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BLOW-UP AND GLOBAL SOLUTIONS FOR SOME PARABOLIC SYSTEMS UNDER NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, blows-up and global solutions for a class of nonlinear divergence form parabolic equations with the abstract form of $(\varrho(u))_t$ and time dependent coefficients are considered. The conditions are established for the existence of a solution globally and also the conditions are established for the blow up of the solution at some finite time. Moreover, the lower bound and upper bound of the blow-up time are derived if blow-up occurs.

1. Introduction

In this paper we consider the following parabolic system subject to initial and boundary value conditions:

$$\begin{cases} (\varrho(u))_t = \sum_{i,j=1}^{N} (a^{ij}(x)u_{x_i})_{x_j} + \mu(t)h(u), \ \Omega \times (0,t^\star), \\ \sum_{i,j=1}^{N} a^{ij}(x)u_{x_i}n_j = p(u), \ (x,t) \in \partial\Omega \times (0,t^\star), \\ u(x,0) = u_0(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$, $\mu(t)$ is a non-negative function, ϱ is a $C^2(\mathbb{R}^+)(\mathbb{R}^+ = [0,+\infty))$ function with $\varrho'(s) > 0$ for all $s > 0$, $h \in C(\mathbb{R})$, p is a nonnegative $C(\mathbb{R}^+)$ function, μ is a $C^1(\mathbb{R}^+,\mathbb{R}^+)$ function, u_0 is a nonnegative $C^1(\overline{\Omega})$ function, $\frac{\partial}{\partial n}$ is the normal derivative directed outward on $\partial\Omega$, $t^\star$ is the blow-up time if blow-up occurs, or else $t^\star = +\infty$, and $(a^{ij}(x))_{N\times N}$ is a differentiable positive definite matrix.

Mathematical investigations of the phenomenon of blow-up of solutions in initial-boundary-value problems of partial differential equations have received

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much attention in the literature. Many results have been achieved on the bounds for blow-up time in nonlinear parabolic problems. Payne and many other mathematical researchers have done a lot of work on the blow-up solutions of parabolic problems and the author obtained a large number of outstanding achievements, and for more details we refer the reader to [1,3,6,7,9,11–21] and the references therein.

In [8], Payne and Philippin studied the blow-up solution of the following equation

$$\begin{cases} u_t = \Delta u + k(t)f(u), \ \Omega \times (0, t^\star), \\ \dfrac{\partial u(x,t)}{\partial n} = au, \ (x,t) \in \partial\Omega \times (x, t^*), \\ u(x,t) = v(x,t) = 0, \ x \in \Omega, \end{cases}$$

where Δ is the Laplace operator, Ω is a bounded star-shaped region of $\mathbb{R}^3$ with boundary $\partial\Omega$, $k(t)$ is a non-negative function, a is an arbitrary constant, and $\frac{\partial}{\partial n}$ is the normal derivative directed outward on $\partial\Omega$.

In [5], Li and Li dealt with the initial-boundary value problem

$$\begin{cases} u_t = \displaystyle\sum_{i,j=1}^{N} (a^{ij}(x)u_{x_i})_{x_j} + f(u), \ \Omega \times (0, t^\star), \\ \displaystyle\sum_{i,j=1}^{N} a^{ij}(x)u_{x_i} n_j = g(u), \ (x,t) \in \partial\Omega \times (0, t^\star), \\ u(x,0) = u_0(x), \ x \in \Omega, \end{cases}$$

where Ω is a bounded star-shaped region of $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, n is the unit outward normal on $\partial\Omega$, $t^\star$ is the blow-up time if blow-up occurs, or else $t^\star = +\infty$, and $(a^{ij}(x))_{N\times N}$ is a differentiable definite matrix.

In [2], Ding and Hu studied the following reaction diffusion equations under Dirichlet boundary condition

$$\begin{cases} (g(u))_t = \nabla \cdot (\rho(|\nabla u|^2)\nabla u) + k(t)f(u), \ \Omega \times (0, t^\star), \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times (0, t^\star), \\ u(x,0) = u_0(x) \geq 0, \ x \in \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$. By construction of some appropriate auxiliary functions and using a first-order differential inequality technique, the authors established the upper and lower bounds for the blow-up time when blow-up occurs. Moreover, the authors also established the condition to ensure that the solution exists globally.

