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A UNIFIED STABILIZED FINITE VOLUME METHOD FOR STOKES AND DARCY EQUATIONS

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Abstract. In this paper, we present and analyze a cell-centered collocated finite volume scheme for incompressible flows to compute solutions simultaneous to Stokes and Darcy equations by applying a pressure jump stabilization term to avoid locking. We prove that the new stabilized FV formulation satisfies a discrete inf-sup condition and error estimates for both problems. Finally, we present some numerical examples confirming this analysis.

1. Introduction

Finite difference and finite element methods have been extensively applied in computational fluid dynamics. Although the finite difference methods, which are locally mass conserving, are simple to implement, they are not flexible with complicated geometry and general boundary conditions. On the other hand, the finite element methods possess the mesh flexibility to handle complex geometrical settings, but they do not conserve mass at element level. Furthermore, the approximation finite element spaces for the primitive variables (velocity and pressure) must be chosen appropriately to ensure numerical schemes stability and convergence. It is widely understood that many computationally appealing pairs of low-order approximations fail to satisfy the so-called stability inf-sup condition (see [6, 7] and the references therein). The finite volume methods (FVM) were initially developed as an efficient middle ground between the finite difference and finite element methods. Recently, several finite volume schemes have been proposed in many mathematical and engineering studies (see [3–5, 14, 15] and the references therein). Different FV schemes were attemptedly developed by the use of finite element ideas in order to achieve a more rigourous FV methodology. Among these new approaches, the collocated FV schemes seem to attract CFD researchers attention for several reasons. Unfortunately, one crucial drawback was noticed from the very start. The

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well-known lack of discrete coercivity caused some instability in the approximate mixed solutions for the Stokes and Navier-Stokes problems. Nonetheless, one can make the use of some stabilization procedures which were developed earlier for unstable finite element methods (see [5]).

In fluid infiltration problems, like the ones encountered in surface/groundwater or biological flows, the considered simplified problem is solved for a Stokes flow in one part of the domain and a Darcy flow in the other part. It then becomes adequate to work with a FVM that may solve both equations successfully and simultaneously and yield the same optimal convergence rates in both regimes. Similar studies were undertaken in [1,2,11] for finite element methods and in [13] using a finite volume method.

In this paper, we construct a unified treatment of Stokes and Darcy equations using a collocated FVM by approximating velocity and pressure solutions with piecewise constant functions on the cells of a 2D or 3D mesh. Thus, no dual grid is needed and the only requirement on the mesh is a geometrical assumption for the consistency of the approximate diffusion flux [4]. We propose to analyze stability and convergence properties of this discretization method. It is crucial to note that we are not aware of any research work having analyzed and applied the same discrete operators as proposed presently. As it was mentioned above, the resulting scheme is unstable unless some stabilization technique is used. For the latter, we have opted to add to the incompressibility equation a (consistent) symmetric pressure jump stabilization term across over the cell edges. This stabilization was first introduced in the context of Stokes equation [9] in a global form, and later considered in a local form in [10]. This mixed stabilized technique is applied to both Stokes and Darcy flows in a unified manner. We prove optimal a priori error estimates in the energy norm for both cases. Furthermore, some results of numerical tests are provided to check the efficiency of the proposed FV scheme. It is anticipated here that the coupled Stokes-Darcy version of the paper is presently under investigatyion.

The paper is organized as follows. In the next section, we briefly describe the mathematical setting of the incompressible flow problem governed by the Stokes and Darcy equations. We also recall some notation and the classical variational formulation. Section 3 is devoted to the derivation of the approximate problem. First, we define admissible discretizations for the FVM. Then, the proposed FV schemes are discussed and recast under a discrete variational form. The core of the paper is Section 4 in which we present a thorough study of the proposed FV scheme with stability and error estimates results. In the following section we report and discuss numerical results obtained for two test problems. Finally, some conclusions are drawn.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded domain with polygonal or polyhedral boundary $\partial\Omega$.

We consider the following equations:

$$A(\overline{\mathbf{u}}) + \nabla \overline{p} = \mathbf{f} \quad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot \overline{\mathbf{u}} = 0 \quad \text{in } \Omega, \tag{1b}$$

where

$$A(\overline{\mathbf{u}}) = \begin{cases} -2\mu \nabla \cdot \varepsilon(\overline{\mathbf{u}}) & \text{for Stokes,} \\ \alpha \overline{\mathbf{u}} & \text{for Darcy,} \end{cases} \tag{2}$$

with $\varepsilon(\overline{\mathbf{u}}) = \frac{1}{2}\left(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T\right)$, $\overline{\mathbf{u}}$ denoting the velocity vector, $\overline{p}$ the pressure and $\mathbf{f} \in \left[L^2(\Omega)\right]^d$ some given source term. α and μ are two positive parameters.

For simplicity, we assume Dirichlet and Neumann conditions on the boundary:

$$\overline{\mathbf{u}} = 0 \quad \text{on} \quad \partial\Omega \qquad \text{for Stokes,} \tag{3a}$$

$$\overline{\mathbf{u}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \qquad \text{for Darcy.} \tag{3b}$$

Here, $\mathbf{n}$ denotes the unitary vector of the normal to $\partial\Omega$.

In order to formulate the FVM scheme we first introduce the weak formulation of the problem (1a)-(1b). Let us define the function Hilbert spaces:

$$W^D = \{\overline{\mathbf{v}} \in \boldsymbol{H}_{div}(\Omega) : \overline{\mathbf{v}} \cdot \mathbf{n} = 0 \;\text{ on } \partial\Omega\}, \quad W^S = \left\{\overline{\mathbf{v}} \in H_0^1(\Omega)^d\right\}, \tag{4}$$

and

$$L_0^2 = \left\{\overline{q} \in L^2 : \int_\Omega \overline{q} = 0\right\}. \tag{5}$$

Likewise, we denote the product space $W^X \times L_0^2$ by $\mathcal{W}^X$ where X is chosen to be D or S depending on the choice of equation (2).

Now, let $a(\overline{\mathbf{u}}, \overline{\mathbf{v}})$ be the bilinear form corresponding to the weak formulation of $A(\overline{\mathbf{u}})$. We have for Stokes:

$$a(\overline{\mathbf{u}}, \overline{\mathbf{v}}) = \mu \left\{ \int_\Omega \nabla \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{v}} dx + \int_\Omega (\nabla \cdot \overline{\mathbf{u}})(\nabla \cdot \overline{\mathbf{v}}) dx \right\} \tag{6}$$

and for Darcy:

$$a(\overline{\mathbf{u}}, \overline{\mathbf{v}}) = \alpha \int_\Omega \overline{\mathbf{u}} \cdot \overline{\mathbf{v}} dx. \tag{7}$$

Consider also the global bilinear form:

$$B[(\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})] = a(\overline{\mathbf{u}}, \overline{\mathbf{v}}) - (\overline{p}, \nabla \cdot \overline{\mathbf{v}})_{0,\Omega} + (\overline{q}, \nabla \cdot \overline{\mathbf{u}})_{0,\Omega}. \tag{8}$$

The weak formulation of (1a)-(1b) now takes the form:

Find $(\overline{\mathbf{u}}, \overline{p}) \in \mathcal{W}^X$ such that:

$$B[(\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})] = (\mathbf{f}, \overline{\mathbf{v}})_{0,\Omega} \qquad \forall (\overline{\mathbf{v}}, \overline{q}) \in \mathcal{W}^X. \tag{9}$$

3. The approximate problem

3.1. Spatial discretization and inequalities

First, let us recall the notion of admissible discretization for the FVM which is given in [4].

Definition 3.1. An admissible finite volume mesh of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ where:

- $\mathcal{M}$ is a finite family of disjoint non-empty convex subdomains K of Ω (the "control volumes") such that:
  - If $d = 2$, each control volume is either a rectangle or a triangle with internal angles strictly lower than $\frac{\pi}{2}$;
  - If $d = 3$, each control volume is a rectangular parallelepiped;
  - $\overline{\Omega} = \bigcup_{K\in\mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K}\backslash K$ be the boundary of K and $|K| > 0$ denotes the measure of K.
- $\mathcal{E}$ is a finite family of disjoint subsets σ of $\overline{\Omega}$ (the "edges" of the mesh) such that, for all $\sigma \in \mathcal{E}$, there exists a hyper-plane E of $\mathbb{R}^d$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is non-empty open subset of E,whose $(d-1)$ dimensional measure $|\sigma| > 0$. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. Furthermore $\mathcal{E} = \bigcup_{K\in\mathcal{M}} \mathcal{E}_K$. We then assume that $\mathcal{E}$ is partitioned into $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$, such that:
  - $\mathcal{E}_{int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$;
  - $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$.
- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$, such that:
  - If $d = 2$, x_K is the intersection of the perpendicular bisectors of each edge.
  - If $d = 3$, x_K is the intersection of the lines issued from the center of the edge and orthogonal to the edge.

Note that any internal edge separating two control volumes K and L is denoted by $\sigma = K|\,L$ and satisfies the orthogonality condition:

$$x_\sigma = [x_K, x_L] \cap K|\,L.$$

Next, let us denote by d_{KL} the distance between x_K and x_L, and $d_{K\sigma}$ the distance between x_K and x_σ.

If h_K denotes the diameter of each control volume K, then h is the maximum of the values of h_K (for $K \in \mathcal{M}$), i.e.,

$$h = \sup_{K\in\mathcal{M}} h_K.$$

The regularity of the meshing is measured through the function regul $(\mathcal{D})$ defined by:

$$\operatorname{regul}(\mathcal{D}) = \inf\left\{\frac{d_{K\sigma}}{diam\ (K)};\ K \in \mathcal{M}, \sigma \in \mathcal{E}_K\right\}$$
$$\cup\left\{\frac{d_{K\sigma}}{d_{KL}}; K, L \in \mathcal{M} : \sigma = K|\, L \in \mathcal{E}_{int} \cap \mathcal{E}_K\right\} \cup \left\{\frac{1}{card\,(\mathcal{E}_K)}\right\},$$

and we select $\theta > 0$ such that regul $(\mathcal{D}) > \theta$.

Below, $C, C_1, C_2, C_3, \ldots$ will be used to denote constants which are independent of h but may depend on d, θ, Ω and μ.

