J. Korean Math. Soc. **56** (2019), No. 4, pp. 1031–1048
https://doi.org/10.4134/JKMS.j180577
pISSN: 0304-9914 / eISSN: 2234-3008

THE RESOLUTION DIMENSIONS WITH RESPECT TO BALANCED PAIRS IN THE RECOLLEMENT OF ABELIAN CATEGORIES

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Abstract. In this paper we study recollements of abelian categories and balanced pairs. The main results are: recollements induce new balanced pairs from the middle category; the resolution dimensions are bounded under certain conditions. As an application, the resolution dimensions with respect to cotilting objects of abelian categories involved in recollements are recovered.

1. Introduction

Thirty-five years ago, Beilinson, Bernstein and Deligne [6] first introduced the notion of recollements in the context of triangulated categories. A recollement of abelian categories appeared in Macpherson and Vilonen' work [15]. A recollement situation between abelian categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ is a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] & \mathscr{B} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \mathscr{C} \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"] \end{tikzcd}, \tag{1.1}$$

satisfying the following conditions:

(R1) (q, i, p) and (l, e, r) are adjoint triples;

(R2) the functors i, l and r are fully faithful;

(R3) $\operatorname{Im} i = \operatorname{Ker} e$,

which plays an important role in algebraic geometry, representation theory, polynomial functor theory, ring theory and so on. Psaroudakis and Vitória [17] observed that a recollement whose terms are module categories is equivalent to one induced by an idempotent element. Han and Qin [18] clarified the relations between the n-recollements of derived categories of algebras and the Cartan determinants, homological smoothness and Gorensteinness of algebras,

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Received August 25, 2018; Accepted January 24, 2019.

2010 *Mathematics Subject Classification.* 16E10, 16E30, 18E10.

Key words and phrases. recollement, resolution dimension, balanced pair.

This work was supported by National Natural Science Foundation of China (Grant No. 11671126).

respectively. In 2014, Psaroudakis [16] investigated global, finitistic, and representation dimensions of recollements of abelian categories. The reader may refer to [2, 6, 8, 16, 17, 19] and references therein.

Inspired by [10, Definition 8.2.13], Chen [7] introduced the notion of balanced pairs of additive subcategories in an abelian category. He also studied various properties of balanced pairs in subcategories. We know that the balanced pairs arise naturally from cotorsion triples. The relative homological dimension theory with respect to contravariantly finite subcategories originated in the works of Dugas [9] in order to generalize some standard theory of projective modules. They proved that the relative homological dimensions have nice properties of the classical homological dimensions. Psaroudakis explicitly investigated how various homological invariants and dimensions of the categories involved in a recollement are related. In particular, he showed that the homological dimensions of $\mathscr{B}$ can be bounded by the homological dimensions of $\mathscr{A}$ and $\mathscr{C}$.

Motivated by the research mentioned above, we aim to study how the resolution dimension with respect to balanced pairs behaves in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ between abelian categories. The main results of this paper are the following two theorems proved in Section 2:

Theorem A. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. If $(\mathcal{X}, \mathcal{Y})$ is a balanced pair in $\mathscr{B}$, then*

(i) *$(e(\mathcal{X}), e(\mathcal{Y}))$ is a balanced pair in $\mathscr{C}$;*
(ii) *$(q(\mathcal{X}), p(\mathcal{Y}))$ is a balanced pair in $\mathscr{A}$.*

Theorem B. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories with an admissible balanced pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{B}$. If $\mathcal{X}$-res.dim$\mathscr{B} \leq k$ and $re(\mathcal{X}) \subseteq \mathcal{X}$, $iq(\mathcal{X}) \subseteq \mathcal{X}$, then*

(i) *$e(\mathcal{X})$-res.dim$(\mathscr{C}) \leq k$;*
(ii) *if the functor $q : \mathscr{B} \to \mathscr{A}$ is exact, then $q(\mathcal{X})$-res.dim$(\mathscr{A}) \leq k$;*
(iii) *$\mathcal{X}$-res.dim$(\mathscr{B}) \geq \max\{e(\mathcal{X})\text{-res.dim}(\mathscr{C}), q(\mathcal{X})\text{-res.dim}(\mathscr{A})\}$.*

Throughout, we denote by $\mathbb{N}$, K and Id the set of nonnegative integers, a fixed field and the identity functor, respectively.

2. Main results

Let $\mathscr{B}$ be an abelian category. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathscr{B}$ be a full additive subcategory which are closed under taking direct summands. For an object B in $\mathscr{B}$, the morphism $f : X \to B$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation of B if $\mathrm{Hom}_{\mathscr{B}}(\mathcal{X}, X) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{X}, B) \longrightarrow 0$ is exact in $\mathscr{B}$. The subcategory $\mathcal{X}$ of $\mathscr{B}$ is said to be contravariantly finite, if each object B in $\mathscr{B}$ has a right $\mathcal{X}$-approximation. Dually, the subcategory $\mathcal{Y}$ of $\mathscr{B}$ is called covariantly finite provided each object B in $\mathscr{B}$ admits a left $\mathcal{Y}$-approximation. That is, for a morphism $g : B \to Y$ with $Y \in \mathcal{Y}$, the sequence

$$\mathrm{Hom}_{\mathscr{B}}(Y, \mathcal{Y}) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(B, \mathcal{Y}) \longrightarrow 0$$

is exact in $\mathscr{B}$.

Recall that for a contravariantly finite subcategory $\mathcal{X} \subseteq \mathscr{B}$ and an object $M \in \mathscr{B}$, an $\mathcal{X}$-resolution of M is a complex

$$\cdots \longrightarrow T_n \xrightarrow{t_n} T_{n-1} \xrightarrow{t_{n-1}} \cdots \longrightarrow T_1 \xrightarrow{t_1} T_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

with each $T_i \in \mathcal{X}$ such that it is acyclic by applying the functor $\mathrm{Hom}(T, -)$ for each $T \in \mathcal{X}$; this is equivalent to that each induced morphism $T_n \to \mathrm{Ker}\, t_{n-1}$ is a right $\mathcal{X}$-approximation. We also recall that the $\mathcal{X}$-resolution dimension $\mathcal{X}$-res.dim(M) of an object M is defined to be the minimal integer $n \geq 0$ such that there is an $\mathcal{X}$-resolution

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0.$$

That is,

$$\mathcal{X}\text{-res.dim}(M) = \min\{n \in \mathbb{N} \mid \Omega_{\mathcal{X}}^{n+1} M = \mathrm{Ker}\, t_n = 0\}.$$

If there is no such an integer, we set $\mathcal{X}$-res.dim$(M) = \infty$. Define the global $\mathcal{X}$-resolution dimension $\mathcal{X}$-res.dim$(\mathscr{B})$ to be the supreme of the $\mathcal{X}$-resolution dimension of all the objects in $\mathscr{B}$. That is,

$$\mathcal{X}\text{-res.dim}(\mathscr{B}) = \sup\{\mathcal{X}\text{-res.dim}(M) \mid \forall M \in \mathscr{B}\}.$$

Dually, let $\mathcal{Y} \subseteq \mathscr{B}$ be another full additive subcategory which is closed under taking direct summands. One also can define $\mathcal{Y}$-coresolution and $\mathcal{Y}$-coresolution dimension $\mathcal{Y}$-cores.dim$N$ of an object N. Furthermore, one has the notion of global $\mathcal{Y}$-coresolution dimension $\mathcal{Y}$-cores.dim$(\mathscr{B})$. That is,

$$\mathcal{Y}\text{-cores.dim}(\mathscr{B}) = \sup\{\mathcal{Y}\text{-cores.dim}(N) \mid \forall N \in \mathscr{B}\}.$$

For more details, see [3, 5, 7, 9, 14] and references therein.

