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A SINGULAR FUNCTION FROM STURMIAN CONTINUED FRACTIONS

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Abstract. For $\alpha \geq 1$, let $s_\alpha(n) = \lceil \alpha n \rceil - \lceil \alpha(n-1) \rceil$. A continued fraction $C(\alpha) = [0; s_\alpha(1), s_\alpha(2), \ldots]$ is considered and analyzed. Appealing to Diophantine approximation, we investigate the differentiability of $C(\alpha)$, and then show its singularity.

1. Introduction

A real function is said to be *singular* provided that its derivative vanishes almost everywhere. A *numeration* is a particular rule of expressing numbers in finite or infinite words. Many singular functions are constructed by applying two different numerations at once. Such function $y = f(x)$ is typically devised as follows. The real number x is encoded into some words via the one numeration, and (after word processing if necessary) this word is interpreted according to the other numeration to produce the real number y. As for the classical Cantor function, ternary expansions and then binary ones are employed [3]. Meanwhile, the Minkowski question mark function $?(x)$ involves continued fractions and alternate dyadic expansions [4, 17]. Among accumulated works on the Minkowski question mark function, Paradís et al. [13] and Dushistova et al. [5] specified where $?'(x) = 0$ and $?'(x) = \infty$ in terms of partial quotients of $x = [0; a_1, a_2, \ldots, a_t, \ldots]$. More precisely, the increasing rate of their sums $S_x(t) = a_1 + a_2 + \cdots + a_t$ does matter.

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively. An arithmetic function $s_\alpha(n) = \lceil \alpha n \rceil - \lceil \alpha(n-1) \rceil$ represents any $\alpha \geq 0$ as an infinite word $s_\alpha := s_\alpha(1)s_\alpha(2)\cdots$ over the alphabet $\{\lceil \alpha \rceil - 1, \lceil \alpha \rceil\}$. The word s_α is known to be lexicographically greatest among all *mechanical words* of *slope* α [2]. Mechanical words of irrational slopes are called *Sturmian words*, but Sturmian words also refer to mechanical words of rational slopes in some literature. The title of the present paper reflects this abuse of terminology.

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For $\beta > 1$, β-expansions [14] turn the Sturmian words into real numbers

$$\Xi(\alpha, \beta) := \sum_{n=1}^{\infty} \frac{s_\alpha(n)}{\beta^n}.$$

Diverse aspects of the function $\Xi(\alpha, \beta)$ including its singularity were investigated in [8, 9]. Following a similar vein, the author studied the singularity of Dirichlet series with Sturmian coefficients

$$Z(\alpha, s) := \sum_{n=1}^{\infty} \frac{s_\alpha(n)}{n^s},$$

where s is a complex number with $\mathrm{Re}(s) > 1$ [10, 11].

The present paper considers continued fractions whose partial quotients are Sturmian words

$$C(\alpha) := [0; s_\alpha(1), s_\alpha(2), \ldots] = \cfrac{1}{s_\alpha(1) + \cfrac{1}{s_\alpha(2) + \cdots}}.$$

Here we assume $\alpha \geq 1$ because every partial quotient is a positive integer. One readily notes that $C(1) = \frac{-1+\sqrt{5}}{2}$, $C(\frac{3}{2}) = \frac{-1+\sqrt{3}}{2}$, and more generally that $C(\alpha)$ is a quadratic irrational number whenever $\alpha \geq 1$ is rational. On the other hand, if $\alpha \geq 1$ is irrational, then $C(\alpha)$ is known to be a transcendental number [1].

Appealing to a close connection between Sturmian words and Diophantine approximations, we prove that the function $C(\alpha)$ is singular.

2. Sturmian words and Diophantine approximations

First, we briefly review basic facts on continued fractions, which can be found in standard texts, e.g., [7, 15].

Let $t = [a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of a real number t, where $a_0 \in \mathbb{Z}$ and a_i are positive integers for $i \geq 1$. Recall that, for $k \geq 0$,

$$p_k/q_k := [a_0; a_1, \ldots, a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}, \quad \gcd(p_k, q_k) = 1,$$

is called the k-th *convergent* of t. If we set $p_{-1} = 1$ and $q_{-1} = 0$, then the convergents satisfy

$$p_{k+1} = a_{k+1} p_k + p_{k-1}, \quad q_{k+1} = a_{k+1} q_k + q_{k-1} \quad \text{for } k \geq 0.$$

The k-th tail of the continued fraction,

$$\zeta_k := [a_k; a_{k+1}, a_{k+2}, \ldots] = a_k + \cfrac{1}{a_{k+1} + \cfrac{1}{a_{k+2} + \cdots}},$$

is called the k-th *complete quotient* of t. One immediately sees that t is rational if and only if its continued fraction expansion is finite. If t is irrational, then the sequence of its convergents p_k/q_k converges to t in the following manner.