Definition 3.2. Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 3.1 and let $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ be the space of functions which are piecewise constant over each control volume $K \in \mathcal{M}$. For all $v \in H_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote v_K the constant value of v in K. For $(v, w) \in H_{\mathcal{D}}(\Omega)^2$, we define the following inner product (corresponding to Dirichlet boundary conditions) called "discrete H_0^1 inner product":

$$[v, w]_{\mathcal{D}} = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ (\sigma = K|L)}} \frac{|\sigma|}{d_{KL}} (v_L - v_K)(w_L - w_K) + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ (\sigma \in \varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}} v_K w_K. \tag{10}$$

A norm in $H_{\mathcal{D}}(\Omega)$ called "discrete H_0^1 norm" is obtained as follows:

$$\|v\|_{1,\mathcal{D}} = [v, v]_{\mathcal{D}}^{1/2}.$$

We also define the following bilinear form (corresponding to Neumann boundary conditions):

$$\langle v, w\rangle_{\mathcal{D}} = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ (\sigma = K|L)}} \frac{|\sigma|}{d_{KL}} (v_L - v_K)(w_L - w_K), \tag{11}$$

with the semi norm:

$$|v|_{1,\mathcal{D}} = \langle v, v\rangle_{\mathcal{D}}^{1/2}.$$

These definitions extend naturally to vector valued functions as follows. For $\mathbf{v} = \left(v^{(i)}\right)_{i=1,\ldots,d} \in H_{\mathcal{D}}(\Omega)^d$ and $\mathbf{w} = \left(w^{(i)}\right)_{i=1,\ldots,d} \in H_{\mathcal{D}}(\Omega)^d$, we set:

$$[\mathbf{v}, \mathbf{w}]_{\mathcal{D}} = \sum_{i=1}^{d} \left[v^{(i)}, w^{(i)}\right]_{\mathcal{D}}, \quad \|\mathbf{v}\|_{1,\mathcal{D}} = \left(\sum_{i=1}^{d} \left[v^{(i)}, v^{(i)}\right]_{\mathcal{D}}\right)^{1/2}.$$

Proposition 3.3. *The following discrete Poincaré inequalities hold*:

$$\begin{aligned} &\|v\|_{0,\Omega} \leq \operatorname{diam}(\Omega)\, \|v\|_{1,\mathcal{D}} && \forall v \in H_{\mathcal{D}}(\Omega); \\ &\|v\|_{0,\Omega} \leq C(\Omega)\, |v|_{1,\mathcal{D}} && \forall v \in H_{\mathcal{D}}(\Omega) \quad \text{such that } \textstyle\int_\Omega u = 0, \end{aligned} \tag{12}$$

where $C(\Omega)$ depends only on Ω (cf. [4]*).*

As in [5], let us define the interpolation operator $\pi_{\mathcal{D}} : L^2(\Omega) \to H_{\mathcal{D}}(\Omega)$ by setting $(\pi_{\mathcal{D}}\overline{u})_K = \emptyset_K(x_K)$ for all $K \in \mathcal{M}$ and $\overline{u} \in L^2(\Omega)$ with $\emptyset_K$ being the orthogonal projection of $L^2(\Omega)$ on $\mathbb{P}_1$.

It has a natural extension to vector valued functions. We will keep the same notation.

The operator $\pi_{\mathcal{D}}$ satisfies the following:

Proposition 3.4. *Let $\overline{u}$ be a function in $H^2(\Omega) \cap H_0^1(\Omega)$. Then, there exists $C > 0$ such that he following estimate holds*:

$$\|\pi_{\mathcal{D}}\overline{u}\|_{1,\mathcal{D}} \leq Ch|\overline{u}|_{2,\Omega}. \tag{13}$$

Proof. Here, we make use of results established in [5].

Let $\overline{u} \in H^2(\Omega) \cap H_0^1(\Omega)$. First, we have

$$\begin{aligned}
\|\pi_{\mathcal{D}}\overline{u}\|_{1,\mathcal{D}}^2 &= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}((\pi_{\mathcal{D}}\overline{u})_L - (\pi_{\mathcal{D}}\overline{u})_K)^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}}((\pi_{\mathcal{D}}\overline{u})_K)^2 \\
&= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(\emptyset_L(x_L) - \emptyset_K(x_K))^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}}\emptyset_K(x_K)^2 \\
&\leq 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(\emptyset_K(x_L) - \emptyset_K(x_K))^2 \\
&\quad + 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(\emptyset_L(x_L) - \emptyset_K(x_L))^2 \\
&\quad + 2\sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}}(\emptyset_K(x_K) - \emptyset_K(x_\sigma))^2 + 2\sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}}\emptyset_K(x_\sigma)^2 \\
&= T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

To proceed, we must now bound each term of the right hand side of this relation. The first summation in the above relation reads:

$$T_1 = 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(\nabla\emptyset_K \cdot (x_L - x_K))^2 = 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma|\, d_{KL}(\nabla\emptyset_K \cdot n_{KL})^2.$$

We define w_K as the convex hull of $K \cup \left(L; \overline{L} \cap \overline{K} \in \mathcal{E}_K\right)$. Denote by $\overline{h}_K$ the diameter of w_K. The quantity $|\sigma|\, d_{KL}$ can be seen as the measure of a domain included in $K \cup L$, and so is lower than the measure of w_K. We get

$$\begin{aligned}
T_1 &\leq 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\emptyset_K|_{1,w_K}^2 \leq 2C_1\overline{h}_K^2 \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\overline{u}|_{2,w_K}^2 \\
&\leq C_2 h^2 |\overline{u}|_{2,\Omega}^2. \qquad (14)
\end{aligned}$$

The second summation can be estimated as follows:

$$T_2 \le 2C_3^2 \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}\,|L|} \|\emptyset_L - \emptyset_K\|_{0,L}^2$$
$$\le 2C_3^2 \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}\,|L|} \left[\|\emptyset_L - u\|_{0,L}^2 + \|\emptyset_K - u\|_{0,L}^2\right].$$

Then, since L is included in both w_L and w_K, we get

$$T_2 \le 2C_3^2 \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}\,|L|} \left[\|\emptyset_L - u\|_{0,w_L}^2 + \|\emptyset_K - u\|_{0,w_K}^2\right]$$
$$\le 2C_3^2 \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}\,|L|} \left[\overline{h}_L^4 |\overline{u}|_{2,w_L}^2 + \overline{h}_K^4 |\overline{u}|_{2,w_K}^2\right]$$
$$\le C_4 h^2 |\overline{u}|_{2,\Omega}^2. \tag{15}$$

Using the same arguments as for the bound of the term T_1 (replace d_{KL} by $d_{K\sigma}$), a similar result can be obtained for the third term:

$$T_3 \le C_5 h^2 |\overline{u}|_{2,\Omega}^2. \tag{16}$$

Finally, using the linearity of $\emptyset_K$ and the fact that u vanishes on $\partial\Omega$, we have

$$T_4 = 2 \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}} \emptyset_K(x_\sigma)^2 = \frac{1}{|\sigma|\,d_{K\sigma}} \left(\int_\sigma \emptyset_K - u\right)^2 \le \frac{1}{d_{K\sigma}} \|\emptyset_K - u\|_{0,\sigma}^2.$$

It follows that

$$T_4 \le d \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{K\sigma}\,|K|} \left[\|\emptyset_K - u\|_{0,K} + \|\emptyset_K - u\|_{1,K}\right]^2$$
$$\le C_6 h^2 |\overline{u}|_{2,\Omega}^2. \tag{17}$$

Gathering (14)-(17) yields the result. □

3.2. The natural scheme

Let us begin by defining the discrete divergence operator $(\nabla_{\mathcal{D}}\cdot) : H_{\mathcal{D}}(\Omega)^d \to H_{\mathcal{D}}(\Omega)$ by:

$$\forall \mathbf{u} \in H_{\mathcal{D}}(\Omega)^d \quad (\nabla_{\mathcal{D}} \cdot \mathbf{u})_K = \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma| \frac{\mathbf{u}_L + \mathbf{u}_K}{2} \cdot \mathbf{n}_\sigma \quad \forall K \in \mathcal{M}. \tag{18}$$

Likewise, define the discrete gradient $\nabla_{\mathcal{D}} p \in H_{\mathcal{D}}(\Omega)^d$, for any $p \in H_{\mathcal{D}}(\Omega)$ by:

$$(\nabla_{\mathcal{D}} p)_K = \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma| \frac{p_L+p_K}{2}\mathbf{n}_\sigma + \sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K} |\sigma| \, p_K \mathbf{n}_\sigma\right). \tag{19}$$

Since $\sum_{\sigma\in\mathcal{E}_K} |\sigma| \, \mathbf{n}_\sigma = 0$ for all $K \in \mathcal{M}$, this discrete gradient equivalently can be rewritten as:

$$(\nabla_{\mathcal{D}} p)_K = \frac{1}{|K|}\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma| \frac{p_L-p_K}{2}\mathbf{n}_\sigma. \tag{20}$$

Thus, reordering the summations, the adjoint of the discrete divergence with respect to the discrete L^2 inner product defines this discrete gradient $\nabla_{\mathcal{D}}$. For any pressure $q \in H_{\mathcal{D}}(\Omega)$ and velocity $\mathbf{v} \in H_{\mathcal{D}}(\Omega)^d$ we get:

$$\begin{aligned}
\int_\Omega (\nabla_{\mathcal{D}} q)\cdot \mathbf{v} &= \sum_{K\in\mathcal{M}} \mathbf{v}_K \cdot \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma| \frac{q_L-q_K}{2}\mathbf{n}_K \\
&= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ (\sigma=K|L)}} |\sigma| \frac{q_L-q_K}{2}\mathbf{v}_K\cdot\mathbf{n}_K + |\sigma| \frac{q_K-q_L}{2}\mathbf{v}_L\cdot\mathbf{n}_L \\
&= -\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ (\sigma=K|L)}} |\sigma| \frac{q_K-q_L}{2}(\mathbf{v}_K+\mathbf{v}_L)\cdot\mathbf{n}_K \\
&= -\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ (\sigma=K|L)}} q_K |\sigma| \frac{\mathbf{v}_K+\mathbf{v}_L}{2}\cdot\mathbf{n}_K - q_L |\sigma| \frac{\mathbf{v}_K+\mathbf{v}_L}{2}\cdot\mathbf{n}_K \\
&= -\int_\Omega q\,(\nabla_{\mathcal{D}}\cdot\mathbf{v}). \qquad (21)
\end{aligned}$$

Now, it remains to give a FV discretization of the viscous stress tensor $\varepsilon(\overline{\mathbf{u}})$.

To this end, let us note that the divergence of the stress tensor can be written as:

$$\nabla\cdot\varepsilon(\overline{\mathbf{u}}) = \frac{1}{2}(\Delta\overline{\mathbf{u}} + \nabla(\nabla\cdot\overline{\mathbf{u}})). \tag{22}$$

So, for any given function $\mathbf{u} \in H_{\mathcal{D}}(\Omega)^d$, let $(\Delta_{\mathcal{D}}\mathbf{u}) \in H_{\mathcal{D}}(\Omega)$ be the function defined by

$$(\Delta_{\mathcal{D}}\mathbf{u})_K = \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} \frac{|\sigma|}{d_{LK}}(\mathbf{u}_L-\mathbf{u}_K) + \sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K} \frac{|\sigma|}{d_{K\sigma}}(-\mathbf{u}_K)\right), \tag{23}$$

which is called the discrete Laplace operator (corresponding to the homogeneous Dirichlet boundary conditions).

For discretizing the second summation in (22), we have taken a version of a discrete operator $\nabla_{\mathcal{D}}(\nabla_{\mathcal{D}} \cdot \mathbf{u})$ introduced in [8]. This can be formulated as follows:

$$(\nabla_{\mathcal{D}}(\nabla_{\mathcal{D}} \cdot \mathbf{u}))_K = \frac{1}{|K|} \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} |\sigma| \frac{(\nabla_{\mathcal{D}} \cdot \mathbf{u})_L - (\nabla_{\mathcal{D}} \cdot \mathbf{u})_K}{2} \mathbf{n}_{\sigma,K}. \tag{24}$$

As already mentioned in the introduction, it is very important to emphasize that this version has not been analyzed and applied elsewhere.