Remark 1.1. Our system of equations is the generalization of the systems of equations in [4,8]. If

$$a^{ij}(x) = \delta^{ij} = \begin{cases} 1, \ i = j, \\ 0, \ i \neq j, \end{cases}$$

and $p(u) = au$, $\varrho(u) = u$, then system (1.1) reduces to the corresponding system in [8]; if $\varrho(u) = u$, then system (1.1) reduces to the corresponding system in [4]. Compared with [5], the nonlinear term in our equation has time-dependent coefficients and the abstract form of $(\varrho(u))_t$ is contained in the parabolic systems. The conditions are established respectively on the nonlinearities to get the lower and upper bound of the blow-up solution or the global solution for the nonlinear divergence form parabolic equation with time dependent coefficients in the nonlinear boundary conditions.

2. The global existence of solutions

In this section, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded star-sharped region with smooth boundary $\partial\Omega$, and then the conditions are established on the nonlinearities to guarantee that $u(x,t)$ exists globally.

Just like the same way the authors introduced functions Φ and G in reference [2], for any fixed u, we introduce functions ϕ_1 and $\Re$ as follows:

$$\phi_1(t) = \int_\Omega \Re(u(x,t))dx, t \in (0,t^\star), \quad \Re(s) = 2\int_0^s y\varrho'(y)dy. \tag{2.1}$$

In order to get the main results, we first give the following lemma.

Lemma 2.1 ([5]). *Let Ω be a bounded star-sharped region in $\mathbb{R}^N$, $N \geq 2$. Then for any nonnegative C^1 function w in $\overline{\Omega}$ and real number $r > 0$, we have*

$$\int_{\partial\Omega} w^r dS \leq \frac{N}{\rho_0}\int_\Omega w^r dx + \frac{rd}{\rho_0}\int_\Omega w^{r-1}|\nabla w|dx,$$

where

$$\rho_0 = \min_{x\in\partial\Omega}(x\cdot n), \ \ d = \max_{x\in\Omega}|x|.$$

Theorem 2.1. *Assume that functions $h \in C(\mathbb{R}^+, (-\infty, 0])$ and $p \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy*

$$\begin{aligned} &h(x) \leq -k_1 x^{\overline{q}}, \ \ p(x) \leq k_2 x^q, \ \ x \in \mathbb{R}^+, \\ &\varrho''(s) \geq 0, \ \ s \in \mathbb{R}^+, \end{aligned} \tag{2.2}$$

where $k_1, k_2 \geq 0$, $\overline{q} > q > 1$, $2q < \overline{q} + 1$. Then the nonnegative solution $u(x,t)$ of problem (1.1) *does not blow up, that is $u(x,t)$ exists for all time $t > 0$.*

Proof. Since $(a^{ij}(x))_{N\times N}$ is a positive definite matrix, there exists a real number $\theta_1 > 0$ such that for all $\eta \in \mathbb{R}^n, x \in \Omega$,

$$\sum_{i,j=1}^N a^{ij}(x)\eta_i\eta_j \geq \theta_1|\eta|^2. \tag{2.3}$$

Differentiating (2.1) and using the divergence theorem, combining (1.1), for all $t \in (0,t^\star)$, we have

$$\phi_1'(t) = 2\int_\Omega u\varrho'(u)u_t dx \tag{2.4}$$

$$= 2\int_\Omega u\left(\sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} + \mu(t)h(u)\right)dx$$
$$= 2\int_\Omega u\sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j}dx + 2\mu(t)\int_\Omega uh(u)dx.$$
$$= 2\int_{\partial\Omega} u\sum_{i,j=1}^N a^{ij}(x)u_{x_i}n_j dS - 2\int_\Omega \sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx$$
$$+ 2\mu(t)\int_\Omega uh(u)dx.$$
$$\leq 2k_2\int_{\partial\Omega} u^{q+1}dS - 2\theta_1\int_\Omega |\nabla u|^2dx - 2\mu(t)\int_\Omega k_1u^{\overline{q}+1}dx.$$

By Lemma 2.1, we have

$$\int_{\partial\Omega} u^{q+1}dS \leq \frac{N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx. \tag{2.5}$$