Proposition 2.1. *Let t be an irrational number. Then, for any $k \geq 1$,*

$$t - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(q_k\zeta_{k+1} + q_{k-1})},$$

and hence

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| t - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.$$

For real $\alpha \geq 0$ and $\rho \in [0, 1]$, two arithmetic functions $s_{\alpha,\rho}, s'_{\alpha,\rho}$ are defined by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$
$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

Then two infinite words $s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots$ and $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$ are called *lower* and *upper mechanical word* respectively with *slope* α and *intercept* ρ. For general theory on mechanical words, the readers are referred to [12]. This paper studies continued fractions whose partial quotients are from the upper mechanical word $s'_{\alpha,0}$ with $\alpha \geq 1$ and $\rho = 0$. In fact, continued fractions with partial quotients from the lower mechanical word $s_{\alpha,0}$ can be also analyzed by a similar technique in this paper. But the cases of general mechanical words $s_{\alpha,\rho}$ or $s'_{\alpha,\rho}$ are much more complicated to analyze. This is why inhomogeneous Diophantine approximations are necessarily involved if $\rho \notin \alpha\mathbb{Z} + \mathbb{Z}$.

If $\alpha \geq 0$ is a rational number, then both $s_{\alpha,\rho}$ or $s'_{\alpha,\rho}$ are obviously purely periodic words over the alphabet $\{\lceil\alpha\rceil - 1, \lceil\alpha\rceil\}$. The case where $\rho = 0$ is of our special interest. For a positive rational $\alpha = p/q$ with $\gcd(p, q) = 1$, let $b = a + 1 = \lceil\alpha\rceil$. Then $s_{\alpha,0}$ and $s'_{\alpha,0}$ comprise a factor $z_{p,q}$ of length $q - 2$ in common, as in the following manner,

$$s_{\alpha,0} = (az_{p,q}b)^\infty, \quad s'_{\alpha,0} = (bz_{p,q}a)^\infty.$$

Here, $z_{p,q}$ coincides with the reversal of itself, i.e., is a *palindrome* [12, Corollary 2.2.9]. If $q = 2$, then $z_{p,q}$ is, of course, the empty word. Moreover, if $q = 1$, or if $\alpha = p$ is an integer, then both $az_{p,q}b$ and $bz_{p,q}a$ should read α by convention. Endowing $\{\lceil\alpha\rceil - 1, \lceil\alpha\rceil\}^{\mathbb{N}}$ with product topology, we can state the continuities of functions $\alpha \mapsto s_{\alpha,0}$ and $\alpha \mapsto s'_{\alpha,0}$. At every positive rational, these functions are continuous in the one direction while discontinuous in the other direction, as the next lemma says.

Lemma 2.2. *Let $p/q > 0$ be rational with $\gcd(p, q) = 1$, and set $b = a + 1 = \lceil p/q \rceil$. Then*

- $\lim_{\alpha \to (p/q)-} s_{\alpha,0} = a(z_{p,q}ab)^\infty, \quad \lim_{\alpha \to (p/q)+} s_{\alpha,0} = (az_{p,q}b)^\infty,$
- $\lim_{\alpha \to (p/q)-} s'_{\alpha,0} = (bz_{p,q}a)^\infty, \quad \lim_{\alpha \to (p/q)+} s'_{\alpha,0} = b(z_{p,q}ba)^\infty.$

When $q = 1$, the infinite word $a(z_{p,q}ab)^\infty$ is understood as ab^∞ with $b = a + 1 = p$, while $b(z_{p,q}ba)^\infty$ as ba^∞ with $b = a + 1 = p + 1$. Lemma 2.3(b) and (c) below prove the case of upper mechanical words. The proof for lower mechanical words is similar.

Now, for typographical convenience, we also write $s_\alpha(n) := s'_{\alpha,0}(n-1)$, and so $s_\alpha = s_\alpha(1)s_\alpha(2)\cdots = s'_{\alpha,0}(0)s'_{\alpha,0}(1)\cdots = s'_{\alpha,0}$. For any real t, we mean by $\|t\|$ the distance from t to the nearest integer, and by $\{t\}$ the fractional part of t, i.e., $t = \lfloor t \rfloor + \{t\}$. The following notation is also useful:

$$\langle t \rangle := \begin{cases} \{t\}, & \text{if } t \notin \mathbb{Z}, \\ 1, & \text{if } t \in \mathbb{Z}. \end{cases}$$

The next lemma enables us to estimate truncations of Sturmian continued fractions.

Lemma 2.3. *Let N be a positive integer.*

(a) *For an irrational $\alpha_0 > 0$, we set*

$$\delta_N := \min\left\{ \frac{\|\alpha_0 n\|}{n} : 1 \le n \le N \right\}. \tag{1}$$

If $|\alpha - \alpha_0| < \delta_N$, then two Sturmian words s_α and s_{α_0} have a common prefix of length $\ge N$.

(b) *For a rational $p/q > 0$ with $\gcd(p,q) = 1$, let us set $b = a + 1 = \lceil p/q \rceil$, and define*

$$\delta_N := \min\left\{ \frac{\langle (p/q)\cdot n \rangle}{n} : 1 \le n \le N \right\}. \tag{2}$$

If $0 \le p/q - \alpha < \delta_N$, then both s_α and $s_{p/q} = (bz_{p,q}a)^\infty$ have a common prefix of length $\ge N$.

