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RIEMANNIAN MANIFOLDS WITH A SEMI-SYMMETRIC METRIC *P*-CONNECTION

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Abstract. We define a class of semi-symmetric metric connection on a Riemannian manifold for which the conformal, the projective, the concircular, the quasi conformal and the m-projective curvature tensors are invariant. We also study the properties of semisymmetric, Ricci semisymmetric and Eisenhart problems for solving second order parallel symmetric and skew-symmetric tensors on the Riemannian manifolds equipped with a semi-symmetric metric P-connection.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold and ∇ denotes the Levi-Civita connection corresponding to the Riemannian metric g on M^n. A linear connection $\tilde{\nabla}$ defined on M^n is said to be symmetric if the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ defined by $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$ is zero identically on M^n. Otherwise, it is non-symmetric connection for X, $Y \in \chi(M^n)$, where $\chi(M^n)$ denotes the set of all smooth vector fields of M^n. In 1924, Friedmann and Schouten [12] introduced the idea of a semi-symmetric linear connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on M^n is said to be semi-symmetric if

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y \tag{1}$$

for X, $Y \in \chi(M^n)$, where π is a 1-form associated with the vector field P and satisfies

$$\pi(X) = g(X,P). \tag{2}$$

The idea of a metric connection $\tilde{\nabla}$ on a Riemannian manifold was introduced by Hayden [14] in 1932, and later named such connection as a Hayden connection. After a long gap, Pak [20] considered Hayden connection $\tilde{\nabla}$ equipped with the torsion tensor $\tilde{T}$ defined as (1) and proved that it is a semi-symmetric metric connection. A linear connection $\tilde{\nabla}$ is said to be metric on M^n if $\tilde{\nabla} g = 0$,

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otherwise it is non-metric. A systematic study of a semi-symmetric metric connection $\tilde{\nabla}$ on a Riemannian manifold was initiated by Yano [31] in 1970. He showed that the Riemannian curvature tensor with respect to a semi-symmetric metric connection vanishes if and only if the manifold is conformally flat. Since then, many geometers studied the geometrical properties of a semi-symmetric metric connection by considering different spaces, for instance ([8], [15], [16], [35]). Barua et al. [1] established that the conformal curvature tensor is invariant corresponding to the semi-symmetric metric connection on a Riemannian manifold under certain assumptions. In [10], authors proved that conformal and projective curvature tensors on an almost contact metric manifold are invariant with respect to the semi-symmetric metric connection under certain restrictions. Also the invariant properties of conformal and conharmonic curvature tensors have been studied in [6]. Several authors studied invariant properties of different curvature tensors by considering distinct restrictions. After considering all these facts, a natural question is arising in our mind that

Question 1. Does there exist a linear connection on a Riemannian manifold (M^n, g) of dimension n for which the projective, the conformal, the concircular, the quasi conformal and the m-projective curvature tensors are invariant?

Eisenhart [11] in 1923 showed that if a positive definite Riemannian manifold admits a second order parallel symmetric covariant tensor other than a constant metric tensor g, then it is reducible. Levy [17] in 1925 established that a second order parallel symmetric non-degenerated tensor of type $(0, 2)$ in a space form is proportional to the metric tensor. The properties of a second order parallel tensor have been studied by Eisenhart and Levy locally, whereas Sharma [23] studied the properties of second order parallel tensor globally on complex space forms. Since then, the properties of a second order parallel symmetric tensor on different geometrical structures have been studied by many geometers, for instant ([2], [7], [9], [24–26]).

The study of the properties of symmetric spaces plays a significant role in the differential geometry and allied areas. A Riemannian manifold (M^n, g) is said to be locally symmetric if its curvature tensor is parallel, that is, $\nabla R = 0$, where R is the Riemannian curvature tensor of M^n. If $R \cdot R = 0$ holds on M^n, then M^n is said to be semisymmetric. It is well known that semisymmetric manifolds include the set of locally symmetric manifolds as a local set but the converse is not true, in general. A Ricci semisymmetric manifolds $(R \cdot S = 0)$ includes the set of Ricci symmetric manifold $(\nabla S = 0)$ as a proper subset.

The main object of this paper is to gives an affirmative answer of the Question 1 and to study the Eisenhart problem of finding the properties of second order parallel tensors on a Riemannian manifold. To attain the goal, we organize the present work as: We give the definition of a semi-symmetric metric P-connection on a Riemannian manifold and an affirmative answer of Question 1 in Section 2. In Section 3, we show that a semisymmetric Riemannian manifold with respect to the Levi-Civita connection and semi-symmetric metric

P-connection coincide if and only if the manifold is Einstein. The properties of Ricci semisymmetric Riemannian manifolds equipped with a semi-symmetric metric P-connection are found in Section 4. Section 5 deals with the study of the Eisenhart problems on Riemannian manifolds admitting a semi-symmetric metric P-connection of finding the properties of second order parallel symmetric and skew-symmetric tensors.

