)\exp\delta_5\exp\delta_5^*\\ &= \cosh\delta_2\exp\delta_3\sinh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4;\end{aligned}$$

$$\begin{aligned}(50)\qquad & \Big(\cosh\delta_2\exp\delta_3\sinh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4\Big)\\ &\times\Big(\cosh\delta_4^*\exp\delta_3^*\sinh\delta_2^*+\sinh\delta_4^*\exp(-\delta_3)\cosh\delta_2^*\Big)=-1;\end{aligned}$$

$$(51)\qquad \sinh\delta_2\exp\delta_3\sinh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\cosh\delta_4=0,$$

where $\alpha = \cos\theta_0 + \sin\theta_0 e_1 e_2$.

Remark 3.6. If the half rotation angle θ_0 is 0, Equation (48) is the same as Equation (28). We also note that Equations (49), (50) and (51) are the same as (29), (30) and (31), respectively.

Proof. Without loss of generality, we may assume that the common ideal vertex of L_1 and L_5 is ∞, $L_1 = L_{[\infty,0]}$, $L_5 = L_{[1,\infty]}$, $C_0 = \{-x_0 + e_3 : x_0 \in \mathbb{R}\}$, $L_0 = L_{[1,-1]}$, $p_{01} = e_3$, $L_6 = L_{[2,0]}$, $p_{50} = 1 + e_3$ and $\Pi_1 = \Pi_{[\infty,0]\vee[-e_2,e_2]}$ (see Figure 8).

Let η_i $(i = 1, \ldots, 5)$ be an isometry such that $\mathrm{Axis}(\eta_i) = L_i$ and

$$(52)\qquad \begin{aligned}&\eta_i(L_{i-1}) = L_{i+1},\ i = 1, \ldots, 5,\\ &\eta_i(\Pi_i) = \Pi_i,\ i = 1, 3, 5,\\ &\eta_i(\Pi_{i-1}) = \Pi_{i+1},\ i = 2, 4.\end{aligned}$$

Let $\eta_6 = I_h \circ P$ where P is a screw parabolic isometry which maps a PLP-configuration $(1 + e_3, [1, \infty], \Pi_5)$ to a PLP-configuration $(e_3, [0, \infty], \Pi_1)$. Note that P has a Vahlen matrix of the form $\pm\left(\begin{smallmatrix}\alpha^* & -\alpha^*\\ 0 & \alpha^*\end{smallmatrix}\right)$ where $\alpha = \cos\theta_0 + \sin\theta_0 e_1 e_2$ for some real number $\theta_0 \in \mathbb{R}$. Then, the isometry $\eta_6 \cdots \eta_2\eta_1$ maps the PLP-configuration (e_3, L_1, Π_1) onto itself. In addition, since $\eta_6 \cdots \eta_2\eta_1$ maps L_0 onto itself, it is the identity map *id*.

Define $\iota_0 = \text{id}$ and for $i = 1, \dots, 6$,

$$\begin{aligned} \iota_i &= (\eta_i \cdots \eta_2\eta_1)^{-1}, \\ \psi_i &= \iota_{i-1}\eta_i\iota_{i-1}^{-1}. \end{aligned} \tag{53}$$

Then

$$\begin{aligned} &\iota_i(L_i) = L_0,\ i = 1,3,5, \\ &\iota_i : (L_{i+1}, \Pi_{i+1}) \mapsto (L_1, \Pi_1),\ i = 2,4. \end{aligned} \tag{54}$$

In particular, $\psi_i (i = 1,3,5)$ fixes 0 and ∞, and $\psi_i (i = 2,4)$ fixes 1 and -1. So, their Vahlen matrices are of the form

$$\psi_i = \begin{cases} \begin{pmatrix} \exp\delta_i & 0 \\ 0 & \exp(-\delta_i^*) \end{pmatrix} & \text{if } i = 1,3,5, \\ \begin{pmatrix} \cosh\delta_i & \sinh\delta_i \\ \sinh\delta_i & \cosh\delta_i \end{pmatrix} & \text{if } i = 2,4. \end{cases} \tag{55}$$