(c) *With a rational $p/q > 0$ and integers a, b as in (b), we define*

$$\delta_N := \min\left\{ \frac{1 - \{(p/q)\cdot n\}}{n} : 1 \le n \le N \right\}. \tag{3}$$

If $0 < \alpha - p/q \le \delta_N$, then both s_α and $b(z_{p,q}ba)^\infty$ have a common prefix of length $\ge N$.

Proof. (a) Since α_0 is irrational, $\alpha_0 n$ is an integer only if $n = 0$. Suppose, on the contrary, that the k-th letters of s_α and s_{α_0} are different for some $1 \le k \le N$. We may assume that k is the smallest such integer. Then $\lceil \alpha i \rceil = \lceil \alpha_0 i \rceil$ for all $i = 1, 2, \ldots, k-1$. If $s_\alpha(k) < s_{\alpha_0}(k)$, then

$$\alpha k \le \alpha_0 k - \{\alpha_0 k\} \le \alpha_0 k - \delta_N k,$$

where we have used $\frac{\{\alpha_0 k\}}{k} \ge \frac{\|\alpha_0 k\|}{k} \ge \delta_N$ in the last inequality. Now we have a contradiction $\alpha \le \alpha_0 - \delta_N$. On the other hand, assume $s_{\alpha_0}(k) < s_\alpha(k)$. Noting $\frac{1-\{\alpha_0 k\}}{k} \ge \frac{\|\alpha_0 k\|}{k} \ge \delta_N$, one deduces that

$$\alpha k > \alpha_0 k + (1 - \{\alpha_0 k\}) \ge \alpha_0 k + \delta_N k,$$

which yields a contradiction $\alpha > \alpha_0 + \delta_N$.

(b) If the k-th letters of s_α and $s_{p/q}$ are different for some $1 \le k \le N$, and if k is the smallest such integer, then $s_\alpha(k) < s_{p/q}(k)$, and therefore,

$$\alpha k \le (p/q) \cdot k - \langle (p/q) \cdot k \rangle \le (p/q) \cdot k - \delta_N k \;\Rightarrow\; p/q - \alpha \ge \delta_N.$$

(c) See [9, Lemma 2.3]. □

Let α be an irrational number. Recall that the *irrationality exponent* of α is defined by

$$\mu(\alpha) := \sup\{\nu : \liminf_{q\to\infty} q^{\nu-1}\|q\alpha\| = 0\}.$$

One immediately checks that $\mu(\alpha) \ge 2$ for every irrational α. And the well-known Khinchin's theorem states that $\mu(\alpha) = 2$ for almost every α in the sense of Lebesgue measure. In particular, every algebraic irrational number has irrationality exponent 2 by Roth's theorem [16]. In view of finer measures, for $c \ge 2$, the Hausdorff dimension of the set $\{\alpha \in \mathbb{R} : \mu(\alpha) \ge c\}$ is $2/c$, which is due to Jarník [6]. Consequently, the set $\{\alpha \in \mathbb{R} : \mu(\alpha) = \infty\}$ of *Liouville numbers* has Hausdorff dimension zero.

The irrationality measure crucially required in our analysis is

$$\theta(\alpha) := \sup\{\lambda : \liminf_{q\to\infty} \lambda^q \|q\alpha\| = 0\},$$

which is termed the *irrationality base* of α, and was firstly coined by Sondow [18]. The definitions tell us that $\mu(\alpha) < \infty$ implies $\theta(\alpha) = 1$, and that $\theta(\alpha) > 1$ implies $\mu(\alpha) = \infty$. But there are uncountably many α such that both $\mu(\alpha) = \infty$ and $\theta(\alpha) = 1$ hold. The irrationality base can be computed via the convergent of α.

Proposition 2.4 ([8,18])**.** *Let p_k/q_k be the k-th convergent of an irrational α. Then*

$$\log\theta(\alpha) = -\liminf_{k\to\infty} \frac{\log\|q_k\alpha\|}{q_k} = \limsup_{k\to\infty} \frac{\log q_{k+1}}{q_k}.$$

The next lemma will play a key role to establish the main theorem. See [8, Theorem 4.12] for its proof.

Lemma 2.5. *For irrational α_0, let δ_N be as in* (1)*.*

(a) *If $\theta(\alpha_0) < \lambda$, then there exists a positive integer N_0 for which*

$$\bigcup_{n=N_0}^{\infty} (\lambda^{-n}, \delta_n) = (0, \delta_{N_0}).$$

Furthermore, two consecutive intervals in the union have a nonempty intersection.

(b) *If $1 < \lambda < \theta(\alpha_0)$, then a set*

$$Q = \left\{ n \in \mathbb{N} : \delta_n \le \frac{\|\alpha_0 n\|}{n} < \lambda^{-n} \right\}$$

contains infinitely many denominators of the convergents of α_0.