Under all above considerations, the natural (classical) stabilized finite volume scheme for the solution of problem (1a)-(1b) consists in finding $(\mathbf{u}, p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ such that for each control volume K of $\mathcal{M}$:

$$\int_K (A_{\mathcal{D}}(\mathbf{u}))_K + \int_K (\nabla_{\mathcal{D}} p)_K = \int_K \mathbf{f}, \tag{25a}$$

$$\int_K (\nabla_{\mathcal{D}} \cdot \mathbf{u})_K + \delta \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} |\sigma| \, h_{\partial K} [p] = 0, \tag{25b}$$

where

$$(A_{\mathcal{D}}(\mathbf{u}))_K = \begin{cases} -\mu (\Delta_{\mathcal{D}} \mathbf{u})_K - \mu (\nabla_{\mathcal{D}} (\nabla_{\mathcal{D}} \cdot \mathbf{u}))_K & \text{for Stokes,} \\ \alpha |K| \, \mathbf{u}_K & \text{for Darcy,} \end{cases} \tag{26}$$

and $[\cdot]$ denoting the jump through interior edges. For insuring pressure uniqueness, the system (25a)-(25b) is supplemented by the relation:

$$\sum_{K \in \mathcal{M}} |K| \, p_K = 0.$$

Next, denote by $\mathcal{W}_{\mathcal{D}}^X$ the product space $H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ with $\int_\Omega p = 0$, where the upper index X is chosen to be S or D depending on the choice of Stokes or Darcy's equation. $\mathcal{W}_{\mathcal{D}}^X$ is equipped with the following norm:

$$\|(\mathbf{u}, p)\|_{\mathcal{W}_{\mathcal{D}}^X}^2 = \|\mathbf{u}\|_l^2 + \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2,$$

with

$$l = \begin{cases} 1, \mathcal{D} & \text{for Stokes,} \\ 0, \Omega & \text{for Darcy.} \end{cases}$$

3.3. Finite volume scheme for the weak formulation

The proposed schemes (25a)-(25b) may be recast under following "discrete variational form" closer to that used in the finite element setting: Find $(\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^X$ such that:

$$a(\mathbf{u}, \mathbf{v}) - (p, \nabla_{\mathcal{D}} \cdot \mathbf{v})_{0,\Omega} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_D(\Omega)^d, \tag{27a}$$

$$(q, \nabla_{\mathcal{D}} \cdot \mathbf{u})_{0,\Omega} + J(p, q) = 0 \qquad \forall q \in H_D(\Omega), \tag{27b}$$

where

$$(28)\quad a\left(\mathbf{u},\mathbf{v}\right)=\begin{cases}\mu[\mathbf{u},\mathbf{v}]_{1,\mathcal{D}}+\mu\int_{\Omega}\left(\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)\left(\nabla_{\mathcal{D}}\cdot\mathbf{v}\right) & \text{for Stokes,}\\ \alpha\sum_{K\in\mathcal{M}}|K|\,\mathbf{u}_K\cdot\mathbf{v}_K & \text{for Darcy,}\end{cases}$$

and

$$J\left(p,q\right)=\delta\sum_{K\in\mathcal{M}}\int_{\partial K\backslash\partial\Omega}h_{\partial K}\left[p\right]\left[q\right]ds.$$

Indeed, let $\mathbf{v}\in H_{\mathcal{D}}(\Omega)^d$ and $q\in H_{\mathcal{D}}(\Omega)$. Multiplying (25a) and (25b) by $\mathbf{v}_K$ and q_K respectively, and summing over $K\in\mathcal{M}$, the above formulation is straightforward.

For Stokes we have

$$\int_K\left(A_{\mathcal{D}}\left(\mathbf{u}\right)\right)_K\cdot\mathbf{v}_K=A+B,$$

where

$$\begin{aligned}A&=\sum_K\mathbf{v}_K\cdot\int_K-\mu\left(\Delta_{\mathcal{D}}\mathbf{u}\right)_K\\ &=-\mu\sum_K\mathbf{v}_K\cdot\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}}\frac{|\sigma|}{d_{KL}}\left(\mathbf{u}_L-\mathbf{u}_K\right)+\sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K}\frac{|\sigma|}{d_{K\sigma}}\left(-\mathbf{u}_K\right)\right)\\ &=-\mu\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ (\sigma=K|L)}}\frac{|\sigma|}{d_{KL}}\left(\mathbf{u}_L-\mathbf{u}_K\right)\cdot\left(\mathbf{v}_K-\mathbf{v}_L\right)+\sum_{\substack{\sigma\in\mathcal{E}_{ext}\\ (\sigma\in\mathcal{E}_K)}}\frac{|\sigma|}{d_{K\sigma}}\left(-\mathbf{u}_K\right)\cdot\mathbf{v}_K\right)\\ &=\mu[\mathbf{u},\mathbf{v}]_{\mathcal{D}},\end{aligned}$$

and

$$\begin{aligned}B&=-\mu\sum_K\mathbf{v}_K\cdot\int_K\left(\nabla_{\mathcal{D}}\left(\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)\right)_K\\ &=-\,\mu\sum_K\mathbf{v}_K\cdot\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}}|\sigma|\,\frac{\left(\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)_L-\left(\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)_K}{2}\mathbf{n}_{\sigma,K}.\end{aligned}$$

Using the same technique as in (21), we get

$$B=\mu\int_{\Omega}\left(\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)\left(\nabla_{\mathcal{D}}\cdot\mathbf{v}\right).$$

Now, we are in a position to introduce the following bilinear form which we will be the base for our FVM:

$$(29)\quad B_{\mathcal{D}}\left[\left(\mathbf{u},p\right),\left(\mathbf{v},q\right)\right]=a\left(\mathbf{u},\mathbf{v}\right)-\left(p,\nabla_{\mathcal{D}}\cdot\mathbf{v}\right)_{0,\Omega}+\left(q,\nabla_{\mathcal{D}}\cdot\mathbf{u}\right)_{0,\Omega}+J\left(p,q\right),$$

in a such way that our proposed FV formulation reads: Find $(\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^{X}$ such that

$$B_{\mathcal{D}}\left[(\mathbf{u}, p), (\mathbf{v}, q)\right] = (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall (\mathbf{v}, q) \in \mathcal{W}_{\mathcal{D}}^{X}. \tag{30}$$

4. FV scheme study

4.1. Discrete solution regularity

Let us first state a boundedness result in the $L^2(\Omega)$ norm of the discrete divergence for any element $\mathbf{u} \in H_{\mathcal{D}}(\Omega)^d$.

Lemma 4.1. *Let $\mathcal{D}$ be an admissible finite volume discretization of Ω in the sense of Definition* (3.1). *Then, there exists C such that for all $\mathbf{u} \in H_{\mathcal{D}}(\Omega)^d$*

$$\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega} \leq C\|\mathbf{u}\|_{1,\mathcal{D}.} \tag{31}$$

Proof. For any $\mathbf{u} \in H_{\mathcal{D}}(\Omega)^d$ we have

$$\begin{aligned}
\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 &= \sum_K \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{\mathbf{u}_L+\mathbf{u}_K}{2}\cdot\mathbf{n}_\sigma\right)^2 \\
&= \sum_K \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{\mathbf{u}_L+\mathbf{u}_K}{2}\cdot\mathbf{n}_\sigma - \sum_{\substack{\sigma\in\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\mathbf{u}_K\cdot\mathbf{n}_\sigma\right)^2 \\
&= \sum_K \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{\mathbf{u}_L}{2}\cdot\mathbf{n}_\sigma + \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{\mathbf{u}_K}{2}\cdot\mathbf{n}_\sigma\right. \\
&\qquad\left. - \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\mathbf{u}_K\cdot\mathbf{n}_\sigma - \sum_{\substack{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\mathbf{u}_K\cdot\mathbf{n}_\sigma\right)^2 \\
&= \sum_K \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{\mathbf{u}_L-\mathbf{u}_K}{2}\cdot\mathbf{n}_\sigma - \sum_{\substack{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\mathbf{u}_K\cdot\mathbf{n}_\sigma\right)^2 \\
&\leq 4\sum_K \frac{1}{|K|}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} \frac{|\sigma|^2}{4}(\mathbf{u}_L-\mathbf{u}_K)^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|^2(\mathbf{u}_K)^2\right)
\end{aligned}$$

$$= \sum_{K} \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} \frac{|\sigma|^2}{|K|} (\mathbf{u}_L - \mathbf{u}_K)^2 + 4 \sum_{K} \sum_{\substack{\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K \\ \sigma = K|L}} \frac{|\sigma|^2}{|K|} (\mathbf{u}_K)^2$$

$$= 2 \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|^2}{|K|} (\mathbf{u}_L - \mathbf{u}_K)^2 + 4 \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma = K|L}} \frac{|\sigma|^2}{|K|} (\mathbf{u}_K)^2.$$

The usual regularity yields

$$\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 \leq C_1 \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{d_{KL}} (\mathbf{u}_L - \mathbf{u}_K)^2 + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma = K|L}} \frac{|\sigma|}{d_{K\sigma}} (\mathbf{u}_K)^2 \right)$$

as required. □

Now, we can state the results for the discrete solution regularity.

Proposition 4.2 (Velocity regularity). *Let* $(\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^X$ *be the solution to* (30). *Then, there exists* C *which depends on* d, Ω *such that the following bounds hold*:

$$\|\mathbf{u}\|_l^2 \leq C \|\mathbf{f}\|_{0,\Omega}^2. \tag{32}$$

Moreover,

$$J(p,p) \leq C \|\mathbf{f}\|_{0,\Omega}^2. \tag{33}$$

Proof. Setting $\mathbf{v} = \mathbf{u}$ and $q = p$ in (30), we get

$$a(\mathbf{u}, \mathbf{u}) + J(p,p) = (\mathbf{f}, \mathbf{u})_{0,\Omega}.$$

Hence, for Stokes we have

$$\mu \|\mathbf{u}\|_{1,\mathcal{D}}^2 + \mu \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 + J(p,p) = (\mathbf{f}, \mathbf{u})_{0,\Omega}.$$

Using a Young inequality followed by the Poincaré inequality (12), we get

$$\mu \|\mathbf{u}\|_{1,\mathcal{D}}^2 + J(p,p) \leq \frac{\operatorname{diam}(\Omega)^2}{2\varepsilon} \|\mathbf{f}\|_{0,\Omega}^2 + \frac{\varepsilon}{2} \|\mathbf{u}\|_{1,\mathcal{D}}^2,$$

which leads to

$$\left(\mu - \frac{\varepsilon}{2}\right) \|\mathbf{u}\|_{1,\mathcal{D}}^2 + J(p,p) \leq \frac{\operatorname{diam}(\Omega)^2}{2\varepsilon} \|\mathbf{f}\|_{0,\Omega}^2$$

if $\varepsilon < 2\mu$.