Then

$$\begin{aligned} \psi_1\psi_2\cdots\psi_6 &= \eta_1(\iota_1\eta_2\iota_1^{-1})\dots(\iota_5\eta_6\iota_5^{-1}) \\ &= \eta_6\cdots\eta_2\eta_1 = id \end{aligned} \tag{56}$$

and $\psi_6 = \eta_6$. Since $\psi_1\cdots\psi_5 = \psi_6^{-1}$,

$$\begin{aligned} &\begin{pmatrix} \exp\delta_1 & 0 \\ 0 & \exp(-\delta_1^*) \end{pmatrix} \begin{pmatrix} \cosh\delta_2 & \sinh\delta_2 \\ \sinh\delta_2 & \cosh\delta_2 \end{pmatrix} \begin{pmatrix} \exp\delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix} \\ &\begin{pmatrix} \cosh\delta_4 & \sinh\delta_4 \\ \sinh\delta_4 & \cosh\delta_4 \end{pmatrix} \begin{pmatrix} \exp\delta_5 & 0 \\ 0 & \exp(-\delta_5^*) \end{pmatrix} \\ = \ &\pm P^{-1}I_h^{-1} = \pm \begin{pmatrix} \alpha e_2 & \alpha e_2 \\ \alpha e_2 & 0 \end{pmatrix} \\ = \ &\begin{pmatrix} \exp\delta_1\cosh\delta_2\exp\delta_3 & \exp\delta_1\sinh\delta_2\exp(-\delta_3^*) \\ \exp(-\delta_1^*)\sinh\delta_2\exp\delta_3 & \exp(-\delta_1^*)\cosh\delta_2\exp(-\delta_3^*) \end{pmatrix} \\ &\begin{pmatrix} \cosh\delta_4\exp\delta_5 & \sinh\delta_4\exp(-\delta_5^*) \\ \sinh\delta_4\exp\delta_5 & \cosh\delta_4\exp(-\delta_5^*) \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\exp\delta_1\Big(\cosh\delta_2\exp\delta_3\cosh\delta_4 + \sinh\delta_2\exp(-\delta_3^*)\sinh\delta_4\Big)\exp\delta_5 \\ &= \pm\,\alpha e_2, \end{aligned} \tag{57}$$

$$\begin{aligned} &\exp\delta_1\Big(\cosh\delta_2\exp\delta_3\sinh\delta_4 + \sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4\Big)\exp(-\delta_5^*) \\ &= \pm\,\alpha e_2, \end{aligned} \tag{58}$$

$$\begin{aligned} &\exp(-\delta_1^*)\Big(\sinh\delta_2\exp\delta_3\cosh\delta_4 + \cosh\delta_2\exp(-\delta_3^*)\sinh\delta_4\Big)\exp\delta_5 \\ &= \pm\,\alpha e_2, \end{aligned} \tag{59}$$

$$\exp(-\delta_1^*)\left(\sinh\delta_2\exp\delta_3\sinh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\cosh\delta_4\right)\exp(-\delta_5^*) = 0. \tag{60}$$

Equation (57) and (60) give us Equation (48) and (51) respectively. Equation (49) comes from that fact that the left hand sides of (57) and (58) are the same. Using(58) and (59), we have (50):

$$\begin{aligned}&\exp\delta_1\left(\cosh\delta_2\exp\delta_3\sinh\delta_4+\sinh\delta_2\exp(-\delta_3^*)\cosh\delta_4\right)\exp(-\delta_5^*)\\&\quad\times\left(\exp(-\delta_1^*)\left(\sinh\delta_2\exp\delta_3\cosh\delta_4+\cosh\delta_2\exp(-\delta_3^*)\sinh\delta_4\right)\exp\delta_5\right)^*\\&=(\alpha e_2)(e_2\alpha^*)=-1.\end{aligned}$$

□

4. Two-generator subgroups of Isom($\mathbb{H}^4$)

A group $\langle A,B,C \mid ABC=1\rangle$ is said to be *a linked two-generator group* if there exist three involutions P,Q,R so that $A=PQ$, $B=QR$ and $C=RP$. Depending on the dimension of hyperbolic space where the isometries act, involutions may be orientation-reversing. A reflection in a geodesic in hyperbolic 2-space is an orientation-reversing involution. A π-rotation around a geodesic in hyperbolic 3-space is an orientation-preserving involution. Finally, a π-rotation around a plane in hyperbolic 4-space is also an orientation-preserving involution which is called as a half turn in this paper.