2. Semi-symmetric metric P-connections

Let M^n be an n-dimensional Riemannian manifold with a Riemannian metric g. If ∇ denotes the Levi-Civita connection corresponding to the metric g on M^n, then a linear connection $\tilde{\nabla}$ on M^n is defined as

$$\tilde{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P \tag{3}$$

for $X, Y \in \chi(M^n)$, where P is a vector field and π is the 1-form satisfies (2). It can be easily proved that the connection $\tilde{\nabla}$ satisfies (1) and $\tilde{\nabla}g = 0$ on (M^n, g) and hence, it is the semi-symmetric metric connection on (M^n, g) [31]. Mishra et al. [18] and Chaubey et al. [4] considered $P = \xi$ and $\tilde{\nabla}\xi = 0$ on an almost contact metric manifold and proved many interesting geometrical results. Motivated by these particular studies, the notion of a semi-symmetric metric ξ-connection on a Riemannian manifold is generalized in this paper.

Definition 1. A linear connection $\tilde{\nabla}$ defined on a Riemannian manifold (M^n, g) is called a semi-symmetric metric P-connection if it satisfies (1), (3), $\tilde{\nabla}g = 0$ and $\tilde{\nabla}P = 0$.

From (3) and Definition 1, it is obvious that

$$\tilde{\nabla}_X P = 0 \Longleftrightarrow \nabla_X P = \pi(X)P - \pi(P)X. \tag{4}$$

It is not hard to prove, from (2) and (3), that

$$(\tilde{\nabla}_X \pi)(Y) = (\nabla_X \pi)(Y) + \pi(P)g(X,Y) - \pi(X)\pi(Y) \tag{5}$$

holds for $X, Y \in \chi(M^n)$. Now we state the following curvature restriction with respect to the Levi-Civita connection ∇ as follows:

Proposition 2.1. *If an n-dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric P-connection $\tilde{\nabla}$, then*
(i) $R(X,Y)P = \pi(P)\{\pi(X)Y - \pi(Y)X\}$,
(ii) $R(P,X)Y = \pi(P)\{\pi(Y)X - g(X,Y)P\}$,
(iii) $\pi(R(X,Y)Z) = \pi(P)\{\pi(Y)g(X,Z) - \pi(X)g(Y,Z)\}$
for X, Y, $Z \in \chi(M^n)$, where R denotes the Riemannian curvature tensor with respect to the Levi-Civita connection ∇.

Proof. In consequence of (2), (4) and (5), the Riemannian curvature tensor takes the form

$$R(X,Y)P = \nabla_X\{\pi(Y)P - \pi(P)Y\} - \nabla_Y\{\pi(X)P - \pi(P)X\}$$

$$
\begin{aligned}
&- \pi([X,Y])P + \pi(P)[X,Y] \\
&= \{(\nabla_X \pi)(Y) - (\nabla_Y \pi)(X)\}P + \pi(\nabla_X Y - \nabla_Y X - [X,Y])P \\
&\quad - \pi(P)\{\nabla_X Y - \nabla_Y X - [X,Y]\} + \pi(Y)\{\pi(X)P - \pi(P)X\} \\
&\quad - \pi(X)\{\pi(Y)P - \pi(P)Y\} - (\nabla_X \pi)(P)Y + (\nabla_Y \pi)(P)X \\
&\quad - \pi(\nabla_X P)Y + \pi(\nabla_Y P)X \\
&= \pi(P)\{\pi(X)Y - \pi(Y)X\}.
\end{aligned}
$$

The inner product of the last expression with U gives

$$'R(X,Y,P,U) = \pi(P)\{\pi(X)g(Y,U) - \pi(Y)g(X,U)\},$$

where $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. By considering the symmetric properties of a curvature tensor and the above equation, we can easily find the proposition (ii) and (iii). Hence the statement is completed. □

The Riemannian curvature tensor R with respect to the Levi-Civita connection ∇ is connected by $\tilde{R}$ as

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \theta(Y,Z)X + \theta(X,Z)Y - g(Y,Z)AX + g(X,Z)AY \tag{6}$$

for $X,Y,Z \in \chi(M^n)$, where $\tilde{R}$ denotes the curvature tensor corresponding to $\tilde{\nabla}$ on M^n and θ is a $(0,2)$ type tensor such that

$$\theta(X,Y) = g(AX,Y) = (\nabla_X \pi)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(P)g(X,Y), \tag{7}$$

which is equivalent to

$$AX = \nabla_X P - \pi(X)P + \frac{1}{2}\pi(P)X. \tag{8}$$

In consequence of (2) and (4), (7) and (8) reduce to

$$\theta(X,Y) = -\frac{1}{2}\pi(P)g(X,Y) \quad \text{and} \quad AX = -\frac{1}{2}\pi(P)X. \tag{9}$$

Changing the values of θ and A from (9) into (6), we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \pi(P)\{g(Y,Z)X - g(X,Z)Y\}. \tag{10}$$

Contracting (10) along the vector field X, we conclude that

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\pi(P)g(Y,Z) \tag{11}$$

equivalent to

$$\tilde{Q}Y = QY + (n-1)\pi(P)Y. \tag{12}$$

Again contraction of (12) along Y gives

$$\tilde{r} = r + n(n-1)\pi(P), \tag{13}$$

where $\tilde{r}\,;\, r$ and $\tilde{Q}\,;\, Q$ are the scalar curvatures and Ricci operators corresponding to $\tilde{\nabla}$ and ∇, respectively. The Ricci tensors $\tilde{S}$ and S with respect to the connections $\tilde{\nabla}$ and ∇ are connected with the Ricci operators $\tilde{Q}$ and Q, respectively as $\tilde{S}(X,Y) = g(\tilde{Q}X,Y)$ and $S(X,Y) = g(QX,Y)$.