3. Calculus on Sturmian continued fractions

This section studies the continuity and differentiability of the function $C : [1,\infty) \to \mathbb{R}$ defined by

$$C(\alpha) := [0; s_\alpha(1), s_\alpha(2), \ldots].$$

If $w = a_1 a_2 \cdots$ is an infinite word over positive integers, then we also write $[0; w] := [0; a_1, a_2, \ldots]$ for continued fraction. If $\alpha = p/q$ with $\gcd(p,q) = 1$, and if $b = a + 1 = \lceil \alpha \rceil$, for example, then $C(\alpha) = [0; s_\alpha(1), s_\alpha(2), \ldots] = [0; (bz_{p,q}a)^\infty]$.

We explore first the behavior of $C(\alpha)$ at rational points, where it is left-continuous but not right-continuous. This fact originally comes from the left-continuity but not right-continuity of the ceiling function $\lceil \cdot \rceil$ at every integer. The exact manner of this phenomenon is dealt with in the next theorem. In what follows, we coherently use upper case for convergents P_k/Q_k of $C(\alpha)$, while lower case for convergents p_k/q_k of α.

Theorem 3.1. *Let $p/q \geq 1$ be rational with $\gcd(p,q) = 1$ and let $b = a + 1 = \lceil p/q \rceil$. Then the following hold.*

(a) *At every rational greater that 1, the function $C(\alpha)$ is left-continuous but not right-continuous.*
(b) $\lim_{\alpha \to (p/q)+} C(\alpha) = [0; b(z_{p,q}ba)^\infty] =: C(p/q+)$.
(c) *If q is odd, then $C(p/q) > C(p/q+)$.*
(d) *If q is even, then $C(p/q) < C(p/q+)$.*

Proof. Let P_k/Q_k be the convergents of $C(p/q)$. Given $\varepsilon > 0$, we choose a positive integer N so that $Q_N > 1/\sqrt{\varepsilon}$.

(a) Define δ_N as in (2). Then s_α and $s_{p/q} = (bz_{p,q}a)^\infty$ have a common prefix of length $\geq N$ whenever $0 \leq p/q - \alpha < \delta_N$. Thus, the first N convergents of $C(\alpha) = [0; s_\alpha]$ coincide with those of $C(p/q) = [0; s_{p/q}]$, respectively. One deduces from Proposition 2.1 that

$$\begin{aligned} |C(p/q) - C(\alpha)| &= \left| \left(C(p/q) - \frac{P_N}{Q_N} \right) - \left(C(\alpha) - \frac{P_N}{Q_N} \right) \right| \\ &< \frac{1}{Q_N Q_{N+1}} < \frac{1}{(Q_N)^2} < \varepsilon, \end{aligned}$$

where the first inequality in the second line uses the fact that $C(p/q) - \frac{P_N}{Q_N}$ and $C(\alpha) - \frac{P_N}{Q_N}$ have the same sign, i.e., their product is positive.

The remaining parts below will prove that $C(\alpha)$ is not right-continuous at every rational.

(b) Define δ_N as in (3). If $0 < \alpha - p/q < \delta_N$, then both s_α and $b(z_{p,q}ba)^\infty$ have a common prefix of length $\geq N$. A similar argument to (a) proves $|C(\alpha) - C(p/q+)| < \varepsilon$.

(c, d) Continued fractions obey the *alternating lexicographic order*:

$[c_0; c_1, c_2, \dots, \dots] < [d_0; d_1, d_2, \dots, \dots]$ if and only if, for some $k \geq 0$,

$$\begin{cases} c_0 = d_0, c_1 = d_1, \dots, c_{2k-1} = d_{2k-1}, c_{2k} < d_{2k}, \text{ or} \\ c_0 = d_0, c_1 = d_1, \dots, c_{2k} = d_{2k}, c_{2k+1} > d_{2k+1}. \end{cases}$$

Since the word $bz_{p,q}a$ has length q, this fact proves the claim. □

If $b = a + 1 = \lceil \alpha \rceil$, then the denominators Q_k of the convergents of $C(\alpha) = [0; s_\alpha]$ satisfy $Q_{k+1} = s_\alpha(k+1)Q_k + Q_{k-1}, k \geq 1$ with $Q_0 = 1$ and $Q_1 = s_\alpha(1)$. Define sequences $\{A_k\}_{k\geq 0}$ and $\{B_k\}_{k\geq 0}$ by $A_0 = 1$, $A_1 = a$, $B_0 = 1$, $B_1 = b$, and

$$A_{k+1} = aA_k + A_{k-1}, \quad B_{k+1} = bB_k + B_{k-1} \text{ for } k \geq 1,$$

respectively. Then we find

$$A_k \leq Q_k \leq B_k \text{ for all } k \geq 0. \tag{4}$$

A folklore to solve linear recurrences brings us the following.

Lemma 3.2. *Let a sequence $\{R_k\}_{k\geq 0}$ be defined by $R_0 = 1$, $R_1 = c \geq 1$, and $R_{k+1} = cR_k + R_{k-1}$ for $k \geq 1$. Then,*

$$R_k = \frac{\gamma^{k+1} - \overline{\gamma}^{k+1}}{\sqrt{c^2+4}}, \text{ where } \gamma = \frac{c + \sqrt{c^2+4}}{2}, \ \overline{\gamma} = \frac{c - \sqrt{c^2+4}}{2}.$$

Note that $\gamma > 1$ and $-1 < \overline{\gamma} < 0$.