Throughout this section, we consider an augmented ideal right-angled pentagon $(\Pi_1,L_2,\Pi_3,L_4,\Pi_5)$. That is, we augment an ideal right-angled pentagon $(L_1,\dots,L_5)$ with three planes $\Pi_1\supset L_1$, $\Pi_3\supset L_3$ and $\Pi_5\supset L_5$ in $\mathbb{H}^4$ such that Π_1 and Π_5 are only required to be perpendicular to L_2 and L_4 respectively, and Π_3 is generically uniquely chosen to be perpendicular to L_2 and L_4 (see Figure 7). This gives us a linked two-generator subgroup of Isom($\mathbb{H}^4$) as follows.

Lemma 4.1. *For an augmented ideal right-angled pentagon* $(\Pi_1,L_2,\Pi_3,L_4,\Pi_5)$*, let* I_i $(i=1,3,5)$ *be a half turn around* Π_i *in* $\mathbb{H}^4$ *and* $A=I_5I_1$, $B=I_1I_3$, $C=I_3I_5$*. Then* $\langle A,B,C\rangle$ *is a two-generator subgroup of* Isom($\mathbb{H}^4$) *such that* A *is parabolic and* B,C *are loxodromic.*

The proof is a straightforward application of the following Propositions.

Proposition 4.1 ([6])**.** *Let* I_1 *and* I_2 *be involutions with fixed point sets* P_1 *and* P_2 *in* $\mathbb{H}^4$ *respectively and let* $f=I_2I_1$*. Then* f *is elliptic if and only if* $P_1\cap P_2\neq\emptyset$*;* f *is parabolic if and only if* $P_1\cap P_2=\emptyset$ *and the hyperbolic distance* $d(P_1,P_2)=0$*;* f *is loxodromic if and only if the hyperbolic distance* $d(P_1,P_2)>0$*.*

Applying the above proposition to the boundary at infinity $\partial\mathbb{H}^4$, we have:

Proposition 4.2. *Let P_i $(i = 1, 2)$ be a circle or a line in $\hat{\mathbb{R}}^3$ and I_i be a half turn around P_i. Then the isometry $f = I_2 I_1$ is*

(1) *elliptic if P_1 and P_2 are disjoint and linked (i.e., a unique fixed point in $\mathbb{H}^4$),*
(2) *loxodromic if P_1 and P_2 are disjoint and non-linked,*
(3) *parabolic if P_1 and P_2 intersect at a single point of $\hat{\mathbb{R}}^3$,*
(4) *elliptic if P_1 and P_2 intersect at two points of $\hat{\mathbb{R}}^3$ (i.e., an element in $\mathrm{SO}(2)$).*

Proof. To show (1), suppose that P_1 and P_2 are disjoint and linked. Without loss of generality, we map assume $\infty \in P_2$. Applying a dilation, a rotation or a translation to P_1 and P_2 in $\hat{\mathbb{R}}^3$ if necessary, we may normalize so that P_1 is a unit circle belonging to $\langle 1, e_1\rangle$ and centered at the origin, and P_2 is a line. Since P_1 and P_2 are disjoint and linked, P_2 must intersect the open unit 2-dimensional disk whose boundary is P_1 in $\langle 1, e_1\rangle$. Let Q be the intersection point in $\partial\mathbb{H}^4$. Then there exist $x_0, y_0 \in \mathbb{R}$ such that $x_0^2 + y_0^2 < 1$ and $Q = x_0 + y_0 e_1$. Thus, we can write $P_1 = \{x + y e_1 \in \mathbb{R}^3 : x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$ and $P_2 = \{t(v_0 + v_1 e_1 + v_2 e_2) + x_0 + y_0 e_1 \in \mathbb{R}^3 : t \in \mathbb{R}\} \cup \{\infty\}$ for a unit vector $v_0 + v_1 e_1 + v_2 e_2$. In $\mathbb{H}^4$, let HP_i $(i = 1, 2)$ be the plane whose boundary at infinity is P_i :

$$HP_1 = \{x + y e_1 + u e_3 \in \mathbb{H}^4 \mid x, y, u \in \mathbb{R},\ x^2 + y^2 + u^2 = 1,\ u > 0\},$$

$$HP_2 = \{t(v_0 + v_1 e_1 + v_2 e_2) + x_0 + y_0 e_1 + s e_3 \in \mathbb{H}^4 \mid t, s \in \mathbb{R}, s > 0\}.$$

We see that HP_1 and HP_2 intersect at a point $E = x_0 + y_0 e_1 + s_0 e_3 \in \mathbb{H}^4$ where $s_0 = \sqrt{1 - x_0^2 - y_0^2} > 0$. Then $E \in \mathbb{H}^4$ is the fixed point of the isometry f, so f is elliptic.