The projective curvature tensor $\tilde{P}$ with respect to the semi-symmetric metric P-connection on (M^n, g) is defined as [33]

$$\tilde{\mathcal{P}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{14}$$

for $X, Y, Z \in \chi(M^n)$. In consequence of (10) and (11), (14) implies the form

$$\tilde{\mathcal{P}}(X,Y)Z = \mathcal{P}(X,Y)Z, \tag{15}$$

where $\mathcal{P}$ denotes the projective curvature tensor corresponding to the Levi-Civita connection ∇ and is defined as [33]

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \quad \forall\ X,Y,Z \in \chi(M^n). \tag{16}$$

It is obvious from (15) that the projective curvature tensor with respect to the semi-symmetric metric P-connection $\tilde{\nabla}$ on (M^n, g) is invariant. Thus we have the following theorem:

Theorem 2.2. *Let (M^n, g) be an n-dimensional Riemannian manifold endowed with a semi-symmetric metric P-connection $\tilde{\nabla}$. Then the projective curvature tensors with respect to $\tilde{\nabla}$ and ∇ coincide.*

In particular if $\tilde{R} = 0$ on (M^n, g), then from Theorem 2.2, (10) and (11) we have $\mathcal{P} = 0$, that is (M^n, g) is projectively flat. Thus we state:

Corollary 2.3. *If an n-dimensional Riemannian manifold (M^n, g) admits a semi-symmetric metric P-connection $\tilde{\nabla}$ whose curvature tensor vanishes, then it is projectively flat.*

We next prove the invariant properties of a concircular curvature tensor under the certain transformation in the following theorem as:

Theorem 2.4. *If (M^n, g) is an n-dimensional Riemannian manifold endowed with a semi-symmetric metric P-connection $\tilde{\nabla}$, then the concircular curvature tensors corresponding to $\tilde{\nabla}$ and ∇ are identically equal on M^n.*

Proof. Let $\tilde{\breve{C}}$ be a concircular curvature tensor with respect to the semi-symmetric metric P-connection $\tilde{\nabla}$ on (M^n, g) [32]. Then it is given by

$$\tilde{\breve{C}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{17}$$

for $X, Y, Z \in \chi(M^n)$. Changing the value of $\tilde{R}$ and $\tilde{r}$ from (10) and (13) in (17), we find

$$\tilde{\breve{C}} = \breve{C}, \tag{18}$$

where $\breve{C}$ is a concircular curvature tensor field of type $(1,3)$, defined as [32]:

$$\breve{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{19}$$

for $X, Y, Z \in \chi(M^n)$. Thus (18) state the statement of the theorem. □

In particular, if the scalar curvature corresponding to the semi-symmetric metric P-connection $\tilde{\nabla}$ vanishes, then from (13) and Theorem 2.4 we have $\breve{C} = \tilde{R}$. Thus we have:

Corollary 2.5. *Let (M^n, g) be an n-dimensional Riemannian manifold equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$. If the scalar curvature of $\tilde{\nabla}$ vanishes on (M^n, g), then the concircular curvature tensor $\breve{C}$ coincides with the curvature tensor $\tilde{R}$.*

The study of conformal invariant plays an important role in differential geometry as well as it has many applications in other branches of Sciences, especially, applied physics. These factors motivate us to study the properties of invariant conformal curvature tensors. Therefore we prove the following theorem:

Theorem 2.6. *If (M^n, g) is an n-dimensional Riemannian manifold equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$, then the conformal curvature tensors corresponding to $\tilde{\nabla}$ and ∇ are equal on (M^n, g).*

Proof. The conformal curvature tensor $\tilde{C}$ with respect to the connection $\tilde{\nabla}$ is defined as [30]

$$\begin{aligned} \tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z & \\ & - \frac{1}{n-2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\} \\ & + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{20}$$

for X, Y, $Z \in \chi(M^n)$. In consequence of (10)-(13), (20) takes the form

$$\tilde{C} = C,$$

where C is a conformal curvature tensor field of type $(1, 3)$ given by [30]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z & \\ & - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{21}$$

for X, Y, $Z \in \chi(M^n)$. Thus the statement of the theorem is proved. □

Theorem 2.7. *The m-projective curvature tensor on an n-dimensional Riemannian manifold endowed with a semi-symmetric metric P-connection is invariant with respect to the connection $\tilde{\nabla}$.*

Proof. Let (M^n, g) be an n-dimensional Riemannian manifold equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$. The m-projective curvature tensor [5, 21] with respect $\tilde{\nabla}$ is defined as

$$\tilde{W^*}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y] \tag{22}$$

for $X, Y, Z \in \chi(M^n)$. In view of (10)-(13), (22) becomes

$$\tilde{W^*} = W^*, \tag{23}$$

where W^* is an m-projective curvature tensor with respect to the Levi-Civita connection ∇. Thus the theorem is proved. □

Theorem 2.8. *Let (M^n, g) be an n-dimensional Riemannian manifold admitting a semi-symmetric metric P-connection $\tilde{\nabla}$. Then the quasi-conformal curvature tensor $\tilde{\bar{C}}$ is invariant with respect to $\tilde{\nabla}$.*

Proof. Consider a quasi-conformal curvature tensor $\bar{C}$, expressed as [34]

$$\begin{aligned}\bar{C}(X,Y)Z &= aR(X,Y)Z \\ &\quad + b\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{24}$$

for $X, Y, Z \in \chi(M^n)$. Here a and b are constants.