For Darcy, the choice of $\varepsilon < 2\alpha$ would give

$$\alpha \|\mathbf{u}\|_{0,\Omega}^2 + J(p,p) \leq \frac{1}{2\varepsilon} \|\mathbf{f}\|_{0,\Omega}^2 + \frac{\varepsilon}{2} \|\mathbf{u}\|_{0,\Omega}^2.$$

Consequently, the combined choice $\varepsilon < 2\min\{\mu, \alpha\}$ yields the required bounds. □

Proposition 4.3 (**Pressure regularity**). *Let* $(\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^{X}$ *be a solution to* (30). *Then, there exists* C, *depending only on* d, Ω, μ *and* θ, *such that the following bound holds*:

$$\|p\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}. \tag{34}$$

Proof. Let $p \in H_{\mathcal{D}}(\Omega)$ be given. We apply a classical consequence of a lemma due to Nečas [12]. So, there exists $\overline{\mathbf{w}} \in H_0^1(\Omega)^d$ such that

$$\nabla \cdot \overline{\mathbf{w}}\,(\boldsymbol{x}) = p(x) \quad \text{and} \quad \|\overline{\mathbf{w}}\|_{1,\Omega} \leq C_1\|p\|_{0,\Omega}. \tag{35}$$

Next, we set:

$$w_\sigma^{(i)} = \frac{1}{|\sigma|}\int_\sigma \overline{w}^{(i)}\,(x)\,d\gamma\,(x)\,, \quad \forall \sigma \in \mathcal{E}, \quad i = 1, \ldots, d. \tag{36}$$

Note that $w_\sigma^{(i)} = 0$ for all $\sigma \in \mathcal{E}_{ext}$ and $i = 1, \ldots, d$. We also define $\mathbf{w} \in H_{\mathcal{D}}(\Omega)^d$ by

$$w_K^{(i)} = \frac{1}{|K|}\int_K \overline{w}^{(i)}\,(x)\,dx, \quad \forall K \in \mathcal{M}, \quad i = 1, \ldots, d. \tag{37}$$

Applying the results given in [4], there exists $C_2 > 0$ such that

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \quad |\mathbf{w}_K - \mathbf{w}_\sigma|^2 \leq C_2\frac{h_K}{|\sigma|}\int_K |\nabla\overline{\mathbf{w}}\,(\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{38}$$

In addition, by the continuity of the interpolation operator $\pi_{\mathcal{D}}$ (13), there exists another real number $C_3 > 0$ such that

$$\|\mathbf{w}\|_{1,\mathcal{D}} \leq C_3\|\overline{\mathbf{w}}\|_{1,\Omega} \leq C_4\|p\|_{0,\Omega}. \tag{39}$$

Next,

$$\begin{aligned}
(p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} &= \sum_K p_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\frac{(\mathbf{w}_L + \mathbf{w}_K)}{2}\cdot \mathbf{n}_\sigma \\
&= \sum_K p_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\,\mathbf{w}_\sigma \cdot \mathbf{n}_\sigma \\
&\quad + \sum_K p_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} |\sigma|\left(\frac{\mathbf{w}_L + \mathbf{w}_K}{2} - \mathbf{w}_\sigma\right)\cdot \mathbf{n}_\sigma \\
&\quad = A + B.
\end{aligned}$$

We have

$$\begin{aligned}
A &= \sum_K p_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} \int_\sigma \overline{\mathbf{w}}\,(\boldsymbol{x})\cdot \mathbf{n}_\sigma d\gamma\,(\boldsymbol{x}) \\
&= \sum_K p_K \int_{\partial K} \overline{\mathbf{w}}\,(\boldsymbol{x})\cdot \mathbf{n}_\sigma d\gamma\,(\boldsymbol{x})
\end{aligned}$$

$$= \sum_K p_K \int_K \nabla \cdot \overline{\mathbf{w}}\,(\boldsymbol{x})\, d\boldsymbol{x}$$
$$= \|p\|_{0,\Omega}^2,$$

and

$$B = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma|\,(p_K - p_L)\left(\frac{\mathbf{w}_L + \mathbf{w}_K}{2} - \mathbf{w}_\sigma\right)\cdot \mathbf{n}_\sigma .$$

Applying Cauchy-Schwarz inequality gives

$$|B|^2 \le \left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} h\,|\sigma|\,(p_K-p_L)^2\right)\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(\frac{\mathbf{w}_L+\mathbf{w}_K}{2}-\mathbf{w}_\sigma\right)^2\right)$$
$$= \left(\frac{1}{2}\sum_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} h\,|\sigma|\,(p_K-p_L)^2\right)\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(\frac{\mathbf{w}_L+\mathbf{w}_K}{2}-\mathbf{w}_\sigma\right)^2\right)$$
$$= \left(\frac{1}{2}\sum_K \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\ \sigma=K|L}} \int_\sigma h\,[p]^2\right)\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(\frac{\mathbf{w}_L+\mathbf{w}_K}{2}-\mathbf{w}_\sigma\right)^2\right)$$
$$= \left(\frac{1}{2}\sum_K \int_{\partial K\setminus\partial\Omega} h\,[p]^2\right)\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(\frac{\mathbf{w}_L+\mathbf{w}_K}{2}-\mathbf{w}_\sigma\right)^2\right).$$

Applying inequality (38) and the obvious inequality:

$$\left(\frac{\mathbf{w}_L+\mathbf{w}_K}{2}-\mathbf{w}_\sigma\right)^2 \le \frac{1}{2}\left((\mathbf{w}_K-\mathbf{w}_\sigma)^2 + (\mathbf{w}_L-\mathbf{w}_\sigma)^2\right),$$

we get

$$|B|^2 \le \frac{1}{4\delta} J(p,p) \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left((\mathbf{w}_K-\mathbf{w}_\sigma)^2 + (\mathbf{w}_L-\mathbf{w}_\sigma)^2\right)$$
$$\le \frac{1}{4\delta} J(p,p) \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(C_2\frac{h_K}{|\sigma|}\int_K |\nabla\overline{\mathbf{w}}\,(\boldsymbol{x})|^2\, dx + C_2\frac{h_L}{|\sigma|}\int_L |\nabla\overline{\mathbf{w}}\,(\boldsymbol{x})|^2 dx\right)$$
$$\le J(p,p) \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{h}\left(C_5\frac{h}{|\sigma|}\int_{K\cup L} |\nabla\overline{\mathbf{w}}\,(\boldsymbol{x})|^2\, dx\right) \le C_5 J(p,p)\|\overline{\mathbf{w}}\|_{1,\Omega}^2$$

so that

$$|B| \le C_6 J\,(p,p)^{\frac{1}{2}}\,\|p\|_{0,\Omega}.$$

Collecting all estimated terms, we obtain

$$(p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} \geq \|p\|_{0,\Omega}^2 - C_6 J(p,p)^{\frac{1}{2}} \|p\|_{0,\Omega}.$$

Next, let us set $\mathbf{v} = \mathbf{w}$ in (27a). We get

$$(p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} = a(\mathbf{u}, \mathbf{w}) - (\mathbf{f}, \mathbf{w})_{0,\Omega}$$

so that

$$\|p\|_{0,\Omega}^2 - C_6 J(p,p)^{\frac{1}{2}} \|p\|_{0,\Omega} \leq a(\mathbf{u}, \mathbf{w}) - (\mathbf{f}, \mathbf{w})_{0,\Omega}.$$

Now, it remains to bound each term of the right hand side of this relation. To this end, we have

$$(\mathbf{f}, \mathbf{w})_{0,\Omega} \leq \|\mathbf{f}\|_{0,\Omega} \operatorname{diam}(\Omega) \|\mathbf{w}\|_{1,\mathcal{D}} \leq C_7 \|\mathbf{f}\|_{0,\Omega} \|p\|_{0,\Omega}.$$

On the other hand, for Stokes we have

$$a(\mathbf{u}, \mathbf{w}) \leq \mu \|\mathbf{u}\|_{1,\mathcal{D}} \|\mathbf{w}\|_{1,\mathcal{D}} + \mu \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega} \|\nabla_{\mathcal{D}} \cdot \mathbf{w}\|_{0,\Omega}.$$

Applying inequalities (31), (32) and (39), we get

$$a(\mathbf{u}, \mathbf{w}) \leq C_8 \|\mathbf{f}\|_{0,\Omega} \|p\|_{0,\Omega}.$$

For Darcy, applying Cauchy-Schwarz inequality, (12), (32) and (39), we get

$$a(\mathbf{u}, \mathbf{w}) \leq C_9 \|\mathbf{f}\|_{0,\Omega} \|p\|_{0,\Omega}$$

which yields

$$\|p\|_{0,\Omega}^2 - C_6 J(p,p)^{\frac{1}{2}} \|p\|_{0,\Omega} \leq C_{10} \|\mathbf{f}\|_{0,\Omega} \|p\|_{0,\Omega}.$$

Consequently,

$$\|p\|_{0,\Omega} \leq C_{10} \|\mathbf{f}\|_{0,\Omega} + C_{11} J(p,p)^{\frac{1}{2}}.$$

Finally, applying inequality (33) gives the claimed estimate. □

4.2. Scheme stability

The crucial point is to show that the stabilizing term $J(p,p)$ enhances sufficiently the degrees of freedom in the pressure field so that an inf-sup condition is satisfied. In the analysis, we will use the following composite norm:

$$|||\,(\mathbf{u}, p)\,|||^2 = \|(\mathbf{u}, p)\|_{\mathcal{W}_{\mathcal{D}}^X}^2 + J(p,p) \qquad \forall (\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^X.$$

The main result of this subsection is the following theorem which assures the wellposedness of our FV scheme.

Theorem 4.4. *The FV formulation* (30) *satisfies the following inf-sup condition*:

$$\gamma\, |||\,(\mathbf{u}, p)\,||| \leq \sup_{(\mathbf{v},q) \in \mathcal{W}_{\mathcal{D}}^X} \frac{B_{\mathcal{D}}[(\mathbf{u}, p), (\mathbf{v}, q)]}{|||\,(\mathbf{v}, q)\,|||} \qquad \forall (\mathbf{u}, p) \in \mathcal{W}_{\mathcal{D}}^X. \tag{40}$$

Proof. 1) Control of $\|\mathbf{u}\|_l^2$:

Taking $(\mathbf{v}, q) = (\mathbf{u}, p)$ in (29) we get

$$B_{\mathcal{D}}\left[(\mathbf{u}, p), (\mathbf{u}, p)\right] = a(\mathbf{u}, \mathbf{u}) + J(p, p).$$

Hence, we have for Stokes:

$$a(\mathbf{u}, \mathbf{u}) \geq \mu \|\mathbf{u}\|_{1,\mathcal{D}}^2,$$

and for Darcy:

$$a(\mathbf{u}, \mathbf{u}) = \alpha \|\mathbf{u}\|_{0,\Omega}^2.$$

Thus,

$$B_{\mathcal{D}}\left[(\mathbf{u}, p), (\mathbf{u}, p)\right] \geq C_a \|\mathbf{u}\|_l^2 + J(p, p), \tag{41}$$

where

$$C_a = \begin{cases} \mu & \text{for Stokes,} \\ \alpha & \text{for Darcy.} \end{cases}$$

2) Control of $\|p\|_{0,\Omega}^2$:

Here, we use the same technique as in Proposition 4.3. Let $p \in H_{\mathcal{D}}(\Omega)$ be given. We apply a classical consequence of a lemma due to Nečas [12]. There exists $\overline{\mathbf{w}} \in H_0^1(\Omega)^d$ such that

$$\nabla \cdot \overline{\mathbf{w}}(\boldsymbol{x}) = -p(x) \quad \text{and} \quad \|\overline{\mathbf{w}}\|_{1,\Omega} \leq C_1 \|p\|_{0,\Omega}.$$

Anew, we define $\mathbf{w} \in H_{\mathcal{D}}(\Omega)^d$ as in (36) and (37). In addition, by the continuity of the interpolation operator $\pi_{\mathcal{D}}$ there exists another real number $C_2 > 0$ such that

$$\|\mathbf{w}\|_{1,\mathcal{D}} \leq C_2 \|\overline{\mathbf{w}}\|_{1,\Omega} \leq C_3 \|p\|_{0,\Omega}. \tag{42}$$