To show (2), now suppose that P_1 and P_2 are disjoint and non-linked. As we did before, we may normalize P_1 and P_2 by isometries so that P_1 is the same unit circle belonging to $\langle 1, e_1\rangle$ and centered at the origin, and P_2 is a line. Only this time, because of the condition that P_1 and P_2 are not linked, P_2 does not intersect with the open unit disk in $\langle 1, e_1\rangle$. This implies that the planes HP_1 and HP_2 are disjoint in $\overline{\mathbb{H}^4}$, and hence they have a non-zero hyperbolic distance. Applying Proposition 4.1 of [6], the isometry f is loxodromic.

Items (3) and (4) come directly from Proposition 4.1 of [6]. □

Therefore, we obtain a linked two-generator subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ from an augmented ideal right-angled pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$. However, Lemma 4.1 does not say anything about the discreteness of the group which is our goal in what follows.

Suppose that $\Gamma = \langle A, B, C \mid ABC = 1\rangle$ is a discrete two-generator subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ such that A is parabolic and B, C are loxodromic. All non-elementary discrete two-generator groups are linked in $\mathbb{H}^2$. That is, we can write $A = I_5 I_1, B = I_1 I_3$ and $C = I_3 I_5$ where I_i $(i = 1, 3, 5)$ is a reflection in a geodesic L_i in $\mathbb{H}^2$. Here, L_1 and L_5 have a common boundary point at infinity which is the fixed point of the parabolic isometry A. L_1 and L_3 (respectively, L_3 and L_5) are disjoint and hence they have a unique common orthogonal geodesic, called L_2 (respectively, L_4) in $\mathbb{H}^2$. In this way, we associate an ideal right-angled pentagon $(L_1, L_2, L_3, L_4, L_5)$ to the group Γ in $\mathbb{H}^2$.

The domain D bounded by L_1, L_3 and L_5 is the fundamental domain for the action of the group generated by three reflections, $\langle I_1, I_3, I_5\rangle$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since the group $\Gamma = \langle A, B, C : ABC = 1\rangle$ is an index 2 subgroup of $\langle I_1, I_3, I_5\rangle$, we can obtain a fundamental domain of Γ by doubling D.

Let $\mathcal{T}(\Gamma)$ be the Teichmüller space of Γ:

$$\{\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^2) : \text{discrete, faithful and type-preserving}\}/\mathrm{Isom}(\mathbb{H}^2)$$

which is the set of discrete, faithful and type-preserving representations into $\mathrm{Isom}(\mathbb{H}^2)$ up to the conjugation action of $\mathrm{Isom}(\mathbb{H}^2)$. It has dimension 2. So, we can parameterize it by a pair of two positive numbers (d_2, d_4) where d_2 (respectively, d_4) is the hyperbolic distance between L_1 and L_3 (respectively, between L_3 and L_5). Applying isometries if necessary, we may normalize the configuration so that $L_1 = L_{[\infty,0]}$, $L_3 = L_{[1,s]}$ and $L_5 = L_{[t,\infty]}$ for some $1 < s < t$. Hence, we may also parameterize $\mathcal{T}(\Gamma)$ by the pair of two boundary points (s, t) $(1 < s < t)$ (see Figure 9).