In the similar fashion, we can define the quasi conformal curvature tensor $\tilde{\bar{C}}$ corresponding to the semi-symmetric metric P-connection $\tilde{\nabla}$ as [34]

$$\begin{aligned}\tilde{\bar{C}}(X,Y)Z &= a\tilde{R}(X,Y)Z \\ &\quad + b\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\} \\ &\quad - \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{25}$$

for $X, Y, Z \in \chi(M^n)$. Using (10)-(13) in (25), we find

$$\begin{aligned}\tilde{\bar{C}}(X,Y)Z &= aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY\} - \frac{r}{n}\left(\frac{a+2b(n-1)}{(n-1)}\right)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + a\pi(P)\{g(Y,Z)X - g(X,Z)Y\} + 2b\pi(P)(n-1)\{g(Y,Z)X \\ &\quad - g(X,Z)Y\} - \pi(P)(a + 2b(n-1))\{g(Y,Z)X - g(X,Z)Y\},\end{aligned}$$

which becomes $\tilde{\bar{C}} = \bar{C}$. □

3. Semisymmetric Riemannian manifolds with a semi-symmetric metric *P*-connection

A Riemannian manifold (M^n, g) of dimension n is said to be semisymmetric with respect to the Levi-Civita connection ∇ if $R(X,Y) \cdot R = 0$ for $X, Y \in \chi(M^n)$. In [27,28], author studied the geometric properties of semisymmetric Riemannian manifolds. It is well known that the set of locally symmetric manifolds ($\nabla R = 0$) lie in the class of semisymmetric manifolds as a proper subset. In [19], authors studies the geometrical properties of a semisymmetric Riemannian manifold endowed with a semi-symmetric metric connection.

Definition 2. Let (M^n, g) be a Riemannian manifold of dimension n endowed with a semi-symmetric metric P-connection $\tilde{\nabla}$. Then (M^n, g) is said to be semisymmetric with respect to $\tilde{\nabla}$ if $\tilde{R}(X,Y) \cdot \tilde{R} = 0$ for $X, Y \in \chi(M^n)$ [27,28].

We have

$$\begin{aligned}(\tilde{R}(X,Y) \cdot \tilde{R})(Z,U)V = {} & \tilde{R}(X,Y)\tilde{R}(Z,U)V - \tilde{R}(\tilde{R}(X,Y)Z,U)V \\ & - \tilde{R}(Z,\tilde{R}(X,Y)U)V - \tilde{R}(Z,U)\tilde{R}(X,Y)V.\end{aligned}$$

In light of (10), the above equation converts into the form

$$\begin{aligned}(\tilde{R}(X,Y) \cdot \tilde{R})(Z,U)V = {} & \tilde{R}(X,Y)R(Z,U)V + \pi(P)g(U,V)\tilde{R}(X,Y)Z \\ & - \pi(P)g(Z,V)\tilde{R}(X,Y)U - \tilde{R}(R(X,Y)Z,U)V \\ & - \pi(P)g(Y,Z)\tilde{R}(X,U)V + \pi(P)g(X,Z)\tilde{R}(Y,U)V \\ & - \tilde{R}(Z,R(X,Y)U)V - \pi(P)g(Y,U)\tilde{R}(Z,X)V \\ & + \pi(P)g(X,U)\tilde{R}(Z,Y)V - \tilde{R}(Z,U)R(X,Y)V \\ & - \pi(P)g(Y,V)\tilde{R}(Z,U)X + \pi(P)g(X,V)\tilde{R}(Z,U)Y.\end{aligned}$$

Again using (10) in the last expression, we have

$$\begin{aligned}& (\tilde{R}(X,Y) \cdot \tilde{R})(Z,U)V \\ = {} & R(X,Y)R(Z,U)V \\ & + \pi(P)\{g(Y,R(Z,U)V)X - g(X,R(Z,U)V)Y\} \\ & + \pi(P)g(U,V)R(X,Y)Z + (\pi(P))^2 g(U,V)\{g(Y,Z)X - g(X,Z)Y\} \\ & - \pi(P)g(Z,V)R(X,Y)U - (\pi(P))^2 g(Z,V)\{g(Y,U)X - g(X,U)Y\} \\ & - R(R(X,Y)Z,U)V - \pi(P)[g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U] \\ & - \pi(P)g(Y,Z)R(X,U)V - (\pi(P))^2 g(Y,Z)\{g(U,V)X - g(X,V)U\} \\ & + \pi(P)g(X,Z)R(Y,U)V + (\pi(P))^2 g(X,Z)[g(U,V)Y - g(Y,V)U] \\ & - R(Z,R(X,Y)U)V - \pi(P)\{g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U\} \\ & - \pi(P)g(Y,U)R(Z,X)V - (\pi(P))^2 g(Y,U)[g(X,V)Z - g(Z,V)X] \\ & + \pi(P)g(X,U)R(Z,Y)V + (\pi(P))^2 g(X,U)[g(Y,V)Z - g(Z,V)Y]\end{aligned}$$

$$- R(Z,U)R(X,Y)V - \pi(P)\{g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U\}$$
$$- \pi(P)g(Y,V)R(Z,U)X - (\pi(P))^2 g(Y,V)[g(U,X)Z - g(Z,X)U]$$
$$+ \pi(P)g(X,V)R(Z,U)Y + (\pi(P))^2 g(X,V)[g(U,Y)Z - g(Z,Y)U].$$