A pair $(\mathcal{X}, \mathcal{Y})$ of additive subcategories in $\mathscr{B}$ is called a balanced pair if the following conditions are satisfied:

(BP1) the subcategory $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite;

(BP2) for each object M, there is an $\mathcal{X}$-resolution $X^\bullet \to M$ such that it is acyclic by applying the functors $\mathrm{Hom}_{\mathscr{B}}(-, Y)$ for all $Y \in \mathcal{Y}$;

(BP3) for each object N, there is a $\mathcal{Y}$-coresolution $N \to Y^\bullet$ such that it is acyclic by applying the functors $\mathrm{Hom}_{\mathscr{B}}(X, -)$ for all $X \in \mathcal{X}$.

Example 1. The pair (Proj-A, Inj-A) in Mod-A over an Artin algebra A is a balanced pair, where Proj-A (resp. Inj-A) is the full subcategory of Mod-A consisting of all projectives (resp. injectives) in Mod-A. Dually, the pair (Proj-C, Inj-C) in Comod-C over a right semiperfect coalgebra C is a balanced pair, where Proj-C (resp. Inj-C) is the full subcategory of Comod-C consisting of all projectives (resp. injectives) in Comod-C. If R is a Gorenstein ring, then the pair $(\mathrm{GP}(R), \mathrm{GI}(R))$ in Mod-R is also a balanced pair, where $\mathrm{GP}(R)$ (resp. $\mathrm{GI}(R)$) is the full subcategory of Mod-R consisting of all Gorenstein projectives (resp. Gorenstein injectives) in Mod-R.

Chen [7] also defined that the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible if $\mathcal{X} \subseteq \mathscr{B}$ is admissible, that is, each right $\mathcal{X}$-approximation is epic. In this case the additive subcategory $\mathcal{Y} \subseteq \mathscr{B}$ is coadmissible, that is, each left $\mathcal{Y}$-approximation is monic.

Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Some properties of a recollement are listed as follows. The reader may refer to [8], [13], [17], [16, Remarks 2.2-2.5] and references therein.

(i) The functors $e : \mathscr{B} \to \mathscr{C}$ and $i : \mathscr{A} \to \mathscr{B}$ are exact. Moreover, $qi \simeq \mathrm{Id}_{\mathscr{A}}$, $\mathrm{Id}_{\mathscr{A}} \simeq pi$, $er \simeq \mathrm{Id}_{\mathscr{C}}$ and $\mathrm{Id}_{\mathscr{C}} \simeq el$.

(ii) If the pair (l, e) is an adjoint functor pair and the functor e is exact, then the left adjoint functor l preserves projective objects.

(iii) If the pair (e, r) is an adjoint functor pair and the functor r is exact, then the left adjoint functor e preserves projective objects.

(iv) If the pair (l, e) is an adjoint functor pair and the functor l is exact, then the right adjoint functor e preserves injective objects.

(v) If the pair (e, r) is an adjoint functor pair and the functor e is exact, then the right adjoint functor r preserves injective objects.

(vi) For any adjoint functor pair, the left adjoint functor preserves the right exactness and commutes with any direct sums; the right adjoint functor preserves the left exactness and commutes with any direct products, such as for the adjoint pair (l, e), we have that $\mathrm{Add}(l(M)) = l(\mathrm{Add}(M))$ and $\mathrm{Prod}(e(N)) = e(\mathrm{Prod}(N))$.

Now we have the following key observations, which are very important for the proof of our main result.

Lemma 2.1. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories.*

(i) *If $\mathcal{X}$ is a contravariantly finite subcategory of $\mathscr{B}$, then $e(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{C}$.*

(ii) *If $\mathcal{Y}$ is a covariantly finite subcategory of $\mathscr{B}$, then $e(\mathcal{Y})$ is a covariantly finite subcategory of $\mathscr{C}$.*

Proof. (i) For each object C in $\mathscr{C}$, we have $r(C) \in \mathscr{B}$ and there exists an object $X_0 \in \mathcal{X}$ such that the morphism $f : X_0 \to r(C)$ is a right $\mathcal{X}$-approximation. We apply the exact functor e to f, then the natural equivalence $er \simeq \mathrm{Id}_{\mathscr{C}}$ can induce a morphism $e(f) : e(X_0) \to C$ in $\mathscr{C}$. Now we claim that $e(f)$ is a right $e(\mathcal{X})$-approximation. Indeed, it suffices to show that the sequence

$$\mathrm{Hom}_{\mathscr{C}}(e(X), e(X_0)) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(e(X), C) \longrightarrow 0$$

is exact in $\mathscr{C}$, for each object $e(X) \in e(\mathcal{X})$, i.e., the morphism $\mathrm{Hom}_{\mathscr{C}}(e(X), e(f))$ is epic. Since (e, r) is an adjoint pair, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(e(X), e(X_0)) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(e(X), e(f))"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{C}}(e(X), C) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{B}}(X, re(X_0)) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(X, re(f))"] & \mathrm{Hom}_{\mathscr{B}}(X, r(C)).
\end{tikzcd}$$

To prove $\mathrm{Hom}_{\mathscr{C}}(e(X), e(f))$ is epic, it suffices to show that there exists a morphism $\alpha : X \to re(X_0)$ such that $re(f)\alpha = \gamma$ for any $\gamma : X \to r(C)$. Suppose that ν is the unit of the adjoint pair (e, r), then the existence of α can be inferred from the following commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, dotted, "s"'] \arrow[ddl, dotted, "\alpha"] \arrow[d, "\gamma"] \\
X_0 \arrow[r, "f"] \arrow[d, "\nu_{X_0}"'] & r(C) \arrow[d, equal] \\
re(X_0) \arrow[r, "re(f)"] & r(C)
\end{tikzcd}$$

where the existence of the morphism s is from the fact that f is a right $\mathcal{X}$-approximation. Here we just need to take $\alpha = \nu_{X_0} \circ s$. Therefore, the morphism $\mathrm{Hom}_{\mathscr{B}}(X, re(f))$ is epic. Equivalently, $\mathrm{Hom}_{\mathscr{C}}(e(X), e(f))$ is epic. Hence, for each object C in $\mathscr{C}$, there exists an object $e(X_0) \in e(\mathcal{X})$ such that $e(f) : e(X_0) \to C$ is a right $e(\mathcal{X})$-approximation. Consequently, $e(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{C}$.