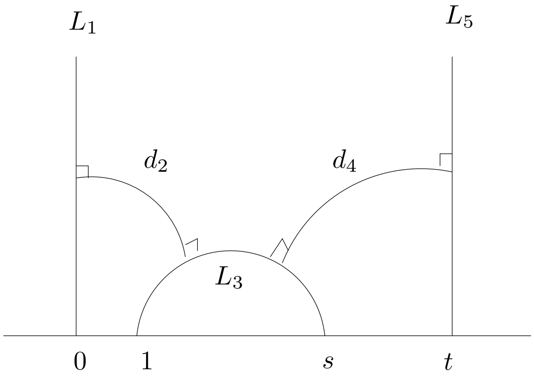

FIGURE 9. An ideal right-angled pentagon associated to a linked two-generator group in $\mathbb{H}^2$

In hyperbolic 4-space, we define the deformation space $\mathcal{D}(\Gamma)$ of Γ by the set of all discrete, faithful and type-preserving representation of Γ into $\mathrm{Isom}(\mathbb{H}^4)$ up to the conjugation action of $\mathrm{Isom}(\mathbb{H}^4)$. That is,

$$\mathcal{D}(\Gamma) = \{\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^4) : \text{discrete, faithful and type-preserving}\}/\mathrm{Isom}(\mathbb{H}^4).$$

By assigning an augmented ideal right-angled pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ to Γ, we may consider Γ itself as an image of the identity representation into $\mathrm{Isom}(\mathbb{H}^4)$ as follows. Without loss of generality, we may assume that the ideal right-angled pentagon $(L_1, L_2, L_3, L_4, L_5)$ associated to Γ is embedded in the plane $\langle 1, e_3\rangle \subset \mathbb{H}^4 = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R},\ x_3 > 0\}$. As we see in Figure 9, we may normalize it so that $L_1 = L_{[\infty,0]}$, $L_3 = L_{[1,s]}$ and $L_5 = L_{[t,\infty]}$ for some $1 < s < t$. We augment the ideal right-angled pentagon with three planes Π_i $(i = 1, 3, 5)$. In particular, we choose Π_1 and Π_5, on the boundary at infinity, to be a line parallel to $\langle e_1\rangle$. Note that Π_1 (respectively Π_5) passes through 0 and ∞ (respectively t and ∞). Let Π_3 be a 1-dimensional circle embedded in $\langle 1, e_1\rangle$ and having the segment between 1 and s a diameter on $\hat{\mathbb{R}}^3$. Now, we extend I_i $(i = 1, 3, 5)$ as a half turn around Π_i in $\mathbb{H}^4$. By construction, the group $\langle I_1, I_3, I_5\rangle$ keeps the plane $\langle 1, e_3\rangle$ invariant, so does Γ. On the plane $\langle 1, e_3\rangle$, the action of group $\langle I_1, I_3, I_5\rangle$ is the same as the original Fuchsian group. This shows how we extend the group Γ into a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^4)$.

For the augmented ideal right-angled pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ in $\mathbb{H}^4$, we have associated half turns to each plane and the group generated by these half turns. Hence, deforming the augmented ideal right-angled pentagon may give us a new point in the deformation space $\mathcal{D}(\Gamma)$. To obtain a new point in $\mathcal{D}(\Gamma)$, we will consider a configuration which is deformed continuously from the initial augmented pentagon and impose a condition that Π_1 passes through $0, \infty$, Π_3 through $1, s$ and Π_5 through t, ∞. In other words, we fix the initial three geodesics $L_1 = L_{[\infty,0]}$, $L_3 = L_{[1,s]}$ and $L_5 = L_{[t,\infty]}$. Since there are 2-dimensional family of choices for each Π_i $(i = 1, 3, 5)$, we have 6-dimensional family of choices for (Π_1, Π_3, Π_5) in total. However, we have to consider the action of a rotation around $\langle 1\rangle$ in $\partial\mathbb{H}^4$ since it fixes $0, 1, s, t$ and ∞. Suppose that one configuration is an image of another configuration under a rotation around $\langle 1\rangle$ in $\partial\mathbb{H}^4$. Then the two configurations might be different, but the two groups associated to each configurations are conjugate to each other by the rotation. Therefore, we have a 5-dimensional parameter space for the configurations of (Π_1, Π_3, Π_5) up to the conjugation action of $\mathrm{Isom}(\mathbb{H}^4)$. Now, considering the parameter s and t we have a 7-dimensional parameter space for the deformation space $\mathcal{D}(\Gamma)$. An arbitrary group constructed in this way might not be discrete. So, we deform the pentagon in a special way to obtain discrete groups in the next theorem.