The behavior of $C(\alpha)$ at irrationals is our main result.

Theorem 3.3. *The function $C(\alpha)$ is continuous at every irrational.*

Proof. The argument used in Theorem 3.1 also works well here. But δ_N should come from (1). □

The differentiability of $C(\alpha)$ at irrationals is more subtle. The current definition of limit yields the following kind of differentiability.

Theorem 3.4. *Let $\alpha_0 > 1$ be irrational with $b = a + 1 = \lceil \alpha_0 \rceil$. If $\theta(\alpha_0) < \left(\frac{a+\sqrt{a^2+4}}{2}\right)^2$, then $C(\alpha)$ is differentiable at $\alpha = \alpha_0$, and $C'(\alpha_0) = 0$.*

Proof. Take λ so that $\theta(\alpha_0) < \lambda < \left(\frac{a+\sqrt{a^2+4}}{2}\right)^2$. Then Lemma 2.5 guarantees $\bigcup_{n=N_0}^{\infty} (\lambda^{-n}, \delta_n) = (0, \delta_{N_0})$ for some $N_0 \geq 1$, where δ_n is adopted as in (1). If $\lambda^{-n} < |\alpha - \alpha_0| < \delta_n$, then the first n convergents of $C(\alpha)$ are equal to those of $C(\alpha_0)$, respectively. Together with (4), Lemma 3.2 derives

$$\begin{aligned} \frac{|C(\alpha) - C(\alpha_0)|}{|\alpha - \alpha_0|} &< \frac{\lambda^n}{Q_n Q_{n+1}} < \frac{\lambda^n}{(Q_n)^2} < \frac{\lambda^n}{(A_n)^2} = \frac{(a^2+4)\lambda^n}{(\gamma_1^{n+1} - \overline{\gamma}_1^{n+1})^2} \\ &= \frac{a^2+4}{(\gamma_1(\gamma_1/\sqrt{\lambda})^n - \overline{\gamma}_1(\overline{\gamma}_1/\sqrt{\lambda})^n)^2} \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $\gamma_1 = \frac{a+\sqrt{a^2+4}}{2}$. Let $\{\alpha_k\}_{k\geq 1}$ be an arbitrary sequence such that $\alpha_k \neq \alpha_0$ and $\alpha_k \to \alpha_0$. Then $|\alpha_k - \alpha_0|$ eventually belongs to some interval (λ^{-n}, δ_n), and moreover $n \to \infty$ as $k \to \infty$. □

By Jarník's theorem, the set $\{\alpha \in \mathbb{R} : \theta(\alpha) > 1\}$ has Hausdorff dimension zero. Now we have the following.

Corollary 3.4.1. *The function $C(\alpha)$ is singular. Its derivative vanishes except on a set of Hausdorff dimension zero.*

There indeed exist uncountably many irrationals at which $C(\alpha)$ is continuous but not differentiable. To extract these numbers, more elaborate Diophantine approximation is required.

Theorem 3.5. *Let $\alpha_0 > 1$ be irrational with $b = a + 1 = \lceil \alpha_0 \rceil$. If $\theta(\alpha_0) > \left(\frac{b+\sqrt{b^2+4}}{2}\right)^2$, then $C(\alpha)$ is not differentiable at $\alpha = \alpha_0$.*

Proof. Fix λ so that $\left(\frac{b+\sqrt{b^2+4}}{2}\right)^2 < \lambda < \theta(\alpha_0)$, and define δ_n by (1). Lemma 2.5 allows us to pick an increasing integer sequence $q_i \nearrow \infty$ from the set $Q = \left\{n \in \mathbb{N} : \delta_n \leq \frac{\|\alpha_0 n\|}{n} < \lambda^{-n}\right\}$, where, furthermore, every q_i is the denominator of the convergent p_i/q_i of α_0. Since convergents are the best rational approximation of irrational numbers [7], we have, for all $i \geq 1$,

$$\delta_{q_i} = \frac{\|\alpha_0 q_i\|}{q_i} < \lambda^{-q_i} \quad \text{and} \quad \delta_{q_i} < \delta_{q_i - 1}.$$

For each $i \geq 1$, we put

$$\alpha_i := \begin{cases} \alpha_0 - \delta_{q_i} & \text{if } \|q_i\alpha_0\| = q_i\alpha_0 - p_i, \\ \alpha_0 + \min\left\{\frac{\delta_{q_i} + \delta_{q_i-1}}{2}, 2\delta_{q_i}\right\} & \text{if } \|q_i\alpha_0\| = p_i - q_i\alpha_0. \end{cases}$$

Then one finds that

$$\lim_{i\to\infty} \alpha_i = \alpha_0 \quad \text{and} \quad |\alpha_i - \alpha_0| \leq 2\delta_{q_i} < \frac{2}{\lambda^{q_i}}.$$