Again, taking $(\mathbf{v}, q) = (\mathbf{w}, 0)$ in (29) we obtain

$$B_{\mathcal{D}}\left[(\mathbf{u}, p), (\mathbf{w}, 0)\right] = a(\mathbf{u}, \mathbf{w}) - (p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega}. \tag{43}$$

To proceed, let us bound each term of the right hand side of this relation. First, for Stokes, we have

$$|a(\mathbf{u}, \mathbf{w})| \leq \mu \|\mathbf{u}\|_{1,\mathcal{D}} \|\mathbf{w}\|_{1,\mathcal{D}} + \mu \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega} \|\nabla_{\mathcal{D}} \cdot \mathbf{w}\|_{0,\Omega}.$$

Applying inequality (31) and (42) yields

$$|a(\mathbf{u}, \mathbf{v})| \leq \frac{2\,\mu^2}{\varepsilon} \|\mathbf{u}\|_{1,\mathcal{D}}^2 + \varepsilon\, C_4 \|p\|_{0,\Omega}^2.$$

Next, for Darcy we have

$$\begin{aligned} |a(\mathbf{u}, \mathbf{w})| &\leq \alpha C_3 \operatorname{diam}(\Omega) \|\mathbf{u}\|_{0,\Omega} \|p\|_{0,\Omega} \\ &\leq \frac{1}{\varepsilon} \|\mathbf{u}\|_{0,\Omega}^2 + \varepsilon C_4 \|p\|_{0,\Omega}^2 \end{aligned}$$

with $C_4 \geq (\alpha C_3 \operatorname{diam}(\Omega))^2$.

In both cases, it follows that

$$a(\mathbf{u},\mathbf{w}) \geq -\frac{C_b}{\varepsilon}\|u\|_l^2 - \varepsilon C_4 \|p\|_{0,\Omega}^2,$$

where

$$C_b = \begin{cases} 2\mu^2 & \text{for Stokes,} \\ 1 & \text{for Darcy.} \end{cases}$$

For the second term in (43), we have

$$\begin{aligned}(p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} = & \sum_K p_K \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} |\sigma|\, \mathbf{w}_\sigma \cdot \mathbf{n} \\ & + \sum_K p_K \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} |\sigma| \left(\frac{\mathbf{w}_L + \mathbf{w}_K}{2} - \mathbf{w}_\sigma \right) \cdot \mathbf{n} \\ \leq & \; - \|p\|_{0,\Omega}^2 + \frac{1}{\varepsilon} J(p,p) + \varepsilon C_5 \|p\|_{0,\Omega}^2 .\end{aligned}$$

This yields

$$-(p, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} \geq (1 - \varepsilon\, C_5) \|p\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J(p,p).$$

Gathering the above results, we finally get

$$B_{\mathcal{D}}[(\mathbf{u},p),(\mathbf{w},0)] \geq -\frac{C_b}{\varepsilon}\|\mathbf{u}\|_l^2 + (1 - \varepsilon\,(\,C_4 + C_5))\|p\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J(p,p). \tag{44}$$

3) Control of $\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2$:
Taking $(\mathbf{v}, q) = (0, \nabla_{\mathcal{D}} \cdot \mathbf{u})$ in (29) gives

$$B_{\mathcal{D}}[(\mathbf{u},p),(0,\nabla_{\mathcal{D}} \cdot \mathbf{u})] = \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 + J(p, \nabla_{\mathcal{D}} \cdot \mathbf{u}).$$

By the definition of discrete divergence operator, $\nabla_{\mathcal{D}} \cdot \mathbf{u} \in H_{\mathcal{D}}(\Omega)$ is constant on each element, and hence we have

$$\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_K^2 = |K|\, |(\nabla_{\mathcal{D}} \cdot \mathbf{u})_K|^2$$

which leads to

$$\frac{|\sigma|}{|K|}\|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_K^2 = |\sigma|\, |(\nabla_{\mathcal{D}} \cdot \mathbf{u})_K|^2 = \|(\nabla_{\mathcal{D}} \cdot \mathbf{u})_K\|_\sigma^2.$$

Thus, for all edges of K we have

$$C_6 \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_K^2 \geq 4\delta h_{\partial K} \|(\nabla_{\mathcal{D}} \cdot \mathbf{u})_K\|_{\partial K}^2 .$$

Using the latter together with, Cauchy-Schwarz and Young inequalities allows the following bound for the stabilization term:

$$|J(p, \nabla_{\mathcal{D}} \cdot \mathbf{u})| \leq \sum_K \int_{\partial K \setminus \partial\Omega} \delta h_{\partial K}\, |[p]|\, |[\nabla_{\mathcal{D}} \cdot \mathbf{u}]| ds$$

$$\leq \sum_K \left\{ \left(\int_{\partial K \backslash \partial\Omega} \delta h_{\partial K} [p]^2 ds \right)^{\frac{1}{2}} \left(\int_{\partial K \backslash \partial\Omega} \delta \ h_{\partial K} |[\nabla_{\mathcal{D}} \cdot \mathbf{u}]|^2 ds \right)^{\frac{1}{2}} \right\}$$

$$= \sum_K \left\{ \left(\int_{\partial K \backslash \partial\Omega} \delta h_{\partial K} [p]^2 ds \right)^{\frac{1}{2}} \delta^{1/2} \left\| h_{\partial K}^{1/2} [\nabla_{\mathcal{D}} \cdot \mathbf{u}] \right\|_{\partial K} \right\}$$

$$\leq \sum_K \left\{ \frac{1}{\varepsilon} \left(\int_{\partial K \backslash \partial\Omega} \delta h_{\partial K} [p]^2 ds \right) + \varepsilon\delta \left\| h_{\partial K}^{1/2} [\nabla_{\mathcal{D}} \cdot \mathbf{u}] \right\|_{0,\partial K}^2 \right\}$$

$$\leq \frac{1}{\varepsilon} J(p,p) + \varepsilon C_7 \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{0,\Omega}^2 .$$

So,

$$B_{\mathcal{D}} [(\mathbf{u}, p), (0, \nabla_{\mathcal{D}} \cdot \mathbf{u})] \geq (1 - \varepsilon C_7) \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|_{\Omega}^2 - \frac{1}{\varepsilon} J(p,p) . \tag{45}$$

Finally, we conclude by setting $(\mathbf{v}, q) = (\beta\mathbf{u} + \mathbf{w}, \beta p + \nabla_{\mathcal{D}} \cdot \mathbf{u})$ with

$$\beta \geq \left(\frac{C_b}{C_a} + 2 \right) \frac{1}{\varepsilon} + \left(\frac{1}{C_a} + 1 \right) (1 - \varepsilon (\ C_4 + C_5)) .$$

By combining the latter with (41), (44) and (45), it follows that

$$\begin{aligned} B_{\mathcal{D}} [(\mathbf{u}, p), (\mathbf{v}, q)] &= \beta B_{\mathcal{D}} [(\mathbf{u}, p), (\mathbf{u}, p)] + B_{\mathcal{D}} [(\mathbf{u}, p), (\mathbf{w}, 0)] \\ &\quad + B_{\mathcal{D}} [(\mathbf{u}, p), (0, \nabla_{\mathcal{D}} \cdot \mathbf{u})] \\ &\geq \min \left\{ \beta C_a - \frac{C_b}{\varepsilon}, 1 - \varepsilon (C_4 + C_5), 1 - \varepsilon C_7 \ , \beta - \frac{2}{\varepsilon} \right\} \\ &\quad ||| \ (\mathbf{u}, p) \ |||^2 . \end{aligned}$$

Therefore,

$$B_{\mathcal{D}} [(\mathbf{u}, p), (\mathbf{v}, p)] \geq (1 - \varepsilon (C_4 + C_5)) \ ||| \ (\mathbf{u}, p) \ |||^2 .$$

We complete the proof by selecting ε sufficiently small, e.g. $\varepsilon < \frac{1}{C_4 + C_5}$, and noting that there also exists C_8 such that

$$||| \ (\mathbf{u}, p) \ ||| \geq C_8 \ ||| \ (\mathbf{v}, q) \ ||| \ .$$

□

4.3. Error estimates

Here, we establish error estimates for the discrete FV solution in the used energy norms. Let us start with the Stokes part. The fundamental result of error estimates is a consequence based on the following intermediate proposition we shall prove first.

Proposition 4.5. *Let* $(\overline{\mathbf{u}}, \overline{p}) \in \left(H^2(\Omega) \cap H_0^1(\Omega) \right)^d \times H^1(\Omega)$ *and* $(\mathbf{u}, p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ *be the respective solutions of* (1a)-(1b) *and* (30) *for Stokes.*

Then, for any $\varepsilon < \frac{1}{2}\min\{\mu, 1\}$ there exists a constant C, depending only on d, μ, Ω and θ, such that

$$\|\mathbf{u} - \pi_{\mathcal{D}}\overline{\mathbf{u}}\|_{1,\mathcal{D}} \leq Ch^2 \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega}\right), \tag{46}$$

$$J(p,p) \leq Ch^2 \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega}\right). \tag{47}$$

Proof. First, let $(\widehat{\mathbf{u}}, \hat{p}) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ be defined by $\widehat{\mathbf{u}} = \pi_{\mathcal{D}}\overline{\mathbf{u}}$ and $\hat{p} = \pi_{\mathcal{D}}\overline{p}$. Integrating (1a) on $K \in \mathcal{M}$ gives

$$-\mu \int_K \nabla \cdot (\nabla \overline{\mathbf{u}}) - \mu \int_K \nabla(\nabla \cdot \overline{\mathbf{u}}) + \int_K \nabla \overline{p} = \int_K \mathbf{f}.$$

The incompressibility equation $\nabla \cdot \overline{\mathbf{u}} = 0$ implies

$$-\mu \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla \overline{\mathbf{u}} \cdot \mathbf{n}_\sigma + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \overline{p}\mathbf{n}_\sigma = \int_K \mathbf{f}. \tag{48}$$

Now, let us introduce for $K \in \mathcal{M}$ the following consistency residuals:

i) $R_{\Delta,KL} = \frac{1}{d_{KL}}(\widehat{\mathbf{u}}_L - \widehat{\mathbf{u}}_K) - \frac{1}{|\sigma|}\int_\sigma \nabla \overline{\mathbf{u}} \cdot \mathbf{n}_\sigma$ for $\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K$ $(\sigma = K|L)$,

ii) $R_{\Delta,\sigma} = \frac{1}{d_{K\sigma}}(0 - \widehat{\mathbf{u}}_K) - \frac{1}{|\sigma|}\int_\sigma \nabla \overline{\mathbf{u}} \cdot \mathbf{n}_\sigma$ for $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$,

iii) $R_{\nabla,KL} = \frac{1}{2}(\hat{p}_L + \hat{p}_K) - \frac{1}{|\sigma|}\int_\sigma \overline{p}$ for $\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K$ $(\sigma = K|L)$,

iv) $R_{\nabla,KL} = \hat{p}_K - \frac{1}{|\sigma|}\int_\sigma \overline{p}$ for $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$.