Theorem 4.3. *Let $\Gamma = \langle A, B, C \mid ABC = 1\rangle$ be a discrete two-generator subgroup of* $\mathrm{Isom}(\mathbb{H}^2)$ *where A is parabolic and B, C are loxodromic. Then there exists a* 6*-dimensional parameter space $\mathcal{P}$ containing the identity representation in the deformation space.*

Proof. We identify the group Γ with the image of the identity representation and consider its associated augmented ideal right-angled pentagon $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ as above: Let $L_1 = L_{[\infty,0]}$, $L_3 = L_{[1,s_0]}$, $L_5 = L_{[t_0,\infty]}$ for some $1 < s_0 < t_0$ and L_2 (respectively L_5) be a line perpendicular to L_1 and L_3 (respectively, L_3 and L_5). In the boundary at infinity $\partial\mathbb{H}^4 = \hat{\mathbb{R}}^3$, Π_1 (respectively Π_5) is a line parallel to $\langle e_1\rangle$ and passes through 0 and ∞ (respectively t and ∞), Π_3 is a

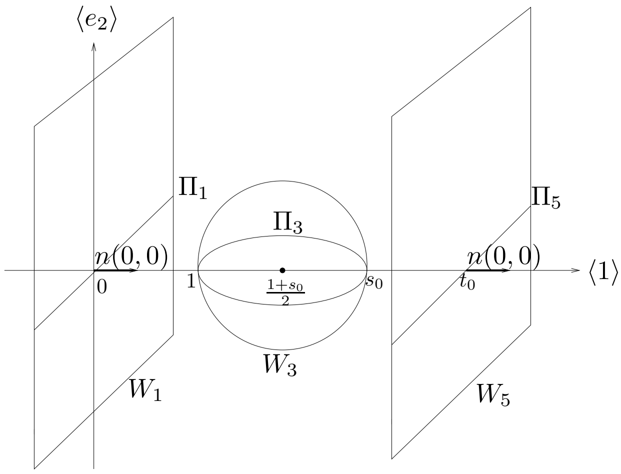

FIGURE 10. D: the boundary of the fundamental domain of $\langle I_1, I_3, I_5\rangle$ in $\partial\mathbb{H}^4$

1-dimensional circle embedded in $\langle 1, e_1\rangle$ and having the segment between 1 and s a diameter, I_i $(i = 1, 3, 5)$ is a half-turn around Π_i, and $A = I_5I_1, B = I_1I_3$ and $C = I_3I_5$. Let L_2 (respectively, L_4) be the unique geodesic orthogonal to L_1 and L_3 (respectively L_3 and L_5). Here, the pentagon $(L_1, \dots, L_5)$ is embedded into $\langle 1, e_3\rangle \subset \mathbb{H}^4$ and the group action of Γ keeps $\langle 1, e_3\rangle$ invariant.

First, we will construct the fundamental domain for the action of $\langle I_1, I_3, I_5\rangle$. In the boundary $\hat{\mathbb{R}^3}$, let W_1 be the 2-dimensional subspace $\langle e_1, e_2\rangle$, W_5 the subspace $t_0 + W_1$ and W_3 the 2-dimensional sphere having Π_3 a great circle. Let D be the domain bounded by W_1, W_3 and W_5 in $\hat{\mathbb{R}^3}$. Since the half turn I_i $(i = 1, 3, 5)$ maps W_i to itself and $I_i(D) \cap D = \emptyset$, D is the fundamental domain of the group $\langle I_1, I_3, I_5\rangle$ (See Figure 10).

Now we will deform the domain D together with the pentagon to obtain a new point in $\mathcal{D}(\Gamma)$ (See Figure 11). Recall that we identify $\mathbb{R}^3$ with the vector space $\{x_0 + x_1e_1 + x_2e_2 \mid x_0, x_1, x_2 \in \mathbb{R}\}$. We denote a unit vector in $\mathbb{R}^3$ by $n(\phi, \psi) = (\cos\phi\cos\psi, \cos\phi\sin\psi, \sin\phi)$ for some real numbers ϕ, ψ. For a unit vector $n(\phi, \psi) \in \mathbb{R}^3$, there is an element $\xi_{\phi,\psi} \in \mathrm{SO}(3)$ such that $\xi_{\phi,\psi}(1) = n(\phi, \psi)$. Note that $\xi_{0,0}(1) = 1$. Let

$$
\begin{aligned}
W_1' &= \langle \xi_{\phi,\psi}(e_1), \xi_{\phi,\psi}(e_2)\rangle = \langle n(\phi,\psi)\rangle^{\perp},\\
\Pi_1' &= \langle \xi_{\phi,\psi}(e_1)\rangle,\\
W_5' &= t_0 + W_1',\\
\Pi_5' &= t_0 + \langle \cos\theta\,\xi_{\phi,\psi}(e_1) + \sin\theta\,\xi_{\phi,\psi}(e_2)\rangle,
\end{aligned}
$$