(ii) For each object C in $\mathscr{C}$, we have $l(C) \in \mathscr{B}$ and there exists an object $Y_0 \in \mathcal{Y}$ such that the morphism $g : l(C) \to Y_0$ is a left $\mathcal{Y}$-approximation. We apply the exact functor e to g, then the natural equivalence $el \simeq \mathrm{Id}_{\mathscr{C}}$ can induce a morphism $e(g) : C \to e(Y_0)$ in $\mathscr{C}$. Now we claim that $e(g)$ is a left $e(\mathcal{Y})$-approximation. Indeed, it suffices to show that the sequence

$$\mathrm{Hom}_{\mathscr{C}}(e(Y_0), e(Y)) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(C, e(Y)) \longrightarrow 0$$

is exact in $\mathscr{C}$, for each object $e(Y) \in e(\mathcal{Y})$, i.e., the morphism $\mathrm{Hom}_{\mathscr{C}}(e(g), e(Y))$ is epic. Since (l, e) is an adjoint pair, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(e(Y_0), e(Y)) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(e(g),e(Y))"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{C}}(C, e(Y)) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{B}}(le(Y_0), Y) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(le(g),Y)"] & \mathrm{Hom}_{\mathscr{B}}(l(C), Y).
\end{tikzcd}$$

To prove $\mathrm{Hom}_{\mathscr{C}}(e(g), e(Y))$ is epic, it suffices to show that there exists a morphism $\beta : le(Y_0) \to Y$ such that $le(g)\beta = \delta$, for any $\delta : l(C) \to Y$. Suppose that μ is the counit of the adjoint pair (l, e), then the existence of β can be inferred from the following commutative diagram

$$\begin{tikzcd}
Y & \\
l(C) \arrow[u, "\delta"] \arrow[r, "le(g)"] & le(Y_0) \arrow[ul, dotted, "\beta"'] \arrow[d, "\mu_{Y_0}"] \\
l(C) \arrow[u, equal] \arrow[r, "g"] & Y_0 \arrow[uul, dotted, "t"]
\end{tikzcd}$$

where the existence of the morphism t is from the fact that g is a left $\mathcal{Y}$-approximation. Here we just need to take $\beta = t \circ \mu_{Y_0}$. Therefore, the morphism $\mathrm{Hom}_{\mathscr{B}}(le(g), Y)$ is epic. Equivalently, $\mathrm{Hom}_{\mathscr{C}}(e(g), e(Y))$ is epic. Hence, for each object C in $\mathscr{C}$, there exists an object $e(Y_0) \in e(\mathcal{Y})$ such that $e(g) : C \to e(Y_0)$ is a left $e(\mathcal{Y})$-approximation. Consequently, $e(\mathcal{Y})$ is a covariantly finite subcategory of $\mathscr{C}$. ☐

Lemma 2.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories.*

- (i) *If $\mathcal{X}$ is a contravariantly finite subcategory of $\mathscr{B}$, then $q(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{A}$.*
- (ii) *If $\mathcal{Y}$ is a covariantly finite subcategory of $\mathscr{B}$, then $p(\mathcal{Y})$ is a covariantly finite subcategory of $\mathscr{A}$.*

Proof. (i) For each object A in $\mathscr{A}$, we have $i(A) \in \mathscr{B}$ and there exists an object $X_0' \in \mathcal{X}$ such that the morphism $f' : X_0' \to i(A)$ is a right $\mathcal{X}$-approximation. We apply the functor q to f, then the natural equivalence $qi \cong \mathrm{Id}_{\mathscr{A}}$ can induce a morphism $q(f') : q(X_0') \to A$ in $\mathscr{A}$. Now we claim that $q(f')$ is a right $q(\mathcal{X})$-approximation. Indeed, it suffices to show that the sequence

$$\mathrm{Hom}_{\mathscr{A}}(q(X), q(X_0')) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(q(X), A) \longrightarrow 0$$

is exact in $\mathscr{A}$, i.e., the morphism $\mathrm{Hom}_{\mathscr{A}}(q(X), q(f'))$ is epic for any $q(X) \in q(\mathcal{X})$. Since (q, i) is an adjoint pair, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(q(X), q(X_0')) \arrow[r, "\mathrm{Hom}_{\mathscr{A}}(q(X), q(f'))"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{A}}(q(X), A) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{B}}(X, iq(X_0')) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(X, iq(f'))"] & \mathrm{Hom}_{\mathscr{B}}(X, i(A)).
\end{tikzcd}$$

To prove the morphism $\mathrm{Hom}_{\mathscr{A}}(q(X), q(f'))$ is epic, it suffices to show $\mathrm{Hom}_{\mathscr{B}}(X, iq(f'))$ is epic, for any $X \in \mathcal{X}$. That is to check that there exists a morphism $\alpha' : X \to iq(X_0')$ such that $iq(f')\alpha' = \gamma'$ for any $\gamma' : X \to i(A)$. Suppose that $\lambda : \mathrm{Id}_{\mathscr{B}} \to iq$ is a unit of the adjoint pair (q, i). The existence of α' can be inferred from the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, dotted, "s'"'] \arrow[ddl, dotted, "\alpha'"] \arrow[d, "\gamma'"] \\
X_0' \arrow[r, "f'"] \arrow[d, "\lambda_{X_0}"'] & i(A) \arrow[d, equal] \\
iq(X_0') \arrow[r, "iq(f')"] & i(A)
\end{tikzcd}$$

where the existence of the morphism s' is from the fact that f' is a right $\mathcal{X}$-approximation. We just need to take $\alpha' = \lambda_{X_0'} \circ s'$. Therefore, $q(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{A}$.

(ii) For each object A in $\mathscr{A}$, we have $i(A) \in \mathscr{B}$ and there exists an object $Y_0' \in \mathcal{Y}$ such that the morphism $g' : i(A) \to Y_0'$ is a left $\mathcal{Y}$-approximation. We apply the functor p to g', then the natural equivalence $pi \simeq \mathrm{Id}_{\mathscr{A}}$ can induce a morphism $p(g') : A \to p(Y_0')$ in $\mathscr{A}$. Now we claim that $p(g')$ is a left $p(\mathcal{Y})$-approximation. Indeed, it suffices to show that the sequence

$$\mathrm{Hom}_{\mathscr{A}}(p(Y_0'), p(Y)) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, p(Y)) \longrightarrow 0$$

is exact in $\mathscr{A}$, for each object $p(Y) \in p(\mathcal{Y})$, i.e., the morphism $\mathrm{Hom}_{\mathscr{A}}(p(g'), p(Y))$ is epic. Since (i, p) is an adjoint pair, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(p(Y_0'), p(Y)) \arrow[r, "\mathrm{Hom}_{\mathscr{A}}(p(g'),p(Y))"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathscr{A}}(A, p(Y)) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{B}}(ip(Y_0'), Y) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(ip(g'),Y)"] & \mathrm{Hom}_{\mathscr{B}}(i(A), Y).
\end{tikzcd}$$

To prove $\mathrm{Hom}_{\mathscr{A}}(p(g'), p(Y))$ is epic, it suffices to show that there exists a morphism $\beta' : ip(Y_0') \to Y$ such that $ip(g')\beta' = \delta'$ for any $\delta' : i(A) \to Y$. Suppose that κ is the counit of the adjoint pair (i, p), then the existence of β' can be inferred from the following commutative diagram

$$\begin{tikzcd}
Y & \\
i(A) \arrow[u, "\delta'"] \arrow[r, "ip(g')"] \arrow[d, equal] & ip(Y_0') \arrow[ul, dotted, "\beta'"'] \arrow[d, "\kappa_{Y_0'}"] \\
i(A) \arrow[r, "g'"] & Y_0' \arrow[uul, dotted, "t'"]
\end{tikzcd}$$

where the existence of the morphism t' is from the fact that g' is a left $\mathcal{Y}$-approximation. Here we just need to take $\beta' = t' \circ \kappa_{Y_0'}$. Therefore, $p(\mathcal{Y})$ is a covariantly finite subcategory of $\mathscr{A}$. □

Next we concentrate on the recollement (1.1) in which $\mathscr{B}$ has a balanced pair. The following result shows that $\mathscr{A}$ and $\mathscr{C}$ also have balanced pairs in this case.