If $\|q_i\alpha_0\| = q_i\alpha_0 - p_i$, then α_i is rational. Accordingly, s_{α_i} is purely periodic, and none of the first $q_i - 1$ letters in s_{α_i} are changed from those in s_{α_0}, but the q_i-th letter of s_{α_i} is smaller than that of s_{α_0}. On the other hand, if $\|q_i\alpha_0\| = p_i - q_i\alpha_0$, then one notes that

$$\delta_{q_i} < \min\left\{\frac{\delta_{q_i} + \delta_{q_i-1}}{2}, 2\delta_{q_i}\right\} < \delta_{q_i-1}.$$

Consequently, the q_i-th letter of s_{α_i} is greater than that of s_{α_0} whereas all the first $q_i - 1$ letters remain untouched. To summarize, let us suppose $s_{\alpha_0} = c_1c_2\cdots$ and $s_{\alpha_i} = d_1d_2\cdots$. Then $c_1\cdots c_{q_i-1} = d_1\cdots d_{q_i-1}$, but

$$d_{q_i} = \begin{cases} c_{q_i} - 1 & \text{if } \|q_i\alpha_0\| = q_i\alpha_0 - p_i, \\ c_{q_i} + 1 & \text{if } \|q_i\alpha_0\| = p_i - q_i\alpha_0. \end{cases}$$

Let ζ_k and η_k be the k-th complete quotients of $C(\alpha_0)$ and $C(\alpha_i)$, respectively. Then one derives

$$\begin{aligned}|\zeta_{q_i} - \eta_{q_i}| &\geq [b;(ba)^\infty] - [a;(ab)^\infty] = 1 + [0;(ba)^\infty] - [0;(ab)^\infty] \\ &> 1 + \frac{1}{b+1} - \frac{1}{a} = 1 - \frac{2}{a(a+2)} \geq \frac{1}{3}.\end{aligned} \tag{5}$$

The first $q_i - 1$ convergents of $C(\alpha_0) = [0; s_{\alpha_0}]$ coincide with those of $C(\alpha_i) = [0; s_{\alpha_i}]$, respectively. With this fact in mind, we find

$$\begin{aligned}&|C(\alpha_0) - C(\alpha_i)| \\ &= \left| \frac{\zeta_{q_i} P_{q_i-1} + P_{q_i-2}}{\zeta_{q_i} Q_{q_i-1} + Q_{q_i-2}} - \frac{\eta_{q_i} P_{q_i-1} + P_{q_i-2}}{\eta_{q_i} Q_{q_i-1} + Q_{q_i-2}} \right| \\ &= \left| \frac{\zeta_{q_i}(P_{q_i-1}Q_{q_i-2} - P_{q_i-2}Q_{q_i-1}) - \eta_{q_i}(P_{q_i-1}Q_{q_i-2} - P_{q_i-2}Q_{q_i-1})}{(\zeta_{q_i} Q_{q_i-1} + Q_{q_i-2})(\eta_{q_i} Q_{q_i-1} + Q_{q_i-2})} \right| \\ &= \frac{|\zeta_{q_i} - \eta_{q_i}|}{(\zeta_{q_i} Q_{q_i-1} + Q_{q_i-2})(\eta_{q_i} Q_{q_i-1} + Q_{q_i-2})} \\ &> \frac{1}{3(b+2)^2 (Q_{q_i-1})^2} \geq \frac{1}{3(b+2)^2 (B_{q_i-1})^2} = \frac{1}{3(b+2)^2} \cdot \frac{b^2+4}{(\gamma_2^{q_i} - \overline{\gamma}_2^{q_i})^2},\end{aligned}$$

where $\gamma_2 = \frac{b+\sqrt{b^2+4}}{2}$. One concludes

$$\begin{aligned}\frac{|C(\alpha_0) - C(\alpha_i)|}{|\alpha_0 - \alpha_i|} &> \frac{b^2+4}{6(b+2)^2} \cdot \frac{\lambda^{q_i}}{(\gamma_2^{q_i} - \overline{\gamma}_2^{q_i})^2} \\ &= \frac{b^2+4}{6(b+2)^2} \cdot \frac{1}{((\gamma_2/\sqrt{\lambda})^{q_i} - (\overline{\gamma}_2/\sqrt{\lambda})^{q_i})^2} \to \infty \text{ as } i \to \infty.\end{aligned}$$

□

The hole between Theorems 3.4 and 3.5 cannot be filled in for the present. In other words, when $\left(\frac{a+\sqrt{a^2+4}}{2}\right)^2 \leq \theta(\alpha_0) \leq \left(\frac{b+\sqrt{b^2+4}}{2}\right)^2$, the technique in this paper cannot tell whether or not $C(\alpha_0)$ is differentiable at $\alpha = \alpha_0$. A more accurate estimation on Q_k is required.