After simplification, it becomes

$$\begin{aligned}(\tilde{R}(X,Y)\cdot\tilde{R})(Z,U)V = {} & (R(X,Y)\cdot R)(Z,U)V \\ & + \pi(P)\{'R(Z,U,V,Y)X -' R(Z,U,V,X)Y \\ & + g(X,V)R(Z,U)Y - g(Y,V)R(Z,U)X \\ & + g(X,U)R(Z,Y)V - g(Y,U)R(Z,X)V \\ & + g(X,Z)R(Y,U)V - g(Y,Z)R(X,U)V\},\end{aligned} \tag{26}$$

where

$$\begin{aligned}(R(X,Y)\cdot R)(Z,U)V = {} & R(X,Y)R(Z,U)V - R(R(X,Y)Z,U)V \\ & - R(Z,R(X,Y)U)V - R(Z,U)R(X,Y)V.\end{aligned}$$

Thus we are in position to state:

Theorem 3.1. *On an n-dimensional Riemannian manifold (M^n, g) equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$, (26) is obtained.*

Theorem 3.2. *Let (M^n, g) be an n-dimensional Riemannian manifold endowed with a semi-symmetric metric P-connection $\tilde{\nabla}$. If (M^n, g) satisfies $\tilde{R}\cdot\tilde{R} = R\cdot R$ and $\pi(P) \neq 0$, then the manifold is Einstein.*

Proof. We suppose that $\tilde{R}\cdot\tilde{R} = R\cdot R$ and $\pi(P) \neq 0$, then from (26), we have

$$\begin{aligned}& 'R(Z,U,V,Y)X -' R(Z,U,V,X)Y + g(X,V)R(Z,U)Y \\ & - g(Y,V)R(Z,U)X + g(X,U)R(Z,Y)V \\ & - g(Y,U)R(Z,X)V + g(X,Z)R(Y,U)V - g(Y,Z)R(X,U)V = 0.\end{aligned}$$

Inner product of the above equation with the vector field P gives

$$\begin{aligned}& 'R(Z,U,V,Y)\pi(X) -' R(Z,U,V,X)\pi(Y) + g(X,V)'R(Z,U,Y,P) \\ & - g(Y,V)'R(Z,U,X,P) + g(X,U)'R(Z,Y,V,P) \\ & - g(Y,U)'R(Z,X,V,P) + g(X,Z)'R(Y,U,V,P) \\ & - g(Y,Z)'R(X,U,V,P) = 0.\end{aligned} \tag{27}$$

Setting $Z = Y = e_i$, where $\{e_i, i = 1,2,3,\ldots,n\}$ be an orthonormal basis of the tangent space at each point of the manifold (M^n, g), in (27) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}& \pi(X)S(U,V) -' R(P,U,V,X) - g(X,V)S(U,P) -' R(V,U,X,P) \\ & -' R(U,X,V,P) +' R(X,U,V,P) - n'R(X,U,V,P) = 0.\end{aligned}$$

Making use of Proposition 2.1 in the above equation, it converts into the form

$$\pi(X)S(U,V) = -(n-1)\pi(P)\pi(X)g(U,V).$$

In general $\pi(X) \neq 0$ on Riemannian manifold (M^n, g), therefore the above equation gives

$$S(U,V) = -(n-1)\pi(P)g(U,V) \Longrightarrow r = -n(n-1)\pi(P). \tag{28}$$

This shows that the manifold (M^n, g) under consideration is an Einstein manifold. Hence proof is completed. □

With the help of (28) and Theorem 3.2, we are in position to write the following corollary as:

Corollary 3.3. *If an n-dimensional Riemannian manifold (M^n, g), $n > 2$, endowed with a semi-symmetric metric P-connection $\tilde{\nabla}$ satisfies $\tilde{R}(X,Y) \cdot \tilde{R} = R(X,Y) \cdot R$, then the following relations are equivalent*:

(a) (M^n, g) *is an Einstein manifold, that is,* (28) *holds.*
(b) $\tilde{R} = \mathcal{P}$.
(c) $C = \tilde{R}$.
(d) *the m-projective curvature tensor of* (M^n, g) *is equal to* $\tilde{R}$.
(e) $\breve{C} = \tilde{R}$.
(f) $\bar{C} = a\tilde{R}$, *where a is defined in* (24).

Proof. By hypothesis (28) holds on (M^n, g). If by hypothesis, the relation (a) holds on (M^n, g), then from (10) and (16) we can easily get (b). In the similar fashion, we can prove the other results by considering (10), (11), (19) and (24). □

4. Ricci semisymmetric manifolds endowed with a semi-symmetric metric *P*-connection

A Riemannian manifold (M^n, g), $n \geq 3$, is said to be Ricci semisymmetric if the tensorial relation $R(X,Y) \cdot S = 0$ holds $\forall\ X, Y \in \chi(M^n)$. The class of a Ricci semisymmetric manifold includes the set of Ricci symmetric manifolds $(\nabla S = 0)$. That is, every Ricci symmetric manifold is a Ricci semisymmetric but the converse is not true in general. Analogous to this, we define:

Definition 3. An n-dimensional Riemannian manifold (M^n, g), $n > 2$, endowed with $\tilde{\nabla}$ is called a Ricci semisymmetric manifold with respect to $\tilde{\nabla}$ if $\tilde{R}(X,Y) \cdot \tilde{S} = 0, \forall\, X, Y \in (M^n)$.