Proposition 2.3. *Let* $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ *be a recollement of abelian categories. If* $(\mathcal{X}, \mathcal{Y})$ *is a balanced pair in* $\mathscr{B}$*, then*

(i) $(e(\mathcal{X}), e(\mathcal{Y}))$ *is a balanced pair in* $\mathscr{C}$;
(ii) $(q(\mathcal{X}), p(\mathcal{Y}))$ *is a balanced pair in* $\mathscr{A}$.

Proof. (i) According to Lemma 2.1, the pair $(e(\mathcal{X}), e(\mathcal{Y}))$ meets the condition (BP1). Now we only check $(e(\mathcal{X}), e(\mathcal{Y}))$ satisfies (BP2) and (BP3).
About BP2: Since $e(\mathcal{X})$ is contravariantly finite in $\mathscr{C}$, it follows that there is a $e(\mathcal{X})$-resolution as follows

$$(2.1) \qquad \cdots \to e(X_n) \xrightarrow{\delta_n} e(X_{n-1}) \to \cdots \to e(X_1) \xrightarrow{\delta_1} e(X_0) \xrightarrow{\delta_0} H \to 0$$

for each object $H \in \mathscr{C}$. It suffices to show that the resolution (2.1) is acyclic by applying the functor $\operatorname{Hom}_{\mathscr{C}}(-, e(Y))$, for each object $e(Y)$ in $e(\mathcal{Y})$. We note that the induced morphism $e(X_n) \to \operatorname{Ker}\delta_{n-1}$ is a right $e(\mathcal{X})$-approximation for arbitrary $n \in \mathbb{N}$. Applying the functor l to (2.1), we have the following complex

$$(2.2) \quad \cdots \to le(X_n) \xrightarrow{l(\delta_n)} le(X_{n-1}) \to \cdots \to le(X_1) \xrightarrow{l(\delta_1)} le(X_0) \xrightarrow{l(\delta_0)} l(H) \to 0.$$

Since $(\mathcal{X}, \mathcal{Y})$ is a balanced pair in $\mathscr{B}$, it follows that (2.2) is acyclic by applying the functor $\operatorname{Hom}_{\mathscr{B}}(-, Y)$ to (2.2) for all $Y \in \mathcal{Y}$. Moreover, since (l, e) is an adjoint pair, there is the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathscr{B}}(l(H), Y) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathscr{B}}(le(X_0), Y) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \operatorname{Hom}_{\mathscr{B}}(le(X_n), Y) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_{\mathscr{C}}(H, e(Y)) \arrow[r] & \operatorname{Hom}_{\mathscr{C}}(e(X_0), e(Y)) \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_{\mathscr{C}}(e(X_n), e(Y)) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, the resolution (2.1) is acyclic by applying the functor $\operatorname{Hom}_{\mathscr{C}}(-, e(Y))$, for each object $e(Y)$ in $e(\mathcal{Y})$.
About BP3: For each object $H' \in \mathscr{C}$, there is a $e(\mathcal{Y})$-coresolution as follows

$$(2.3) \qquad 0 \to H' \xrightarrow{\gamma_0} e(Y_0) \xrightarrow{\gamma_1} e(Y_1) \to \cdots \to e(Y_{n-1}) \xrightarrow{\gamma_n} e(Y_n) \to \cdots$$

since $e(\mathcal{Y})$ is covariantly finite in $\mathscr{C}$. It suffices to show that the coresolution (2.3) is acyclic by applying the functor $\operatorname{Hom}_{\mathscr{C}}(e(X), -)$, for each object $e(X)$ in $e(\mathcal{X})$. We note that the induced morphism $\operatorname{Ker}\gamma_n \to e(Y_{n-1})$ is a left $e(\mathcal{Y})$-approximation for arbitrary $n \in \mathbb{N}$. Applying the functor r to (2.3), we have the following complex

$$(2.4) \quad 0 \to r(H') \xrightarrow{r(\gamma_0)} re(Y_0) \xrightarrow{r(\gamma_1)} re(Y_1) \to \cdots \to re(Y_{n-1}) \xrightarrow{r(\gamma_n)} re(Y_n) \to \cdots .$$

Since $(\mathcal{X}, \mathcal{Y})$ is a balanced pair in $\mathscr{B}$, it follows that (2.4) is acyclic by applying the functor $\operatorname{Hom}_{\mathscr{B}}(X, -)$ to (2.4) for all $X \in \mathcal{X}$. Moreover, since (e, r) is an adjoint pair, there exists the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(X, r(H')) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{B}}(X, re(Y_0)) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(X, re(Y_n)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(e(X), H') \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(e(X), e(Y_0)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(e(X), e(Y_n)) \arrow[r] & \cdots .
\end{tikzcd}$$

Thus, the coresolution (2.3) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{C}}(e(X), -)$, for each object $e(X)$ in $e(\mathcal{X})$. Therefore, the proof of the statement (i) is end.

(ii) By Lemma 2.2, the pair $(q(\mathcal{X}),\ p(\mathcal{Y}))$ meets the condition (BP1). Now we only check $(q(\mathcal{X}),\ p(\mathcal{Y}))$ satisfies (BP2) and (BP3).

About BP2: Since $q(\mathcal{X})$ is contravariantly finite in $\mathscr{A}$, there is a $q(\mathcal{X})$-resolution as follows

$$\cdots \to q(X_n) \xrightarrow{\alpha_n} q(X_{n-1}) \to \cdots \to q(X_1) \xrightarrow{\alpha_1} q(X_0) \xrightarrow{\alpha_0} Z \to 0 \tag{2.5}$$

for each object $Z \in \mathscr{A}$. It suffices to show that the resolution (2.5) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{A}}(-, p(Y))$, for each object $p(Y)$ in $p(\mathcal{Y})$. We note that the induced morphism $q(X_n) \to \mathrm{Ker}\,\alpha_{n-1}$ is a right $q(\mathcal{X})$-approximation for arbitrary $n \in \mathbb{N}$. Applying the functor i to (2.5), we have the following complex

$$\cdots \to iq(X_n) \xrightarrow{i(\alpha_n)} iq(X_{n-1}) \to \cdots \to iq(X_1) \xrightarrow{i(\alpha_1)} iq(X_0) \xrightarrow{i(\alpha_0)} i(Z) \to 0. \tag{2.6}$$

Since $(\mathcal{X},\ \mathcal{Y})$ is a balanced pair in $\mathscr{B}$, it follows that (2.6) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{B}}(-, Y)$ to (2.6) for all $Y \in \mathcal{Y}$. Moreover, since (i, p) is an adjoint pair, there exits the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(i(Z), Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{B}}(iq(X_0), Y) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(iq(X_n), Y) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(Z, p(Y)) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(X_0), p(Y)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(X_n), p(Y)) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, the resolution (2.5) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{A}}(-, p(Y))$, for each object $p(Y)$ in $p(\mathcal{Y})$.