The graph of $C(\alpha)$ is plotted in Figure 1. The upper one lets α vary in an interval $(1, 5)$, and the lower one in $(1, 2)$ to see it in more detail. This graph also allows us to verify the inequalities in Theorem 3.1(c, d). At every integer $n \geq 1$, one observes $C(n) > C(n+)$ in the figure, which is in accordance with Theorem 3.1 because the denominator of $n = n/1$ is odd. As q increases, the discontinuous jump or fall at p/q decreases in magnitude quickly. Moreover, the increasing or decreasing manner at p/q is subject to the parity of q. These make the graph look like a step function.

4. Examples

Owing to Roth's theorem, Theorem 3.4 proves that $C'(\alpha) = 0$ for all algebraic irrational $\alpha > 1$, because $\theta(\alpha) = 1$. Furthermore, only Liouville numbers

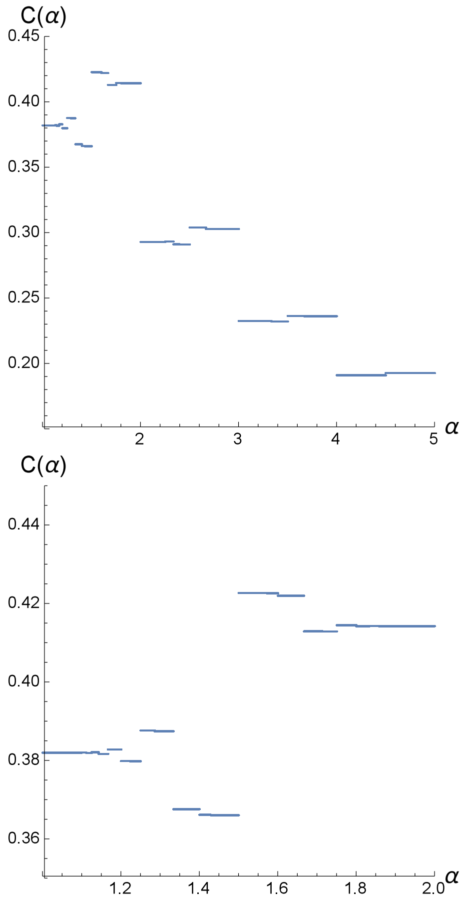

FIGURE 1. Graph of $C(\alpha)$

can be candidates for being continuous but non-differentiable points of $C(\alpha)$. We will see below that such Liouville numbers are very rare.

Example 1. Let $\alpha_0 = 1 + \sum_{n=1}^{\infty} 1/10^{n!}$. In other words, $\alpha_0 - 1$ is *Liouville's constant*. Then $C'(\alpha_0) = 0$ because $\theta(\alpha_0) = 1$. See [8, Example 3].

For a real $x \geq 0$, its n-th *tetration* $\mathcal{E}_n(x)$ is defined by

$$\mathcal{E}_0(x) = 1,\ \mathcal{E}_n(x) = x^{\mathcal{E}_{n-1}(x)} \text{ for } n \geq 1.$$

That is, $\mathcal{E}_1(x) = x$, $\mathcal{E}_2(x) = x^x$, $\mathcal{E}_3(x) = x^{x^x}, \ldots$.

Assume that $t = [a_0; a_1, a_2, \ldots]$ is irrational with convergents $\{p_k/q_k\}_{k\geq 0}$. Since the denominators of convergents satisfy $q_{k+1} = a_{k+1}q_k + q_{k-1}$, we see that $a_{k+1}q_k < q_{k+1} < (a_{k+1}+1)q_k$, and hence

$$\prod_{i=1}^{k} a_i < q_k < \prod_{i=1}^{k}(a_i+1).$$

Example 2. A real $\alpha_0 = [\mathcal{E}_0(2); \mathcal{E}_1(2), \mathcal{E}_2(2), \ldots]$ is a Liouville number, and $C'(\alpha_0) = 0$.

Proof. One readily check that α_0 is a Liouville number. By Proposition 2.4, we get

$$\begin{aligned}\log\theta(\alpha_0) &= \limsup_{k\to\infty}\frac{\log q_{k+1}}{q_k} \le \limsup_{k\to\infty}\frac{\log\prod_{i=1}^{k+1}(a_i+1)}{\prod_{i=1}^{k} a_i}\\ &\le \limsup_{k\to\infty}\frac{\log(2^{k+1}a_{k+1}^{k+1})}{\prod_{i=1}^{k} a_i} = \limsup_{k\to\infty}\frac{(k+1)(1+a_k)\log 2}{\prod_{i=1}^{k} a_i} = 0. \qquad \square\end{aligned}$$

Suppose that an irrational α_0 satisfies the hypothesis of Theorem 3.5, i.e., $\theta(\alpha_0) > \left(\frac{b+\sqrt{b^2+4}}{2}\right)^2$. Then Theorem 3.5 says that $C(\alpha)$ is not differentiable at α_0. But the proof of Theorem 3.5 cannot distinguish the following three cases: $C'(\alpha_0) = \infty$; $C'(\alpha_0) = -\infty$; $C(\alpha)$ is not differentiable at α_0 and $C'(\alpha_0) \neq \pm\infty$. In the next example, however, we can make a more precise statement. We need the following lemmas that can be proved by mathematical induction.