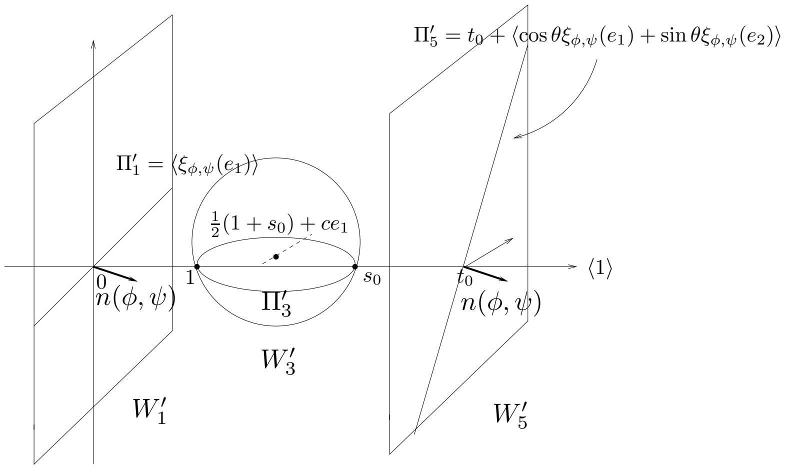

FIGURE 11. The fundamental domain D' in $\mathbb{R}^3$

where $0 \le \theta < \pi$. Then $\Pi'_i \subseteq W'_i$ $(i = 1, 5)$ and W'_1 and W'_5 are parallel in $\mathbb{R}^3$. Let W'_3 be the sphere centered at $\frac{1}{2}(s_0 + 1) + ce_1$ for $c \in \mathbb{R}$ and passing through $1, s_0$, and Π'_3 be the great circle of W'_3 passing through 1 and s_0. Note that $\Pi'_i \subset W'_i$ for $i = 1, 3, 5$. Since $1 < s_0 < t_0$, there are small positive numbers ϵ_0, δ_0 such that W'_1, W'_3 and W'_5 are mutually disjoint in $\mathbb{R}^3$ for any $-\epsilon_0 < \phi, \psi < \epsilon_0$ and $-\delta_0 < c < \delta_0$.

Let D' be the domain bounded by W'_1, W'_3 and W'_5 in $\hat{\mathbb{R}^3}$ for $-\epsilon_0 < \phi, \psi < \epsilon_0$ and $-\delta_0 < c < \delta_0$ and associate the half turns I'_i around Π'_i to each side W'_i. Then, the domain D' is the fundamental domain for the action of the group $\langle I'_1, I'_3, I'_5\rangle$. Since W'_i and $\Pi'_i (i = 1, 3, 5)$ depend on ϕ, ψ, c and θ, let P be the 4-dimensional parameter space,

$$P = \{(\phi, \psi, c, \theta) \in \mathbb{R}^4 \mid -\epsilon_0 < \phi, \psi < \epsilon_0,\ -\delta_0 < c < \delta_0,\ 0 \le \theta < \pi\}.$$

Note that the group associated $(0, 0, 0, 0) \in P$ is the initial group Γ in $\mathbb{H}^4$.