About BP3: For each object $Z' \in \mathscr{A}$, there is a $p(\mathcal{Y})$-coresolution as follows

$$0 \to Z' \xrightarrow{\beta_0} p(Y_0) \xrightarrow{\beta_1} p(Y_1) \to \cdots \to p(Y_{n-1}) \xrightarrow{\beta_n} p(Y_n) \to \cdots \tag{2.7}$$

since $p(\mathcal{Y})$ is covariantly finite in $\mathscr{A}$. It suffices to show that the coresolution (2.7) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{A}}(q(X), -)$, for each object $q(X)$ in $q(\mathcal{X})$. We note that the induced morphism $\mathrm{Ker}\,\beta_n \to p(Y_{n-1})$ is a left $p(\mathcal{Y})$-approximation for arbitrary $n \in \mathbb{N}$. Applying the functor i to (2.7), we have the following complex

$$0 \to i(Z') \xrightarrow{i(\beta_0)} ip(Y_0) \xrightarrow{i(\beta_1)} ip(Y_1) \to \cdots \to ip(Y_{n-1}) \xrightarrow{i(\beta_n)} ip(Y_n) \to \cdots . \tag{2.8}$$

Since $(\mathcal{X},\ \mathcal{Y})$ is a balanced pair in $\mathscr{B}$, it follows that (2.8) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{B}}(X, -)$ to (2.8) for all $X \in \mathcal{X}$. Moreover, since (q, i) is an adjoint pair, there exists the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(X, i(Z')) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{B}}(X, ip(Y_0)) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(X, ip(Y_n)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(X), Z') \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(X), p(Y_0)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(X), p(Y_n)) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, the coresolution (2.7) is acyclic by applying the functor $\mathrm{Hom}_{\mathscr{A}}(q(X), -)$ for each object $q(X)$ in $q(\mathcal{X})$. Consequently, we finish the proof of the statements (i) and (ii). ☐

Corollary 2.4. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. If the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible in $\mathscr{B}$, then the balanced pairs $(e(\mathcal{X}), e(\mathcal{Y}))$ and $(q(\mathcal{X}), p(\mathcal{Y}))$ are admissible in $\mathscr{C}$ and $\mathscr{A}$, respectively.*

Proof. Assume that the balanced pair $(\mathcal{X}, \mathcal{Y})$ is admissible. According to the proof of Lemma 2.1 and Lemma 2.2, we then know that the right $\mathcal{X}$-approximations $f : X_0 \to r(C)$ and $f' : X'_0 \to i(A)$ are epic. Observe that the functors e is exact and q is right exact. This implies that $e(f) : e(X_0) \to C$ and $q(f') : q(X'_0) \to A$ are epic. Thus, the contravariantly finite subcategories $e(\mathcal{X})$ and $q(\mathcal{X})$ are admissible. Then the results follows from [7, Corollary 2.3] immediately. ☐

Here, we have prepared all the ingredients to state our main result in this section.

Theorem 2.5. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories with an admissible balanced pair $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{B}$. If $\mathcal{X}$-res.dim$\mathscr{B} \leq k$ and $re(\mathcal{X}) \subseteq \mathcal{X}$, $iq(\mathcal{X}) \subseteq \mathcal{X}$, then*

(i) $e(\mathcal{X})$-res.dim$(\mathscr{C}) \leq k$;
(ii) *if the functor $q : \mathscr{B} \to \mathscr{A}$ is exact, then* $q(\mathcal{X})$-res.dim$(\mathscr{A}) \leq k$;
(iii) $\mathcal{X}$-res.dim$(\mathscr{B}) \geq \max\{e(\mathcal{X})\text{-res.dim}(\mathscr{C}), q(\mathcal{X})\text{-res.dim}(\mathscr{A})\}$.

Proof. (i) Since $\mathcal{X}$ is a contravariantly finite subcategory of $\mathscr{B}$, it follows from Lemma 2.1(i) that $e(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{C}$. Let C be an object in $\mathscr{C}$. Since the object $r(C) \in \mathscr{B}$ and $\mathcal{X}$-res.dim$(\mathscr{B}) \leq k$, there exists an $\mathcal{X}$-resolution of $r(C)$

$$0 \to T_k \xrightarrow{t_k} T_{k-1} \xrightarrow{t_{k-1}} T_{k-2} \to \cdots \to T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} r(C) \to 0 \tag{2.9}$$

with $T_i (i = 0, 1, \ldots, k) \in \mathcal{X}$ such that the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{B}}(r(C), T) \to \mathrm{Hom}_{\mathscr{B}}(T_0, T) \to \cdots \to \mathrm{Hom}_{\mathscr{B}}(T_k, T) \to 0$$

is exact for all $T \in \mathcal{Y}$. Applying the exact functor $e : \mathscr{B} \to \mathscr{C}$ to (2.9), we obtain an $e(\mathcal{X})$-resolution of C

$$0 \to e(T_k) \xrightarrow{t_k^*} e(T_{k-1}) \xrightarrow{t_{k-1}^*} \cdots \to e(T_1) \xrightarrow{t_1^*} e(T_0) \xrightarrow{t_0^*} C \to 0$$

with $e(T_i)(i = 0, 1, \ldots, k) \in e(\mathcal{X})$. To prove that $e(\mathcal{X})$-res.dim$(\mathscr{C}) \leq k$, it is enough to show that the induced sequence

(2.10)

$$0 \to \mathrm{Hom}_{\mathscr{C}}(C, e(T)) \to \mathrm{Hom}_{\mathscr{C}}(e(T_0), e(T)) \to \cdots \to \mathrm{Hom}_{\mathscr{C}}(e(T_k), e(T)) \to 0$$

is exact for all $e(T) \in e(\mathcal{Y})$. Since (e, r) is an adjoint pair, there is the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(er(C), e(T)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(e(T_0), e(T)) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(e(T_k), e(T)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(r(C), re(T)) \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(T_0, re(T)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(T_k, re(T)) \arrow[r] & 0
\end{tikzcd}$$

Since the morphisms $T_0 \to r(C)$, $T_1 \to \mathrm{Ker}(t_0)$, ..., $T_k \to \mathrm{Ker}\,(t_{k-1})$ are right $\mathcal{X}$-approximations, it follows that the morphisms $e(T_0) \to er(C)$, $e(T_1) \to e(\mathrm{Ker}(t_0))$, $\cdots$, $e(T_k) \to e(\mathrm{Ker}(t_{k-1}))$ are right $e(\mathcal{X})$-approximations. Indeed, the functor r is fully faithful and $re(\mathcal{X}) \subseteq \mathcal{X}$. This implies that the sequence (2.10) is exact. Hence, $e(\mathcal{X})$-res.dim$(\mathscr{C}) \leq k$.