Lemma 4.1. *Suppose that an irrational $t = [a_0; a_1, a_2, \ldots]$ has convergents $\{p_k/q_k\}_{k\ge 0}$. If a_1 is odd, and if a_{2k} is odd and a_{2k+1} is even for every $k \ge 1$, then q_{4k-2} is even and q_{4k} is odd for every $k \ge 1$.*

Lemma 4.2. *For any $k \ge 3$, we have $\mathcal{E}_{k-1}(k+1) > k\mathcal{E}_{k-1}(k)$.*

Proof. Since $\mathcal{E}_{k-1}(k+2) > \mathcal{E}_{k-1}(k+1)$, one obtains $\mathcal{E}_{k-1}(k+2) \ge 1+\mathcal{E}_{k-1}(k+1)$. Suppose that the claim hold for $k = n$. One deduces

$$\begin{aligned}\mathcal{E}_n(n+2) &= (n+2)^{\mathcal{E}_{n-1}(n+2)} > (n+1)^{1+\mathcal{E}_{n-1}(n+1)}\\ &= (n+1)\cdot(n+1)^{\mathcal{E}_{n-1}(n+1)} = (n+1)\mathcal{E}_n(n+1). \qquad \square\end{aligned}$$

Example 3. Let $\alpha_0 = [1; \mathcal{E}_0(0), \mathcal{E}_1(1), \mathcal{E}_2(2), \mathcal{E}_3(3), \ldots] = [1; 1, 1, 2^2, 3^{3^3}, \ldots]$. Then $C(\alpha)$ is continuous but not differentiable at $\alpha = \alpha_0$. Moreover, $C'(\alpha_0) \neq \infty$ and $C'(\alpha_0) \neq -\infty$.

Proof. Proposition 2.4 leads us to compute

$$\begin{aligned}\log\theta(\alpha_0) &= \limsup_{k\to\infty}\frac{\log q_{k+2}}{q_{k+1}} \ge \limsup_{k\to\infty}\frac{\log\prod_{i=0}^{k+1}(\mathcal{E}_i(i))}{\prod_{i=0}^{k}(\mathcal{E}_i(i)+1)}\\ &\ge \limsup_{k\to\infty}\frac{\log\mathcal{E}_{k+1}(k+1)}{2\cdot 2^k(\mathcal{E}_k(k))^k} = \limsup_{k\to\infty}\frac{\log(k+1)}{2\cdot 2^k}\cdot\frac{\mathcal{E}_k(k+1)}{(\mathcal{E}_k(k))^k}\\ &= \limsup_{k\to\infty}\frac{\log(k+1)}{2\cdot 2^k}\cdot\frac{(k+1)^{\mathcal{E}_{k-1}(k+1)}}{k^{k\mathcal{E}_{k-1}(k)}}\end{aligned}$$

$$\geq \limsup_{k\to\infty} \frac{\log(k+1)}{2\cdot 2^k}\cdot\frac{(k+1)^{k\mathcal{E}_{k-1}(k)}}{k^{k\mathcal{E}_{k-1}(k)}} \quad (\because\ \text{Lemma 4.2})$$
$$= \limsup_{k\to\infty} \frac{\log(k+1)}{2\cdot 2^k}\cdot\left(1+\frac{1}{k}\right)^{k\mathcal{E}_{k-1}(k)} = \infty \quad (\because\ e>2).$$

Hence, $\theta(\alpha_0)=\infty$ implies that $C(\alpha)$ is not differentiable at $\alpha=\alpha_0$ by Theorem 3.5.

Let $\{p_k/q_k\}_{k\geq 0}$ be the convergents of α_0. Then $p_k/q_k-\alpha_0<0$ whenever k is even. Thus, when k is even, the first q_k-1 letters of $s_{p_k/q_k}=(bz_{p_k,q_k}a)^\infty$ coincide with those of s_{α_0}, respectively. And their q_k-th letters satisfy $s_{p_k/q_k}(q_k)<s_{\alpha_0}(q_k)$. Now turning to the convergents of $C(p_k/q_k)=[0;(bz_{p_k,q_k}a)^\infty]$ and $C(\alpha_0)=[0;s_{\alpha_0}]$, we find by Lemma 4.1,

$$C(p_{4k-2}/q_{4k-2})<C(\alpha_0) \quad\text{and}\quad C(p_{4k}/q_{4k})>C(\alpha_0),$$

which are followed by

$$\frac{C(p_{4k-2}/q_{4k-2})-C(\alpha_0)}{p_{4k-2}/q_{4k-2}-\alpha_0}>0 \quad\text{and}\quad \frac{C(p_{4k}/q_{4k})-C(\alpha_0)}{p_{4k}/q_{4k}-\alpha_0}<0.$$

But Theorem 3.5 guarantees a sequence $\{\alpha_k\}_{k\geq 1}$ converging α_0 such that

$$\lim_{k\to\infty}\left|\frac{C(\alpha_k)-C(\alpha_0)}{\alpha_k-\alpha_0}\right|=\infty.$$

Therefore, we conclude that neither $C'(\alpha_0)=\infty$ nor $C'(\alpha_0)=-\infty$. □

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