We will show that two points in P give us two distinct groups in $\mathcal{D}(\Gamma)$. Suppose $p_k = (\phi_k, \psi_k, c_k, \theta_k)$, $k = 1, 2$ are two points in P. For $k = 1, 2$, let D_k be the fundamental domain associated to p_k with W^k_i $(i = 1, 3, 5)$, and G_k be the group generated by three half turns I^k_i $(i = 1, 3, 5)$ around Π^k_i associated to each sides of D_k. If G_1 is conjugate to G_2 by an isometry f, then D_1 is the image of D_2 under the isometry f. In particular, $f(\Pi^1_i) = \Pi^2_i$ for $i = 1, 3, 5$. Since we impose the condition that Π^k_1 passes through $0, \infty$, Π^k_3 through $1, s_0$ and Π^k_5 through t_0, ∞ during the deformation, f fixes 0, 1, s_0, t_0 and ∞. So, f is a rotation around $\langle 1\rangle$ by some angle α. However, since $\Pi^k_3 \subset \langle 1, e_1\rangle$ during the deformation, f keeps the 2-dimensional subspace $\langle 1, e_1\rangle$ invariant. Thus,

α should be 0 which implies that f is the identity map. This proves that each point in P gives us a distinct point in the deformation space $\mathcal{D}(\Gamma)$.

Since $1 < s_0 < t_0$ were arbitrary, this locally gives us a 6-dimensional parameter space $\mathcal{P} = \{(s,t) : 1 < s < t\} \times P$ for the deformation space $\mathcal{D}(\Gamma)$. □

References

[1] L. V. Ahlfors, *Old and new in Möbius groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 93–105.

[2] ———, *Möbius transformations and Clifford numbers*, in Differential geometry and complex analysis, 65–73, Springer, Berlin, 1985.

[3] ———, *On the fixed points of Möbius transformations in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 15–27.

[4] A. Basmajian, *Constructing pairs of pants*, Ann. Acad. Sci. Fenn. Ser. A I Math. **15** (1990), no. 1, 65–74.

[5] ———, *Generalizing the hyperbolic collar lemma*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 154–158.

[6] A. Basmajian and B. Maskit, *Space form isometries as commutators and products of involutions*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 5015–5033.

[7] A. F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics, **91**, Springer-Verlag, New York, 1983.

[8] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer-Verlag, Berlin, 1992.

[9] C. Cao and P. L. Waterman, *Conjugacy invariants of Möbius groups*, in Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 109–139, Springer, New York, 1998.

[10] W. Fenchel, *Elementary Geometry in Hyperbolic Space*, De Gruyter Studies in Mathematics, **11**, Walter de Gruyter & Co., Berlin, 1989.

[11] W. M. Goldman, *The complex-symplectic geometry of* $\mathrm{SL}(2,\mathbb{C})$*-characters over surfaces*, in Algebraic groups and arithmetic, 375–407, Tata Inst. Fund. Res., Mumbai, 2004.

[12] L. Keen, *Collars on Riemann surfaces*, in Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), 263–268. Ann. of Math. Studies, **79**, Princeton Univ. Press, Princeton, NJ, 1974.

[13] Y. Kim, *Quasiconformal stability for isometry groups in hyperbolic 4-space*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 175–187.

[14] ———, *Geometric classification of isometries acting on hyperbolic* 4*-space*, J. Korean Math. Soc. **54** (2017), no. 1, 303–317.

[15] C. Kourouniotis, *Complex length coordinates for quasi-Fuchsian groups*, Mathematika **41** (1994), no. 1, 173–188.

[16] B. Maskit, *Kleinian Groups*, Grundlehren der Mathematischen Wissenschaften, **287**, Springer-Verlag, Berlin, 1988.

[17] J. R. Parker and I. D. Platis, *Complex hyperbolic Fenchel-Nielsen coordinates*, Topology **47** (2008), no. 2, 101–135.

[18] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics, **149**, Springer-Verlag, New York, 1994.

[19] C. Series, *An extension of Wolpert's derivative formula*, Pacific J. Math. **197** (2001), no. 1, 223–239.

[20] S. P. Tan, *Complex Fenchel-Nielsen coordinates for quasi-Fuchsian structures*, Internat. J. Math. **5** (1994), no. 2, 239–251.

[21] S. P. Tan, Y. L. Wong, and Y. Zhang, *Delambre-Gauss formulas for augmented, right-angled hexagons in hyperbolic* 4*-space*, Adv. Math. **230** (2012), no. 3, 927–956.

[22] M. Wada, *Conjugacy invariants of Möbius transformations*, Complex Variables Theory Appl. **15** (1990), no. 2, 125–133.

[23] P. L. Waterman, *Möbius transformations in several dimensions*, Adv. Math. **101** (1993), no. 1, 87–113.

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