(ii) Since $\mathcal{X}$ is a contravariantly finite subcategory of $\mathscr{B}$, it follows from lemma 2.2(i) that $q(\mathcal{X})$ is a contravariantly finite subcategory of $\mathscr{A}$. Let A be an object in $\mathscr{A}$. Since $\mathcal{X}$-res.dim$(\mathscr{B}) \leq k$, there exists an $\mathcal{X}$-resolution of $i(A)$

$$0 \to T_k \xrightarrow{s_k} T_{k-1} \xrightarrow{s_{k-1}} T_{k-2} \to \cdots \to T_1 \xrightarrow{s_1} T_0 \xrightarrow{s_0} i(A) \to 0 \tag{2.11}$$

with $T_i (i = 0, 1, \ldots, k) \in \mathcal{X}$ such that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{B}}(i(A), T) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(T_0, T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathscr{B}}(T_k, T) \longrightarrow 0$$

is exact for all $T \in \mathcal{Y}$. Applying the exact functor $q : \mathscr{B} \to \mathscr{A}$ to (2.11), we obtain a $q(\mathcal{X})$-resolution of A

$$0 \longrightarrow q(T_k) \xrightarrow{\quad s_k^* \quad} q(T_{k-1}) \xrightarrow{\quad s_{k-1}^* \quad} \cdots \longrightarrow q(T_1) \xrightarrow{s_1^*} q(T_0) \xrightarrow{s_0^*} A \longrightarrow 0$$

with $q(T_i)(i = 0, 1, \ldots, k) \in q(\mathcal{X})$. To show that $q(\mathcal{X})$-res.dim$(\mathscr{A}) \leq k$, it is enough to prove that the induced sequence

(2.12)

$$0 \to \mathrm{Hom}_{\mathscr{A}}(A, q(T)) \to \mathrm{Hom}_{\mathscr{A}}(q(T_0), q(T)) \longrightarrow \cdots \to \mathrm{Hom}_{\mathscr{A}}(q(T_k), q(T)) \to 0$$

is exact. Since (q, i) is an adjoint pair, there is the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(qi(A), q(T)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{A}}(q(T_0), q(T)) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(q(T_k), q(T)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(i(A), iq(T)) \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(T_0, iq(T)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(T_k, iq(T)) \arrow[r] & 0
\end{tikzcd}$$

Since the morphisms $T_0 \to i(A)$, $T_1 \to \mathrm{Ker}(s_0), \ldots, T_k \to \mathrm{Ker}(s_{k-1})$ are right $\mathcal{X}$-approximations, it follows that the morphisms $q(T_0) \to qi(A)$, $q(T_1) \to q(\mathrm{Ker}(s_0)), \ldots, q(T_k) \to q(\mathrm{Ker}(s_{k-1}))$ are right $q(\mathcal{X})$-approximations. Indeed, i is fully faithful and $iq(\mathcal{X}) \subseteq \mathcal{X}$. This implies that the sequence (2.12) is exact. So, we have $q(\mathcal{X})$-res.dim$(\mathscr{A}) \leq k$.

Finally, the assertion (iii) can be easily obtained from the proofs of (i) and (ii). $\square$

According to the above theorem and [7, Corollary 2.5], we deduce the following corollary.

Corollary 2.6. *Let* $(\mathscr{A},\mathscr{B},\mathscr{C})$ *be a recollement of abelian categories with an admissible balanced pair* $(\mathcal{X},\mathcal{Y})$ *in* $\mathscr{B}$. *If* $\mathcal{Y}$-cores.dim$\mathscr{B} \leq k$ *and* $le(\mathcal{Y}) \subseteq \mathcal{Y}$, $ip(\mathcal{Y}) \subseteq \mathcal{Y}$, *then*

(i) $e(\mathcal{Y})$-cores.dim$(\mathscr{C}) \leq k$;

(ii) *if the functor* $p : \mathscr{B} \to \mathscr{A}$ *is exact, then* $p(\mathcal{Y})$-cores.dim$(\mathscr{A}) \leq k$;

(iii) $\mathcal{Y}$-cores.dim$(\mathscr{B}) \geq \max\{e(\mathcal{Y})\text{-cores.dim}(\mathscr{C}), p(\mathcal{Y})\text{-cores.dim}(\mathscr{A})\}$.

3. Applications

This subsection is devoted to applying the above results to the cotilting cotorsion pairs of special abelian categories, that is, module categories.

For an R-module T, we define the full subcategory ${}^{\perp}T$ of mod(R) as follows

$$^{\perp}T = \{M \in \mathrm{Mod}(R) \,|\, \mathrm{Ext}^n_R(M,T) = 0 \text{ for any } n \in \mathbb{N}\}.$$

Recall from [1, Definition 2.1] that an R-module T is said to be cotilting, if it satisfies the following three conditions:

(CT1) inj.dim$(T) \leq \infty$;

(CT2) $T \in^{\perp} T$;

(CT3) For any $X \in^{\perp} T$, there exists a short exact sequence

$$0 \longrightarrow X \longrightarrow T' \longrightarrow X' \longrightarrow 0$$

with $X' \in^{\perp} T$, $T' \in \mathrm{add}(T)$, where add(T) denotes the class of direct summands of direct sums of copies of T.

There is a well-known example of cotilting modules. For instance, for a finite dimensional algebra R and a field K and a tilting R^{op}-module T, the K-dual of T, DT is a cotilting R-module.

Recall that a pair $(\mathcal{X},\mathcal{Y})$ of subcategories in an abelian category $\mathscr{B}$ is called a cotorsion pair provided that $\mathcal{X} =^{\perp} \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. Furthermore, the cotorsion pair $(\mathcal{X},\mathcal{Y})$ is complete provided that for each object M in $\mathscr{B}$ there exists two short exact sequences

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow M \longrightarrow Y' \longrightarrow X' \longrightarrow 0$$

with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is said to be hereditary provided that $\mathcal{X}$ is resolving, or equivalently, $\mathcal{Y}$ is coresolving. The reader may refer to [7] and [11] for resolving and coresolving subcategories.

Given a cotilting module T in Mod(R), we will write $\mathcal{X}_T$ instead of ${}^{\perp}T$. Moreover, we set

$$\mathcal{Y}_T := \{N \in \mathrm{Mod}(R) \,|\, \mathrm{Ext}^n_R(M, N) = 0 \text{ for any } n \in \mathbb{N} \text{ and any } M \in \mathcal{X}_T\}.$$

We know from [11, Chapter 8] that the pair $(\mathcal{X}_T, \mathcal{Y}_T)$ generated by T is hereditary and complete.

Proposition 3.1. *Let* $\mathrm{Mod}(R)$ *be a module category with a cotilting object* T*. Then the cotilting cotorsion pair* $(\mathcal{X}_T, \mathcal{Y}_T)$ *is an admissible balanced pair in* $\mathrm{Mod}(R)$*.*

Proof. Firstly, we claim that the cotilting cotorsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ is a balanced pair. To show this, it suffices to check that it satisfies (BP1), (BP2) and (BP3) of the balanced pair.

About (BP1): From the above consideration, we know that the cotilting cotorsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ is complete. For each R-module M, there exists two short exact sequences as follows

$$\xi: \ 0 \longrightarrow Y \longrightarrow X \xrightarrow{f} M \longrightarrow 0\ ,$$

$$\gamma: \ 0 \longrightarrow M \xrightarrow{g} Y' \longrightarrow X' \longrightarrow 0$$

with $X \in \mathcal{X}_T$, $Y \in \mathcal{Y}_T$ and $X' \in \mathcal{X}_T$, $Y' \in \mathcal{Y}_T$. It follows from [10, Definition 7.1.6] that the sequence ξ is a special right $\mathcal{X}_T$-approximation, and γ a special left $\mathcal{Y}_T$-approximation. In particular, we have that f is a right $\mathcal{X}_T$-approximation and g is a left $\mathcal{Y}_T$-approximation. Thus $\mathcal{X}_T$ is contravariantly finite and $\mathcal{Y}_T$ is covariantly finite.