Now, we prove the following theorem:

Theorem 4.1. *Let (M^n, g), $n > 2$, be an n-dimensional Riemannian manifold equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$. Then, Ricci semisymmetric manifolds with respect to $\tilde{\nabla}$ and ∇ are identical if and only if (M^n, g) is either Einstein with respect to ∇ or Ricci flat with respect to $\tilde{\nabla}$.*

Proof. We have

$$(\tilde{R}(X,Y) \cdot \tilde{S})(Z,U) = -\tilde{S}(\tilde{R}(X,Y)Z,U) - \tilde{S}(Z,\tilde{R}(X,Y)U) \tag{29}$$

for $X, Y, Z, U \in \chi(M^n)$. In view of (10) and (11), (29) takes the form

$$(\tilde{R}(X,Y)\cdot\tilde{S})(Z,U) = (R(X,Y)\cdot S)(Z,U) - \pi(P)\{g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + S(Z,X)g(Y,U) - g(X,U)S(Z,Y)\}, \tag{30}$$

where $(R(X,Y)\cdot S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U)$.

Let us suppose that $\tilde{R}\cdot\tilde{S} = R\cdot S$, then equation (30) gives

$$g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + S(Z,X)g(Y,U) - g(X,U)S(Z,Y) = 0,$$

provided $\pi(P) \neq 0$. Setting $X = Z = e_i$, where $\{e_i, i = 1, 2, \ldots, n\}$ is a local orthonormal basis of the tangent space, in the last expression and taking summation for i, $1 \leq i \leq n$, we get

$$S(Y,U) = \frac{r}{n}g(Y,U). \tag{31}$$

From Proposition 2.1(i), we can easily find that

$$S(Y,P) = -(n-1)\pi(P)\pi(Y). \tag{32}$$

Replacing U with P in (31) and then using (32), we conclude that $r = -n(n-1)\pi(P)$. Hence (31) takes the form

$$S(Y,U) = -(n-1)\pi(P)g(Y,U). \tag{33}$$

This shows that the manifold is an Einstein manifold. Also (11) and (33) reflect that $\tilde{S} = 0$. Therefore (M^n, g) is Ricci flat with respect to the semi-symmetric metric P-connection $\tilde{\nabla}$. Conversely, it is obvious from (30) and (33). □

5. Second order parallel (skew-)symmetric tensor

This section deals with the study of solutions of the Eisenhart problem on a Riemannian manifold (M^n, g) endowed with a semi-symmetric metric P-connection. A tensor field α of type $(0,2)$ on (M^n, g) is said to be second order parallel symmetric tensor with respect to the Levi-Civita connection ∇ if $\nabla\alpha = 0$.

Theorem 5.1. *A Riemannian metric g on an n-dimensional regular Riemannian manifold M^n equipped with a semi-symmetric metric P-connection is irreducible, i.e., $\alpha = \lambda g$ for some constant λ.*

Proof. Let us suppose that α is a second order parallel symmetric tensor with respect to the Levi-Civita connection ∇ on a regular Riemannian manifold M^n $(R(X,Y)P \neq 0)$, that is, $\nabla\alpha = 0$. Then we have $R(X,Y).\alpha = 0$, $\forall\ X, Y \in \chi(M^n)$. This condition is equivalent to

$$\alpha(R(X,Y)Z,U) + \alpha(Z,R(X,Y)U) = 0$$

for $X, Y, Z, U \in \chi(M^n)$. Taking $X = P$ in the above equation and using Proposition 2.1, we find

$$\pi(P)\{\pi(Z)\alpha(Y,U) - g(Y,Z)\alpha(P,U)$$

$$+ \pi(U)\alpha(Z,Y) - \alpha(Z,P)g(Y,U)\} = 0. \tag{34}$$

Since M^n is regular, therefore $\pi(P) \neq 0$. Thus (34) becomes

$$\pi(Z)\alpha(Y,U) - g(Y,Z)\alpha(P,U) + \pi(U)\alpha(Z,Y) - \alpha(Z,P)g(Y,U) = 0. \tag{35}$$

Setting $Z = P$ in (35), we get

$$\pi(P)\alpha(Y,U) - \pi(Y)\alpha(P,U) + \pi(U)\alpha(P,Y) - \alpha(P,P)g(Y,U) = 0. \tag{36}$$

Again putting $U = P$ in (36), then it becomes

$$\pi(P)\alpha(Y,P) = \alpha(P,P)g(Y,P). \tag{37}$$

In consequence of (37), (36) takes the form

$$\alpha(Y,U) = \alpha'(P,P)g(Y,U), \tag{38}$$

provided $\pi(P) \neq 0$ and $\alpha'(P,P) = \frac{\alpha(P,P)}{\pi(P)} \neq 0$. The covariant derivative of (38) along an arbitrary vector field X of a Riemannian manifold M^n with respect to the Levi-Civita connection ∇ shows that $\alpha'(P,P)$ is constant. Therefore from (38), we can say that the second order parallel symmetric tensor field α is a constant multiple of the metric tensor g on a regular Riemannian manifold M^n. That is $\alpha = \lambda g$, where $\lambda = \alpha'(P,P)$. □