Using these notations and the relation $\sum_{\sigma \in \mathcal{E}_K} |\sigma|\mathbf{n}_\sigma = 0$, from (48) we get

$$\begin{aligned} &-\mu \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ (\sigma = K|L)}} \frac{|\sigma|}{d_{KL}} (\widehat{\mathbf{u}}_L - \widehat{\mathbf{u}}_K) + \sum_{\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K} \frac{|\sigma|}{d_{K\sigma}} (0 - \widehat{\mathbf{u}}_K) \right) \\ &+ \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ (\sigma = K|L)}} |\sigma| \frac{\hat{p}_L - \hat{p}_K}{2} \mathbf{n}_\sigma \\ = &\int_K \mathbf{f} + |K|\, R_K, \end{aligned}$$

with

$$R_K = -\mu \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma|\, R_\Delta + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma|\, R_\nabla \mathbf{n}_\sigma.$$

Set $\mathbf{e} = \widehat{\mathbf{u}} - \mathbf{u}$ and $\epsilon = \hat{p} - p$. Subtracting (25a) from the above equation, we then get

$$\begin{aligned} &-\mu \textstyle\sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ (\sigma = K|L)}} \frac{|\sigma|}{d_{KL}} (\mathbf{e}_L - \mathbf{e}_K) - \mu \sum_{\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K} \frac{|\sigma|}{d_{K\sigma}} (0 - \mathbf{e}_K) \\ &+\mu \left(\nabla_{\mathcal{D}} (\nabla_{\mathcal{D}} \cdot \mathbf{u})\right)_K + \textstyle\sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ (\sigma = K|L)}} |\sigma| \frac{\epsilon_L - \epsilon_K}{2} \mathbf{n}_\sigma = |K|\, R_K. \end{aligned}$$

For all $\mathbf{v} \in H_{\mathcal{D}}(\Omega)^d$, we get

$$\mu[\mathbf{e},\mathbf{v}]_{1,\mathcal{D}} - \mu\int_\Omega (\nabla_{\mathcal{D}}\cdot\mathbf{u})(\nabla_{\mathcal{D}}\cdot\mathbf{v}) - \int_\Omega \epsilon(\nabla_{\mathcal{D}}\cdot\mathbf{v}) = \int_\Omega R\cdot\mathbf{v}. \tag{49}$$

By setting $\mathbf{v}=\mathbf{e}$ in this last equation, we get:

$$\mu\|\mathbf{e}\|_{1,\mathcal{D}}^2 + \mu\|\nabla_{\mathcal{D}}\cdot\mathbf{u}\|_{0,\Omega}^2 - \int_\Omega \epsilon(\nabla_{\mathcal{D}}\cdot\mathbf{e}) = \int_\Omega R\cdot\mathbf{e} + \mu\int_\Omega (\nabla_{\mathcal{D}}\cdot\mathbf{u})(\nabla_{\mathcal{D}}\cdot\widehat{\mathbf{u}}). \tag{50}$$

Now, let us integrate (1b) on $K\in\mathcal{M}$. This gives

$$\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \overline{\mathbf{u}}\cdot\mathbf{n}_\sigma = 0.$$

Since $\overline{\mathbf{u}}$ vanishes on the boundary of Ω, we obtain

$$\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}} \frac{|\sigma|}{2}(\widehat{\mathbf{u}}_L+\widehat{\mathbf{u}}_K)\cdot\mathbf{n}_\sigma = \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}} |\sigma|\,R_{div,KL} \qquad \forall K\in\mathcal{M}$$

with

$$R_{div,KL} = \left(\frac{1}{2}(\widehat{\mathbf{u}}_L+\widehat{\mathbf{u}}_K) - \frac{1}{|\sigma|}\int_\sigma \overline{\mathbf{u}}\right)\cdot\mathbf{n}_\sigma.$$

Then, subtracting the relation (25b) from the above equation gives

$$\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}} \frac{|\sigma|}{2}(\mathbf{e}_L+\mathbf{e}_K)\cdot\mathbf{n}_\sigma = \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}} |\sigma|\,R_{div,KL} + 2\delta\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}} |\sigma|\;\, h\;\, [p].$$

This yields

$$\int_\Omega q(\nabla_{\mathcal{D}}\cdot\mathbf{e}) = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma|\,R_{div,KL}(q_K-q_L) + J(p,q),$$

and setting $q=\epsilon$ in this relation gives

$$\int_\Omega \epsilon(\nabla_{\mathcal{D}}\cdot\mathbf{e}) = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma|\,R_{div,KL}(\epsilon_K-\epsilon_L) + J(p,\epsilon). \tag{51}$$

Gathering (50) and (51), we get

$$\begin{aligned} &\mu\|\mathbf{e}\|_{1,\mathcal{D}}^2 + \mu\|\nabla_{\mathcal{D}}\cdot\mathbf{u}\|_{0,\Omega}^2 + J(p,p) \\ &= \int_\Omega R\cdot\mathbf{e} + \mu\int_\Omega (\nabla_{\mathcal{D}}\cdot\mathbf{u})(\nabla_{\mathcal{D}}\cdot\widehat{\mathbf{u}}) \\ &\quad + \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma|\,R_{div,KL}(\epsilon_K-\epsilon_L) + J(p,\hat{p}). \end{aligned} \tag{52}$$

Next, let us study the terms at the right-hand side of (52). The first term is

$$\begin{aligned}\int_\Omega R\cdot\mathbf{e} &= \sum_K |K|\, R_K\cdot\mathbf{e}_K \\ &= -\mu\sum_K\sum_{\sigma\in\mathcal{E}_K}|\sigma|\, R_\Delta\cdot\mathbf{e}_K + \sum_K\sum_{\sigma\in\mathcal{E}_K}|\sigma|\, R_\nabla\mathbf{n}_\sigma\cdot\mathbf{e}_K \\ &= A+B.\end{aligned} \tag{53}$$

By virtue of interpolation results proven in [5], we obtain

$$\begin{aligned}A = &-\mu\sum_K\mathbf{e}_K\cdot\sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K\\(\sigma=K|L)}}\frac{|\sigma|}{d_{KL}}\left(\widehat{\mathbf{u}}_L-\widehat{\mathbf{u}}_K\right)-\frac{1}{|\sigma|}\int_\sigma\nabla\overline{\mathbf{u}}\cdot\mathbf{n}_\sigma \\ &-\mu\sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K}\frac{|\sigma|}{d_{K\sigma}}\left(0-\widehat{\mathbf{u}}_K\right)-\frac{1}{|\sigma|}\int_\sigma\nabla\overline{\mathbf{u}}\cdot\mathbf{n}_\sigma \\ = &-\mu\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}}\left(\mathbf{e}_K-\mathbf{e}_L\right)\cdot\left(\frac{|\sigma|}{d_{KL}}\left(\widehat{\mathbf{u}}_L-\widehat{\mathbf{u}}_K\right)-\frac{1}{|\sigma|}\int_\sigma\nabla\overline{\mathbf{u}}\cdot\mathbf{n}_\sigma\right) \\ &-\mu\sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}}\mathbf{e}_K\cdot\left(\frac{|\sigma|}{d_{K\sigma}}\left(0-\widehat{\mathbf{u}}_K\right)-\frac{1}{|\sigma|}\int_\sigma\nabla\overline{\mathbf{u}}\cdot\mathbf{n}_\sigma\right).\end{aligned}$$

By using Cauchy-Schwarz inequality and the obvious fact that $\sqrt{a}\sqrt{b}+\sqrt{c}\sqrt{d}\le\sqrt{a+c}\sqrt{b+d}$, it follows that

$$|A|\le\mu\|\mathbf{e}\|_{1,\mathcal{D}}\left\{\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}}\frac{d_{KL}}{|\sigma|}\left(|\sigma|\, R_{\Delta,KL}\right)^2+\sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}}\frac{d_{K\sigma}}{|\sigma|}\left(|\sigma|\, R_{\Delta,\sigma}\right)^2\right\}^{\frac{1}{2}}.$$

Now, consistency error estimates shown in [5], yield

$$|A|\le C_2 h\|\mathbf{e}\|_{1,\mathcal{D}}\|\overline{\mathbf{u}}\|_{2,\Omega}.$$

Repeating the same steps as above, reordering the summations and using the Cauchy-Schwarz inequality and consistency error estimates (cf. [5]), we get

$$|B|\le C_3 h\ \|\mathbf{e}\|_{1,\mathcal{D}}\|\overline{p}\|_{1,\Omega}.$$

Getting back to (53), it follows that

$$\int_\Omega R\cdot\mathbf{e}\le C_2\frac{h^2}{\varepsilon}\|\overline{\mathbf{u}}\|^2_{2,\Omega}+2\varepsilon\|\mathbf{e}\|^2_{1,\mathcal{D}}+C_3\frac{h^2}{\varepsilon}\|\overline{p}\|^2_{1,\Omega} \tag{54}$$

for all $\varepsilon>0$.

Using Cauchy-Schwarz inequality, (31) and (13) on the second term in (52), we get

$$\mu \int_\Omega (\nabla_\mathcal{D} \cdot \mathbf{u}) (\nabla_\mathcal{D} \cdot \widehat{\mathbf{u}}) \leq C_4 \frac{h^2}{\varepsilon} \|\overline{\mathbf{u}}\|_{2,\Omega}^2 + \varepsilon \|\nabla_\mathcal{D} \cdot \mathbf{u}\|_{0,\Omega}^2. \tag{55}$$

Let us decompose the third term in (52) to get

$$\begin{aligned} &\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, R_{div,KL} \left(\epsilon_K - \epsilon_L\right) \\ &= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, R_{div,KL} \left(\hat{p}_K - \hat{p}_L\right) + \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, R_{div,KL} \left(p_K - p_L\right). \end{aligned}$$

We have

$$\begin{aligned} &\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, R_{div,KL} \left(\hat{p}_K - \hat{p}_L\right) \\ &\leq \left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{|\sigma|}{d_{KL}} \left(\hat{p}_K - \hat{p}_L\right)^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{d_{KL}}{|\sigma|} \left(|\sigma| \, R_{div,KL}\right)^2 \right)^{\frac{1}{2}} \\ &\leq C_5 h^2 |\hat{p}|_{1,\mathcal{D}} \|\overline{\mathbf{u}}\|_{2,\Omega} \\ &\leq C_5 h^2 \left(\frac{1}{\varepsilon} \|\overline{\mathbf{u}}\|_{2,\Omega}^2 + \varepsilon \|\overline{p}\|_{1,\Omega}^2 \right) \qquad (56) \end{aligned}$$

by making use of useful interpolation results and consistency error estimates (cf. [5]).