About (BP2): For each R-module M, we let

(3.1)
$$\begin{tikzcd}
\cdots \arrow[r] & X_n \arrow[r, "\varepsilon_n"] \arrow[d] & \cdots \arrow[r] & X_1 \arrow[r, "\varepsilon_1"] \arrow[d] & X_0 \arrow[r, "\varepsilon_0"] & M \arrow[r] & 0 \\
& \mathrm{Ker}(\epsilon_{n-1}) = K_{n-1} \arrow[ur, hook] & \cdots & \mathrm{Ker}(\epsilon_0) = K_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

be an $\mathcal{X}_T$-resolution of M. The fact that $(\mathcal{X}_T, \mathcal{X}_T \cap \mathcal{Y}_T)$ is complete implies that there is a short exact sequence as follows

$$\delta_0: \ 0 \longrightarrow K_0 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_0 \in \mathcal{X}_T$ and $K_0 \in \mathcal{X}_T \cap \mathcal{Y}_T$. Now we take a short exact sequence

$$0 \longrightarrow K_0 \longrightarrow I \longrightarrow Y_1 \longrightarrow 0$$

with injective envelope I. The fact that the pair $(\mathcal{X}_T, \mathcal{X}_T \cap \mathcal{Y}_T)$ is hereditary implies that $\mathcal{X}_T \cap \mathcal{Y}_T$ is coresolving. Note that $K_0, I \in \mathcal{X}_T \cap \mathcal{Y}_T$, we have that $Y_1 \in \mathcal{X}_T \cap \mathcal{Y}_T$. Observe the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, equal] & X_0 \arrow[r] \arrow[d, dotted, "\alpha"'] & M \arrow[r] \arrow[d, dotted, "\beta"'] & 0 \\
0 \arrow[r] & K_0 \arrow[r] & I \arrow[r] & Y_1 \arrow[r] & 0,
\end{tikzcd}$$

where the existence of the map α is from the injectivity of I and the existence of the map β is from the universal properties of the cokernel. Now applying the functor $\mathrm{Hom}_R(-, Z)$ for all $Z \in \mathcal{Y}_T$ to the above diagram, then we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(Y_1, Z) \arrow[r] \arrow[d] & \mathrm{Hom}_R(I, Z) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K_0, Z) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_R(Y_1, Z) \\
0 \arrow[r] & \mathrm{Hom}_R(M, Z) \arrow[r] & \mathrm{Hom}_R(X_0, Z) \arrow[r] & \mathrm{Hom}_R(K_0, Z). &
\end{tikzcd}$$

Since $\mathrm{Ext}^n_R(Y_1, Z) = 0$ for any $n \in \mathbb{N}$, it follows that the map $\mathrm{Hom}_R(I, Z) \to \mathrm{Hom}_R(K_0, Z)$ is surjective. We also note that the above map can be factors as:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(X_0, Z) \arrow[d] \\
\mathrm{Hom}_R(I, Z) \arrow[ur] \arrow[r, two heads] & \mathrm{Hom}_R(K_0, Z)
\end{tikzcd}$$

Therefore, the map $\mathrm{Hom}_R(X_0, Z) \to \mathrm{Hom}_R(K_0, Z)$ is surjective. Similarly, for the short exact sequences

$$\delta_n (n \geq 1) : \ 0 \longrightarrow K_n \longrightarrow X_n \longrightarrow K_{n-1} \longrightarrow 0$$

we can deduce that the maps $\mathrm{Hom}_R(X_n, Z) \to \mathrm{Hom}_R(K_n, Z)$ $(n \geq 1)$ is surjective. Therefore, the sequence (3.1) is acyclic by applying the functors $\mathrm{Hom}_R(-, Z)$ for all $Z \in \mathcal{Y}_T$.

About (BP3): For each R-module N, we let

(3.2)

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\omega_0"] & Y_0 \arrow[r, "\omega_1"] \arrow[d] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_{n-1} \arrow[r, "\omega_n"] \arrow[d] & Y_n \arrow[r] & \cdots \\
 & & K^0 \arrow[ur, hook] & & \cdots & K^{n-1} \arrow[ur, hook] & &
\end{tikzcd}$$

be a $\mathcal{Y}_T$-coresolution of N. The fact that $(\mathcal{X}_T \cap \mathcal{Y}_T, \mathcal{Y}_T)$ is complete implies that there exists a short exact sequence as follows

$$\delta^0 : \ 0 \longrightarrow N \longrightarrow Y_0 \longrightarrow K^0 \longrightarrow 0$$

with $Y_0 \in \mathcal{Y}_T$ and $K^0 \in \mathcal{X}_T \cap \mathcal{Y}_T$. Here we take a short exact sequence

$$0 \longrightarrow X^1 \longrightarrow P \longrightarrow K^0 \longrightarrow 0$$

with projective cover P and $X^1 = \mathrm{Ker}(P \to K^0)$. The fact that the pair $(\mathcal{X}_T \cap \mathcal{Y}_T, \mathcal{Y}_T)$ is hereditary implies that $\mathcal{X}_T \cap \mathcal{Y}_T$ is resolving. Note that $K^0, P \in \mathcal{X}_T \cap \mathcal{Y}_T$, we have that $X^1 \in \mathcal{X}_T \cap \mathcal{Y}_T$. Observe the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X^1 \arrow[r] \arrow[d, dotted, "\alpha'"'] & P \arrow[r] \arrow[d, dotted, "\beta'"'] & K^0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & Y_0 \arrow[r] & K^0 \arrow[r] & 0,
\end{tikzcd}$$

where the existence of the map β' is from the projectivity of P and the existence of the map α' is from the universal properties of the kernel. Now applying the functor $\mathrm{Hom}_R(X,-)$ for all $X \in \mathcal{X}_T$ to the above diagram, then we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(X,X^1) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,P) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,K^0) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_R(X,X^1) \\
0 \arrow[r] & \mathrm{Hom}_R(X,N) \arrow[r] & \mathrm{Hom}_R(X,Y_0) \arrow[r] & \mathrm{Hom}_R(X,K^0). &
\end{tikzcd}$$

Since $\mathrm{Ext}^n_R(X,X^1)=0$, it follows that the map $\mathrm{Hom}_R(X,P) \to \mathrm{Hom}_R(X,K^0)$ is surjective. We also note that the above map can be factors as

$$\begin{tikzcd}
 & \mathrm{Hom}_R(X,Y_0) \arrow[d] \\
\mathrm{Hom}_R(X,P) \arrow[ur] \arrow[r, two heads] & \mathrm{Hom}_R(X,K^0).
\end{tikzcd}$$

Therefore, the map $\mathrm{Hom}_R(X,Y_0) \to \mathrm{Hom}_R(X,K^0)$ is surjective. Similarly, for the short exact sequences

$$\delta^n (n \geq 1): \ 0 \longrightarrow K^{n-1} \longrightarrow Y_n \longrightarrow K^n \longrightarrow 0$$

we can deduce that the maps $\mathrm{Hom}_R(X,Y_n) \to \mathrm{Hom}_R(X,K^n)$ $(n \geq 1)$ is surjective. Therefore, the sequence (3.2) is acyclic by applying the functors $\mathrm{Hom}_R(X,-)$ for all $X \in \mathcal{X}_T$.