Corollary 5.2. *Every Ricci symmetric Riemannian manifold of dimension $n\,(> 1)$ endowed with a semi-symmetric metric P-connection is an Einstein manifold.*

Proof. Suppose the Riemannian manifold M^n is Ricci symmetric with respect to the Levi-Civita connection ∇, that is, $\nabla S = 0$. Then with the help of Theorem 5.1, we get

$$S(X,Y) = S'(P,P)g(X,Y), \tag{39}$$

where $S'(P,P) = \frac{S(P,P)}{\pi(P)} = -(n-1)\pi(P) \neq 0$, where (32) is used. □

Remark 5.3. The statement of Corollary 5.2 is same as Theorem 5.1 but approaches are different.

We define the following definition as:

Definition 4. A triplet (g, V, λ) on an n-dimensional Riemannian manifold (M^n, g) is said to be a Ricci soliton [13] if

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0 \tag{40}$$

holds on M^n for a vector field V and a real constant λ. Here, $\mathfrak{L}_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton (g, V, λ) is said to be shrinking, expanding or steady if $\lambda <, >$ or $= 0$, respectively. For more details, we cite ([3], [22]).

Theorem 5.4. *Let (M^n, g), $(n > 1)$, be an n-dimensional Ricci symmetric manifold equipped with a semi-symmetric metric P-connection. If $\mathfrak{L}_P g$ is parallel, then the Ricci soliton (g, P, λ) on (M^n, g) is always expanding.*

Proof. We suppose that the Lie derivative of the Riemannian metric g along the vector field P is parallel, that is $\nabla\mathfrak{L}_P g = 0$, and a regular Riemannian manifold of dimension $n\,(>1)$ equipped with a semi-symmetric metric P-connection is Ricci symmetric. From (4) we have

(41) $(\mathfrak{L}_P g)(X,Y) = g(\nabla_X P, Y) + g(X, \nabla_Y P) = 2\{\pi(X)\pi(Y) - \pi(P)g(X,Y)\}$.

We define

$$\alpha_1(X,Y) = \frac{1}{2}(\mathfrak{L}_P g)(X,Y) + S(X,Y) \tag{42}$$

for X, $Y \in \chi(M^n)$. In view of (39) and (41), (42) becomes

$$\alpha_1(X,Y) = \pi(X)\pi(Y) - n\pi(P)g(X,Y). \tag{43}$$

By considering above results, (43) and Theorem 5.1, we get

$$\alpha_1(X,Y) = -(n-1)[\pi(P)]^2 g(X,Y). \tag{44}$$

In consequence of (40) and (44), we conclude that $\lambda = (n-1)(\pi(P))^2$. Hence the proof is completed. □

Definition 5. A vector field X on a Riemannian manifold M^n of dimension n is said to be affine Killing if $\nabla\mathfrak{L}_X g = 0$.

Corollary 5.5. *An affine Killing vector field on an n-dimensional Riemannian manifold M^n, $n > 1$, equipped with a semi-symmetric metric P-connection is Killing.*

Proof. By considering Definition 5 and Theorem 5.1, we conclude that

$$(\mathfrak{L}_X g)(Y,Z) = \nu g(Y,Z),$$

where $\nu = \frac{(\mathfrak{L}_X g)(P,P)}{\pi(P)}$. From (32), we have $QP = -(n-1)\pi(P)P$. From (2) and (4), we get $(\mathfrak{L}_X Q)(P) = 0$. Hence $(\mathfrak{L}_X S)(P,P) = 0$, provided $n > 1$. From (32), we conclude that $(\mathfrak{L}_X S)(P,P) = -2S(\mathfrak{L}_X P, P) = 2(n-1)\pi(P)g(\mathfrak{L}_X P, P)$ $= 0$ for $n > 1$ and $\pi(P) \neq 0$. This implies that $g(\mathfrak{L}_X P, P) = 0$. Therefore, $(\mathfrak{L}_X g)(P,P) = 0$ and hence $(\mathfrak{L}_X g)(Y,Z) = 0$. This reflects that the affine Killing vector field X is a Killing vector field. □

The notion of the weakly Ricci symmetric manifolds have been introduced by L. Tamassy and T. Q. Binh in 1993 [29].

Definition 6. A Riemannian manifold (M^n, g), $n > 2$, of dimension n is said to be weakly Ricci symmetric if its non-vanishing Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) = \sigma_1(X)S(Y,Z) + \sigma_2(Y)S(Z,X) + \sigma_3(Z)S(X,Y) \tag{45}$$

for $X, Y, Z \in \chi(M^n)$, where σ_i, $i = 1,2,3$, are the 1-forms.