Likewise,

$$\begin{aligned} \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, R_{div,KL} \left(p_K - p_L\right) &\leq J\left(p,p\right)^{\frac{1}{2}} \left(\frac{1}{2\delta h} \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{1}{|\sigma|} \left(|\sigma| \, R_{div,KL}\right)^2 \right)^{\frac{1}{2}} \\ &\leq C_6 \frac{h^2}{\varepsilon} \|\overline{\mathbf{u}}\|_{2,\Omega}^2 + \varepsilon J\left(p,p\right). \qquad (57) \end{aligned}$$

On the other hand, Cauchy-Schwarz inequality implies

$$\begin{aligned} J\left(p,\hat{p}\right) &\leq J\left(p,p\right)^{\frac{1}{2}} J\left(\hat{p},\hat{p}\right)^{\frac{1}{2}} \\ &= J\left(p,p\right)^{\frac{1}{2}} \left(2\delta \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} |\sigma| \, h (\hat{p}_K - \hat{p}_L)^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{2\delta} h J\left(p,p\right)^{\frac{1}{2}} |\hat{p}|_{1,\mathcal{D}} \end{aligned}$$

$$\leq \varepsilon J(p,p) + C_7 \frac{h^2}{\varepsilon} \|\overline{p}\|^2_{1,\Omega}. \tag{58}$$

Gathering (54)-(58) yields the control error inequality:

$$(\mu - 2\varepsilon) \|\mathbf{e}\|^2_{1,\mathcal{D}} + (\mu - \varepsilon) \|\nabla_{\mathcal{D}} \cdot \mathbf{u}\|^2_{0,\Omega} + (1 - 2\varepsilon) J(p,p)$$
$$\leq C_8 h^2 \max\left\{\frac{1}{\varepsilon}, \varepsilon\right\} \left(\|\overline{\mathbf{u}}\|^2_{2,\Omega} + \|\overline{p}\|^2_{1,\Omega}\right). \tag{59}$$

It is clear that choosing ε sufficiently small, e.g. $\varepsilon < \frac{1}{2}\min\{\mu, 1\}$, the latter implies (46) and (47). □

Theorem 4.6. *In addition to assumptions of Proposition* (4.5)*, let* $\overline{\mathbf{u}}_{\mathcal{D}}$ *be the function of* $H_{\mathcal{D}}(\Omega)^d$ *defined by:* $\overline{\mathbf{u}}_{\mathcal{D}K} = \overline{\mathbf{u}}(\boldsymbol{x}_K)$ $(K \in \mathcal{M})$. *Then,*

$$\|\mathbf{u} - \overline{\mathbf{u}}_{\mathcal{D}}\|_{1,\mathcal{D}} \leq Ch \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega}\right) \tag{60}$$

and

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{0,\Omega} \leq Ch \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega}\right), \tag{61}$$

$$\|p - \overline{p}\|_{0,\Omega} \leq Ch \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega}\right). \tag{62}$$

Proof. First of all, notice that (60) is straightforwardly deduced from the definition of $\overline{\mathbf{u}}_{\mathcal{D}}$ and (46) by applying the triangular inequality and a classical interpolation result (cf. [5]). Likewise, (61) follows by similar arguments from the triangular inequality, interpolation results and the discrete Poincaré inequality(12). It remains to establish (62).

Here again, we will follow the methodology already used in the proof of Proposition 4.3.

Using $\int_\Omega \hat{p}(x)dx = 0$ and therefore $\int_\Omega \epsilon(x) = 0$, let $\overline{\mathbf{w}} \in H^1_0(\Omega)^d$ be given such that

$$\nabla \cdot \overline{\mathbf{w}}(\boldsymbol{x}) = \epsilon(x) \quad \text{and} \quad \|\overline{\mathbf{w}}\|_{1,\Omega} \leq C_{15} \|\epsilon\|_{0,\Omega}.$$

Next, let us define $\mathbf{w}$ as in (36) and (37). This gives

$$\|\epsilon\|^2_{0,\Omega} \leq (\epsilon, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} + J(\epsilon,\epsilon)^{\frac{1}{2}} C_{16} \|\epsilon\|_{0,\Omega},$$

so that

$$\|\epsilon\|^2_{0,\Omega} \leq (\epsilon, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} + \frac{1}{\varepsilon} J(\hat{p},\hat{p}) + \frac{1}{\varepsilon} J(p,p) + \varepsilon C_1 \|\epsilon\|^2_{0,\Omega}$$
$$\leq (\epsilon, \nabla_{\mathcal{D}} \cdot \mathbf{w})_{0,\Omega} + C_2 \frac{h^2}{\varepsilon} \|\overline{p}\|^2_{1,\Omega} + \frac{1}{\varepsilon} J(p,p) + \varepsilon C_1 \|\epsilon\|^2_{0,\Omega}.$$

Choose $\mathbf{w}$ as a test function in (49). We get

$$\int_\Omega \epsilon (\nabla_{\mathcal{D}} \cdot \mathbf{w}) = \mu[\mathbf{e}, \mathbf{w}]_{1,\mathcal{D}} - \mu \int_\Omega (\nabla_{\mathcal{D}} \cdot \mathbf{u})(\nabla_{\mathcal{D}} \cdot \mathbf{w}) - \int_\Omega R \cdot \mathbf{w}.$$

Gathering the latter with the above relations yields

$$\|\epsilon\|^2_{0,\Omega} \leq C_3 \|\mathbf{e}\|_{1,\mathcal{D}} \|\epsilon\|_{0,\Omega} + C_4 \|\mathbf{u}\|_{1,\mathcal{D}} \|\epsilon\|_{0,\Omega} + C_5 h \|\epsilon\|_{0,\Omega} \left(\|\overline{\mathbf{u}}\|^2_{2,\Omega} + \|\overline{p}\|^2_{1,\Omega}\right)$$

$$
\begin{aligned}
&+ C_2 \frac{h^2}{\varepsilon} \|\overline{p}\|_{1,\Omega}^2 + \frac{1}{\varepsilon} J(p,p) + \varepsilon C_1 \|\epsilon\|_{0,\Omega}^2 \\
\leq\ & C_3 \|\mathbf{e}\|_{1,\mathcal{D}} \|\epsilon\|_{0,\Omega} + C_4 \|\mathbf{u} - \pi_{\mathcal{D}} \overline{\mathbf{u}}\|_{1,\mathcal{D}} \|\epsilon\|_{0,\Omega} + C_4 \|\pi_{\mathcal{D}} \overline{\mathbf{u}}\|_{1,\mathcal{D}} \|\epsilon\|_{0,\Omega} \\
&+ C_5 h \|\epsilon\|_{1,\mathcal{D}} \left(\|\overline{\mathbf{u}}\|_{2,\Omega} + \|\overline{p}\|_{1,\Omega} \right) + C_2 \frac{h^2}{\varepsilon} \|\overline{p}\|_{1,\Omega}^2 \\
&+ \frac{1}{\varepsilon} J(p,p) + \varepsilon C_1 \|\epsilon\|_{0,\Omega}^2.
\end{aligned}
$$

Finally, applying successively Young inequality, (13), (46), (47) and by the continuity of the interpolation operator gives the desired result. □

The corresponding result on error estimates for the Darcy part is now established.

Theorem 4.7. *Let* $(\overline{\mathbf{u}}, \overline{p}) \in \left(H^1(\Omega) \cap H_0^1(\Omega)\right)^d \times H^1(\Omega)$ *and* $(\mathbf{u}, p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ *be the respective solutions of* (1a)-(1b) *and* (30) *for Darcy. Then, for any* $\varepsilon < \frac{1}{2}$ *there exists* C*, depending only on* d, Ω, δ *and* θ*, such that*

$$
\|\mathbf{u} - \pi_{\mathcal{D}} \overline{\mathbf{u}}\|_{0,\Omega} \leq Ch \left(\|\overline{\mathbf{u}}\|_{1,\Omega} + \|\overline{p}\|_{1,\Omega} \right), \tag{63}
$$

$$
\|u - \overline{\mathbf{u}}\|_{0,\Omega} \leq Ch \left(\|\overline{\mathbf{u}}\|_{1,\Omega} + \|\overline{p}\|_{1,\Omega} \right), \tag{64}
$$

$$
\|p - \overline{p}\|_{0,\Omega} \leq Ch \left(\|\overline{\mathbf{u}}\|_{1,\Omega} + \|\overline{p}\|_{1,\Omega} \right). \tag{65}
$$

Proof. To start, a control equation on errors similar to (59) is first shown.

Let us define $(\widehat{\mathbf{u}}, \hat{p}) \in H_{\mathcal{D}}^d \times H_{\mathcal{D}}$ by $\widehat{\mathbf{u}} = \pi_{\mathcal{D}} \overline{\mathbf{u}}$ and $\hat{p} = \pi_{\mathcal{D}} \overline{p}$. Integrating (1a) on $K \in \mathcal{M}$ gives

$$
\alpha \int_K \overline{\mathbf{u}} + \int_K \nabla \overline{p} = \alpha \int_K \overline{\mathbf{u}} + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \overline{p} \mathbf{n}_\sigma = \int_K \mathbf{f}. \tag{66}
$$

Anew, we introduce for any $K \in \mathcal{M}$ the following consistency residuals:

i) $R_{0,K} = \alpha \widehat{\mathbf{u}}_K - \frac{\alpha}{|K|} \int_K \overline{\mathbf{u}}$,
ii) $R_{\nabla,KL} = \frac{1}{d_\sigma} (\hat{p}_L + \hat{p}_K) - \frac{1}{|\sigma|} \int_\sigma \overline{p}$ for $\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K$ $(\sigma = K|L)$,
iii) $R_{\nabla,K\sigma} = \hat{p}_K - \frac{1}{|\sigma|} \int_\sigma \overline{p}$ for $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$.

Using these notations and the relation $\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_\sigma = 0$, from (66) we get

$$
\alpha |K| \widehat{\mathbf{u}}_K + \sum_{\substack{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \\ \sigma = K|L}} |\sigma| \frac{\hat{p}_L - \hat{p}_K}{2} \mathbf{n}_\sigma = \int_K \mathbf{f} + |K| R_K,
$$

with

$$
R_K = R_{0,K} + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| R_\nabla \mathbf{n}_\sigma.
$$

Set $\mathbf{e} = \widehat{\mathbf{u}} - \mathbf{u}$ and $\epsilon = \hat{p} - p$. Subtracting (25a) from the above equation, it follows that

$$\alpha \left|K\right| \mathbf{e}_K + \sum_{\substack{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K \\ \sigma=K|L}} \left|\sigma\right| \frac{\epsilon_L - \epsilon_K}{2} \mathbf{n}_\sigma = \left|K\right| R_K.$$

For all $\mathbf{v} \in H_{\mathcal{D}}\left(\Omega\right)^d$, we have

$$\alpha \int_\Omega \mathbf{e}\cdot\mathbf{v} - \int_\Omega \epsilon\left(\nabla_{\mathcal{D}}\cdot\mathbf{v}\right) = \int_\Omega R\cdot\mathbf{v}, \tag{67}$$

which gives for $\mathbf{v} = \mathbf{e}$

$$\alpha\|\mathbf{e}\|_{0,\Omega}^2 - \int_\Omega \epsilon\left(\nabla_{\mathcal{D}}\cdot\mathbf{e}\right) = \int_\Omega R\cdot\mathbf{e}.$$

Using (51), we get

$$\alpha\|\mathbf{e}\|_{0,\Omega}^2 + J\left(p,p\right) = \int_\Omega R\cdot\mathbf{e} + \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma=K|L}} \left|\sigma\right| R_{div,KL}\left(\epsilon_K - \epsilon_L\right) + J\left(p,\hat{p}\right). \tag{68}$$