Next we will show that the pair $(\mathcal{X}_T,\mathcal{Y}_T)$ is admissible. By the assumption of the subcategory $\mathcal{X}_T$, it is easy to see that $\mathcal{X}_T$ contains all the projective R-modules. Thus, all right $\mathcal{X}_T$-approximations are epic, that is, the contravariantly finite subcategory $\mathcal{X}_T$ is admissible. Following from [10, Corollary 2.3], we obtain that $\mathcal{Y}_T$ is coadmissible. Consequently, the cotilting cotorsion pair $(\mathcal{X}_T,\mathcal{Y}_T)$ is an admissible balanced pair in $\mathrm{Mod}(R)$. $\square$

In order to show our application in this subsection, we will study triangular matrix algebras. Now we recall some facts on triangular matrix algebras. Let $\Lambda = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ be an upper triangular matrix algebra, where A, B are K-algebras and M is an A-B-bimodule. Given a right A-module X, a right B-module Y, and a right B-module morphism $f: X \otimes_A M \to Y$, we define the right Λ-module structure on (X,Y,f) by the following identity

$$(x,y) \circ \left[\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right] = (xa, f(x \otimes m) + yb).$$

The reader is referred to [4, III.2, Proposition 2.1] and [12] for more details.

Lemma 3.2 ([12, Proposition 2.7], [16, Example 2.7], [17, Example 2.10])**.** *Let* $\Lambda = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ *be the triangular matrix algebra defined above. Then there exists a recollement as follows*

$$\begin{tikzcd}
\mathrm{Mod}(A) \arrow[r, "i=inc"] & \mathrm{Mod}(\Lambda) \arrow[l, bend right, "q=A\otimes_\Lambda -"'] \arrow[l, bend left, "p=\mathrm{Hom}_\Lambda(A,-)"] \arrow[r, "e=\mathrm{Hom}_\Lambda(B,-)"] & \mathrm{Mod}(B) \arrow[l, bend right, "l=B\otimes_B -"'] \arrow[l, bend left, "r=\mathrm{Hom}_B(B,-)"]
\end{tikzcd} . \tag{3.3}$$

Lemma 3.3. *Let* $(\mathrm{Mod}(A), \mathrm{Mod}(\Lambda), \mathrm{Mod}(B))$ *be a recollement of module categories with a cotilting object* T *in* $\mathrm{Mod}(\Lambda)$*. Then we have*

- (i) $e(T)$ *is cotilting in* $\mathrm{Mod}(B)$;
- (ii) $p(T)$ *is cotilting in* $\mathrm{Mod}(A)$;
- (iii) $\mathcal{X}_{e(T)} = e(\mathcal{X}_T)$;
- (iv) $\mathcal{Y}_{e(T)} = e(\mathcal{Y}_T)$;
- (v) $\mathcal{X}_{q(T)} = q(\mathcal{X}_T)$;
- (vi) $\mathcal{Y}_{q(T)} = q(\mathcal{Y}_T)$.

Proof. We assume that $T = (X, Y, \varphi)$ is the cotilting right Λ-module. Then it is easy to see that there are three cases for $T = (X, Y, \varphi)$.

$$T = \begin{cases} T_1 = (T_A, 0, 0), \text{where } T_A \text{ is a cotilting right } A\text{-module;} \\ T_2 = (0, T_B, 0), \text{where } T_B \text{ is a cotilting right } B\text{-module;} \\ T_3 = (T_A, T \otimes_A M, 1_{T\otimes_A M}), \text{where } T_A \text{ is a cotilting right } A\text{-module.} \end{cases}$$

Following from [12, Proposition 2.4], we can deduce that

$$e(T) = \begin{cases} T_B, \text{when } T = T_2; \\ T \otimes_A M, \text{when } T = T_3, \end{cases}$$

and $p(T) = T_A$. Clearly, the assertion that $p(T)$ is cotilting holds naturally. And the assertion $e(T)$ also holds for the case $T = T_2$. Now we will check that $e(T) = T \otimes_A M$ satisfies the definition of the cotilting.

About (CT1): Since the injective dimension of T is finite, we assume that $\mathrm{inj.dim}(T_\Lambda) = n < \infty$. There exists an injective coresolution of T_Λ as follows

$$0 \longrightarrow T_\Lambda \longrightarrow I^0_\Lambda \longrightarrow I^1_\Lambda \longrightarrow \cdots \longrightarrow I^{n-1}_\Lambda \longrightarrow I^n_\Lambda \longrightarrow 0 .$$

After applying the exact functor e, we get an injective coresolution of $e(T)$

$$0 \longrightarrow e(T_\Lambda) \longrightarrow e(I^0_\Lambda) \longrightarrow e(I^1_\Lambda) \longrightarrow \cdots \longrightarrow e(I^{n-1}_\Lambda) \longrightarrow e(I^n_\Lambda) \longrightarrow 0 .$$

Indeed, we note that the functor $l = B \otimes_B -$ is exact, we have that the functor e preserves injective objects. Namely, $e(I^i_\Lambda)(i = 0, 1, \ldots, n)$ are injective modules in $\mathrm{Mod}(B)$. So $\mathrm{inj.dim}(e(T)) < \infty$.

About (CT3): For any $M \in^\perp (eT)$, there exists an object L in $\mathrm{Mod}(\Lambda)$ such that $e(L) = M$. Since $\mathrm{Ext}_B^n(e(L), e(T)) = 0$, it follows that $\mathrm{Ext}_\Lambda^n(L, T) = 0$. So $L \in^\perp T$. There exists a short exact sequence

$$0 \longrightarrow L \longrightarrow T' \longrightarrow X' \longrightarrow 0$$

with $T' \in \mathrm{add}(T)$ and $X' \in^\perp T$. Applying the functor e we obtain that the following exact sequence

$$0 \longrightarrow e(L) \longrightarrow e(T') \longrightarrow e(X') \longrightarrow 0$$

with $e(T') \in \mathrm{add}e(T)$ and $e(X') \in^\perp e(T)$ since e preserves direct sums and it is exact.

The fact that $\mathrm{Ext}_\Lambda^n((X, Y, \alpha), (X', Y', \alpha')) = 0$ if and only if $\mathrm{Ext}_A^n(p(X, Y, \alpha), p(X', Y', \alpha')) = \mathrm{Ext}_A^n(X, X') = 0$ and

$$\mathrm{Ext}_B^n(e(X, Y, \alpha), e(X', Y', \alpha')) = \mathrm{Ext}_B^n(Y, Y') = 0$$

for any right Λ-modules (X, Y, α) and (X', Y', α') implies (CT2) and the assertions (iii-vi). ☐

The following result is a direct consequence of Theorem 2.5 and Proposition 3.1.

Corollary 3.4. *Let* $(\mathrm{Mod}(A), \mathrm{Mod}(\Lambda), \mathrm{Mod}(B))$ *be a recollement of module categories with a cotilting object* T *in* $\mathrm{Mod}(\Lambda)$. *Then we have*

(i) *the pair* $(e(\mathcal{X}_T), e(\mathcal{Y}_T))$ *is an admissible balanced pair*;

(ii) *the pair* $(q(\mathcal{X}_T), p(\mathcal{Y}_T))$ *is an admissible balanced pair*;

(iii) $$\mathcal{X}_T\text{-res.dim}(\mathrm{Mod}(\Lambda)) \geq \max\{e(\mathcal{X}_T)\text{-res.dim}(\mathrm{Mod}(B)), q(\mathcal{X}_T)\text{-res.dim}(\mathrm{Mod}(A))\};$$

(iv) $$\mathcal{Y}_T\text{-cores.dim}(\mathrm{Mod}(\Lambda)) \geq \max\{e(\mathcal{Y}_T)\text{-cores.dim}(\mathrm{Mod}(B)), p(\mathcal{Y}_T)\text{-cores.dim}(\mathrm{Mod}(A))\}.$$

Acknowledgements. The authors would like to thank the referee for his/her useful suggestions.

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