Theorem 5.6. *Let M^n, $n > 2$, be an n-dimensional weakly Ricci symmetric manifold with a semi-symmetric metric P-connection and $\alpha_1 = \frac{1}{2}\mathfrak{L}_P g + S$ be parallel. Then the Ricci soliton (g, P, λ) on M^n to be shrinking or expanding accordingly $\pi(P) <$ or > 0.*

Proof. Putting $Z = P$ in (45) and then use of (32) gives

$$(\nabla_X S)(Y,P) = -(n-1)\pi(P)\{\pi(Y)\sigma_1(X)+\sigma_2(Y)\pi(X)\}+\sigma_3(P)S(Y,X). \tag{46}$$

Setting $Z = P$ in $(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$ and then using (2), (4) and (32), we find

$$(\nabla_X S)(Y,P) = (n-1)(\pi(P))^2 g(X,Y) + \pi(P)S(Y,X). \tag{47}$$

From (46) and (47), we obtain

$$S(Y,X) = \frac{(n-1)\pi(P)}{\sigma_3(P)-\pi(P)}\left[\pi(P)g(X,Y) + \pi(Y)\sigma_1(X) + \pi(X)\sigma_2(Y)\right], \tag{48}$$

provided $\pi(P) \neq 0$ and $\sigma_3(P) \neq \pi(P)$. Moreover, (48) turns into

$$[\sigma_3(P) - \pi(P)]S(P,P) = (n-1)\pi(P)^2[\pi(P) + \sigma_1(P) + \sigma_2(P)],$$

which gives

$$\sigma_1(P) + \sigma_2(P) + \sigma_3(P) = 0. \tag{49}$$

Consider now that $\alpha_1 = \frac{1}{2}\mathfrak{L}_P g + S$. Then it becomes

$$\begin{aligned} \alpha_1(X,Y) = {} & \left(\frac{(n-1)\pi(P)}{\sigma_3(P)-\pi(P)} - 1\right)\pi(P)g(X,Y) + \pi(X)\pi(Y) \\ & + \frac{(n-1)\pi(P)}{\sigma_3(P)-\pi(P)}\left[\pi(Y)\sigma_1(X) + \pi(X)\sigma_2(Y)\right]. \end{aligned} \tag{50}$$

In consequence of (40) and (50), we get

$$-\lambda = \frac{(n-1)\pi(P)}{\sigma_3(P)-\pi(P)}[\pi(P) + \sigma_1(P) + \sigma_2(P)]. \tag{51}$$

From (49) and (51), we get

$$\lambda = (n-1)\pi(P). \tag{52}$$

Since $n > 1$ and $\pi(P) \neq 0$, $\lambda \neq 0$. If $\alpha_1 = \frac{1}{2}\mathfrak{L}_P g + S$ is parallel on an n-dimensional weakly Ricci symmetric manifold M^n equipped with a semi-symmetric metric P-connection $\tilde{\nabla}$, then from Theorem 5.1 and (52), we can say that the Ricci soliton (g, P, λ) on M^n to be expanding or shrinking if $\pi(P) >$ or < 0, respectively. □

Theorem 5.7. *Let (M^n, g) be an n-dimensional Riemannian manifold admitting a semi-symmetric metric P-connection $\tilde{\nabla}$. Then there doesn't exist a second order parallel skew-symmetric tensor on M^n.*

Proof. Let α be a skew-symmetric parallel tensor with respect to the Levi-Civita connection ∇ on M^n, that is, $\nabla\alpha = 0$. Thus we have

$$\alpha(R(X,Y)Z,U) + \alpha(Z, R(X,Y)U) = 0$$

for $X, Y, Z, U \in \chi(M^n)$. Setting $Z = X = P$ in the above equation and then using Proposition 2.1, we find

$$\alpha(Y,U) = \frac{1}{\pi(P)}\{\pi(Y)\alpha(P,U) + \pi(U)\alpha(P,Y)\}, \tag{53}$$

provided $\pi(P) \neq 0$. Let ν be a tensor field of type $(1,1)$ and metrically equivalent to α, that is,

$$\alpha(Y,U) = g(\nu Y, U).$$

From (53), we have

$$\nu Y = \frac{1}{\pi(P)}\{\pi(Y)\nu P + g(\nu P, Y)P\}, \tag{54}$$

which gives

$$g(\nu P, P) = 0. \tag{55}$$

From (2) and (4), it is obvious that $X(\pi(P)) = 0$, $\forall\ X \in \chi(M^n)$, and hence $\pi(P)$ is constant on M^n. Since α is parallel, ν is also parallel and therefore

$$\nabla_X(\nu P) = -g(\nu P, X)P. \tag{56}$$

Also,

$$g(\nu X, P) = \frac{1}{\pi(P)}\{\pi(X)g(\nu P, P) + \pi(P)g(\nu P, X)\}.$$

In consequence of (55) and (56), we find that

$$g(\nabla_X(\nu P), \nu P) = 0,$$

which shows that $\|\nu P\| =$ constant on M^n. From above results, we can easily observe that

$$\nu^2 P = -\|\nu P\|^2 P. \tag{57}$$

The covariant derivative of (57) with respect to the Levi-Civita connection ∇ along the vector field X gives

$$\nu^2 X = -\|\nu P\|^2 X.$$

Suppose $\|\nu P\|^2 \neq 0$ on M^n, then the almost complex structure J on M^n assumes the form $J = \frac{1}{\|\nu P\|}\nu$. Hence (J, g) is a Kähler structure on M^n and the fundamental 2-form is $g(JX, Y) = a_1\alpha(X, Y)$, where $a_1 = \frac{1}{\|\nu P\|} =$ constant. But α satisfies the equation (53) and thus it is degenerate, which is inadmissible. Therefore $\|\nu P\| = 0$ and hence $\alpha = 0$ on M^n. □

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