Let us study the terms in the right-hand side of the above equation. The first term is

$$\begin{aligned}\int_\Omega R\cdot\mathbf{e} &= \sum_K \left|K\right| R_{0,K}\cdot\mathbf{e}_K + \sum_K \sum_{\sigma\in\mathcal{E}_K} \left|\sigma\right| R_\nabla \mathbf{n}_\sigma\cdot\mathbf{e}_K \\ &\le \left(\sum_K \frac{1}{\left|K\right|}\left(\left|K\right| R_{0,K}\right)^2\right)^{\frac{1}{2}} \|\mathbf{e}\|_{0,\Omega} + C_2 h\|\mathbf{e}\|_{0,\Omega}\|\overline{p}\|_{1,\Omega}\end{aligned}$$

from the result shown in (54). But,

$$\left|K\right| R_{0,K} \le C_1\left|K\right|^{\frac{1}{2}} h_K\|\overline{\mathbf{u}}\|_{1,w_K},$$

so that

$$\begin{aligned}\int_\Omega R\cdot\mathbf{e} &\le C_3 h\|\mathbf{e}\|_{0,\Omega}\|\overline{\mathbf{u}}\|_{1,\Omega} + C_2 h\|\mathbf{e}\|_{0,\Omega}\|\overline{p}\|_{1,\Omega} \\ &\le C_3\frac{h^2}{\varepsilon}\|\overline{\mathbf{u}}\|_{1,\Omega}^2 + 2\varepsilon\|\mathbf{e}\|_{0,\Omega}^2 + C_2\frac{h^2}{\varepsilon}\|\overline{p}\|_{1,\Omega}^2.\end{aligned} \tag{69}$$

Finally, using like (56), (57) and (58) gives (63). Using the same technique as in Theorem 4.6, we deduce (64) and (65). This completes the proof of the theorem. □

5. Numerical tests

In this section we report some representative results of numerical experiments for Stokes and Darcy models of an incompressible flow. All test computations were performed under Matlab. The main objective is to check the validity of the error estimates established in the theoretical convergence study by means of test problems with given analytical solutions. Two numerical examples are devoted for a study of discrete solution accuracy and convergence rates for both Stokes and Darcy flows. The computational domain is $\Omega =]0,1[\times]0,1[$ and the problem (1a)-(1b) is to be discretized and solved using uniform partitionings of Ω into $n \times n$ equal squares ($n = 10, \ldots, 100$). Moreover, in order to restore a unique pressure field a zero mean pressure is imposed on Ω. In all cases, the source term $\mathbf{f}$ is chosen such that equations (1a)-(1b) hold.

The velocity and pressure errors are defined respectively as:

$$e_K^{(i)} = u_K^{(i)} - \overline{u}_K^{(i)}(x_K), \qquad \epsilon_K = p_K - \overline{p}_K(x_K).$$

The following discrete error norms are used for the investigation of convergence rates:

$$\|\mathbf{e}\|_{1,\mathcal{D}} = \sqrt{\sum_{i=1}^{2} \left[e^{(i)}, e^{(i)}\right]_{\mathcal{D}}},$$

$$\|\mathbf{e}\|_{0,\Omega} = \sqrt{\sum_{i=1}^{2} \int_{\Omega} \left(e^{(i)}\right)^2 d\Omega},$$

$$\|\epsilon\|_{0,\Omega} = \sqrt{\int_{\Omega} \epsilon^2 d\Omega}.$$

5.1. Problem I

The first numerical example is a study of both Stokes and Darcy problems with the same exact velocity and pressure fields:

$$\begin{aligned}
\overline{\mathbf{u}}_1(x,y) &= 2000(x-x^2)^2\left(y-y^2\right)(1-2y),\\
\overline{\mathbf{u}}_2(x,y) &= -2000(y-y^2)^2\left(x-x^2\right)(1-2x),\\
\overline{p}(x,y) &= 100(x^2+y^2-\frac{2}{3}).
\end{aligned}$$

We set $\delta = 0.5$. In Fig. 1 and Fig. 2, the approximate velocity vectors and pressure elevations are shown on the 80×80 partitioning for Stokes with $\mu = 0.1$ and for Darcy with $\alpha = 100$ repectively. The displayed graphs are in excellent agreement with the exact solution plots. The computed convergence rates are presented in Fig. 3. Better results than theoretical rates predicted by the above study have been obtained. For Stokes flow, the convergence rates are near to $3/2$ for the both the velocity in $\|\cdot\|_{1,\mathcal{D}}$ norm and pressure in L^2 norm, while the computed rate is close to 2 for the velocity in L^2 norm. For Darcy flow, the convergence rate is near to $3/2$ for the pressure in L^2 norm, whereas it is close to 1 for the velocity in the same norm. This unexpected fact deserves further investigation.

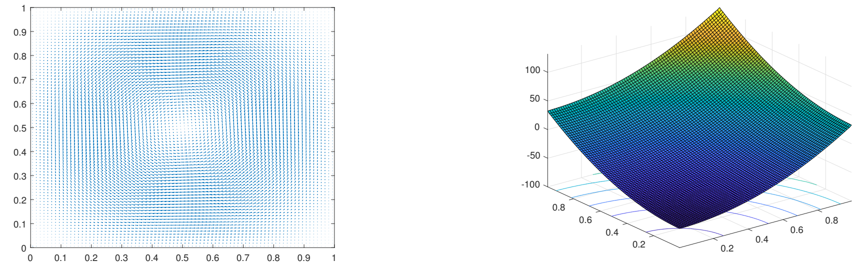

FIGURE 1. Approximate velocity vectors and pressure elevation for Stokes with $\mu = 0.1$.

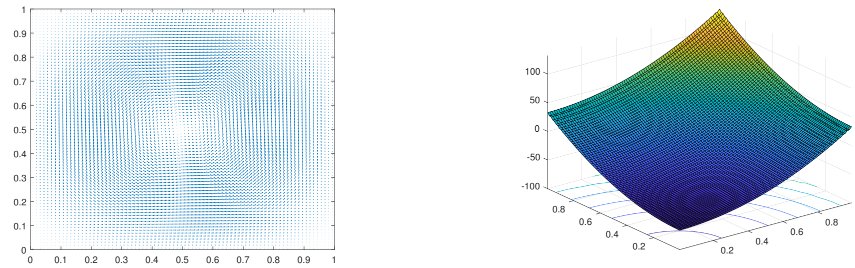

FIGURE 2. Approximate velocity vectors and pressure elevation for Darcy with $\alpha = 100$.

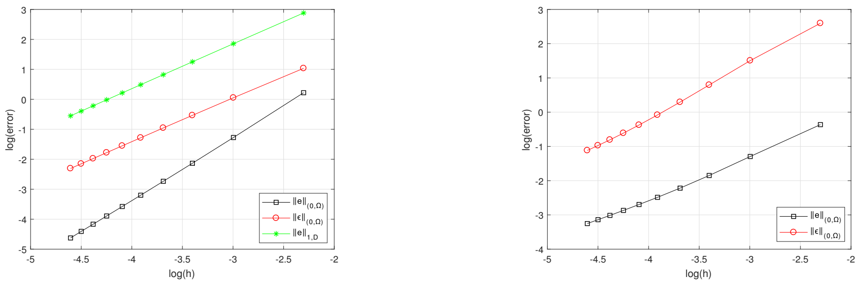

FIGURE 3. Convergence history for Stokes (left) $\mu = 0.1$ and for Darcy (right) with $\alpha = 100$.

5.2. Problem II

The second numerical test is devoted to study the same features when the exact velocity and pressure are given respectively for the Stokes flow by:

$$\overline{\mathbf{u}}_1\left(x, y\right) = \pi \sin(2\pi y) \sin^2\left(\pi x\right),$$

$$\overline{\mathbf{u}}_2\,(x,y) = -\pi \sin(2\pi x)\sin^2(\pi y)\,,$$
$$\overline{p}(x,y) = \sin(2\pi x)\sin(2\pi y)$$

and for the Darcy flow by:

$$\overline{\mathbf{u}}_1\,(x,y) = 1/2\sin(2\pi y)\sin^2(\pi x)\,,$$
$$\overline{\mathbf{u}}_2\,(x,y) = -1/2\sin(2\pi x)\sin^2(\pi y)\,,$$
$$\overline{p}(x,y) = 2x - 4y^3.$$

Now, we set $\delta = 10$. The approximate velocity vectors and pressure elevations on the 80×80 partitioning for Stokes with $\mu = 1$ and for Darcy with $\alpha = 10$ are displayed in Fig. 4 and Fig. 5 respectively. The behavior is again remarkable. According to the convergence error history shown in Fig. 6, predicted optimal rates are widely achieved for both Stokes and Darcy cases and, here also, better results than the predicted rates are attained.

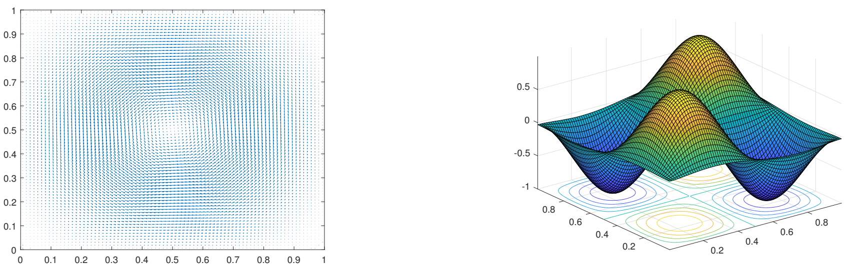

FIGURE 4. Approximate velocity vectors and pressure elevation for Stokes with $\mu = 1$.

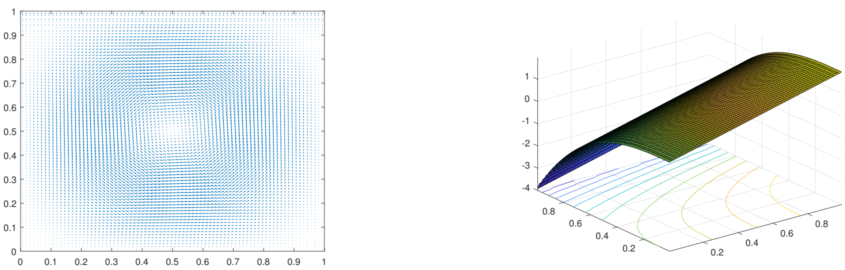

FIGURE 5. Approximate velocity vectors and pressure elevation for Darcy with $\alpha = 10$.

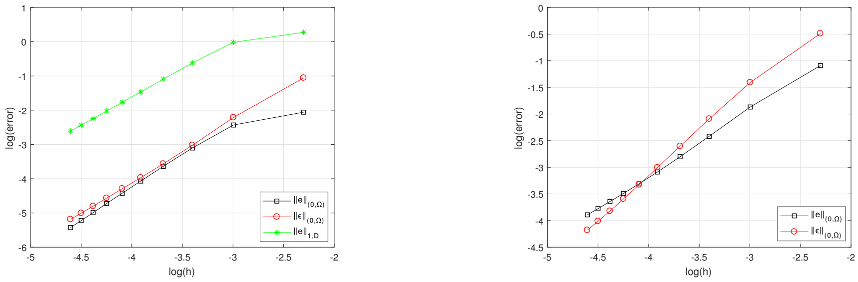

FIGURE 6. Convergence history for Stokes (left) $\mu = 1$ and for Darcy (right) with $\alpha = 10$.

6. Conclusions

The aim of this paper is to solve numerically the uncoupled Stokes and Darcy problems for incompressible fluid with a unique FV scheme. We presented a complete analysis of the proposed cell-centered FVM using pressure jump stabilization. Optimal convergence rates were derived for both Stokes and Darcy cases using the natural norms, results that were confirmed by the numerical experiments. The simplicity and resulting efficiency of the present scheme suggests that it will be a competitive alternative for existing FV techniques.

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