Substituting (2.5) into (2.4), for all $t \in (0, t^\star)$, we have

$$\begin{aligned}\phi_1'(t) \leq{}& \frac{2k_2N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{2k_2(q+1)d}{\rho_0}\int_\Omega u^q|\nabla u|dx\\ &- 2\theta_1\int_\Omega |\nabla u|^2dx - 2k_1\mu(t)\int_\Omega u^{\overline{q}+1}dx.\end{aligned} \tag{2.6}$$

By Cauchy inequality, we have

$$\int_\Omega u^q|\nabla u|dx \leq \frac{a_1}{2}\int_\Omega u^{2q}dx + \frac{1}{2a_1}\int_\Omega |\nabla u|^2dx. \tag{2.7}$$

Choosing $a_1 = \frac{k_2(q+1)d}{2\theta_1\rho_0}$ and substituting (2.7) into (2.6), for all $t \in (0, t^\star)$, we have

$$\phi_1'(t) \leq \frac{2k_2N}{\rho_0}\int_\Omega u^{q+1}dx + 2\theta_1a_1^2\int_\Omega u^{2q}dx - 2k_1\mu(t)\int_\Omega u^{\overline{q}+1}dx. \tag{2.8}$$

By $2q < \overline{q} + 1$, for all $\epsilon > 0$, we have

$$\int_\Omega u^{2q}dx \leq (1-\gamma_0)\epsilon\int_\Omega u^{\overline{q}+1}dx + \gamma_0\epsilon^{\frac{\gamma_0-1}{\gamma_0}}\int_\Omega u^{q+1}dx, \tag{2.9}$$

where $\gamma_0 = \frac{\overline{q}+1-2q}{\overline{q}-q} < 1$. Substituting (2.9) into (2.8), we have

$$\phi_1'(t) \leq M_1\int_\Omega u^{q+1} - M_2(t)\int_\Omega u^{\overline{q}+1}, t \in (0, t^\star), \tag{2.10}$$

where

$$M_1 = \frac{2k_2N}{\rho_0} + 2\theta_1a_1^2\gamma_0\epsilon^{\frac{\gamma_0-1}{\gamma_0}} > 0,\ M_2(t) = 2k_1\mu(t) - 2\theta_1a_1^2(1-\gamma_0)\epsilon > 0$$

for $\epsilon > 0$ sufficiently small. By the Hölder inequality, we have

$$\int_\Omega u^{q+1}dx \le \left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{q+1}{\overline{q}+1}} |\Omega|^{\frac{\overline{q}-q}{\overline{q}+1}}. \tag{2.11}$$

Substituting (2.11) into (2.10), for all $t \in (0, t^\star)$, we get

$$\begin{aligned}\phi_1'(t) &\le M_1\left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{q+1}{\overline{q}+1}} |\Omega|^{\frac{\overline{q}-q}{\overline{q}+1}} - M_2(t)\int_\Omega u^{\overline{q}+1}dx\\ &\le M_1\left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{q+1}{\overline{q}+1}}\left(|\Omega|^{\frac{\overline{q}-q}{\overline{q}+1}} - \frac{M_2(t)}{M_1}\left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{\overline{q}-q}{\overline{q}+1}}\right). \end{aligned}\tag{2.12}$$

Integration by parts, by (2.2), we have

$$\begin{aligned}\Re(u) &= 2\int_0^u y\varrho'(y)dy = \int_0^u \varrho'(y)dy^2\\ &= \varrho'(u)u^2 - \int_0^u \varrho''(y)y^2dy\\ &\le \varrho'(u)u^2.\end{aligned}$$

Using the Hölder inequality, for all $t \in (0, t^\star)$, we have

$$\begin{aligned}\phi_1(t) = &\int_\Omega \Re(u(x,t))dx \le \int_\Omega \varrho'(u)u^2dx\\ &\le \left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{2}{\overline{q}+1}}\left(\int_\Omega (\varrho'(u))^{\frac{\overline{q}+1}{\overline{q}-1}}dx\right)^{1-\frac{2}{\overline{q}+1}},\end{aligned}\tag{2.13}$$

and hence,

$$\left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{\overline{q}-q}{\overline{q}+1}} \ge (\phi_1(t))^{\frac{\overline{q}-q}{2}}\left(\int_\Omega (\varrho'(u))^{\frac{\overline{q}+1}{\overline{q}-1}}dx\right)^{\frac{(\overline{q}-q)(1-\overline{q})}{2(\overline{q}+1)}}. \tag{2.14}$$

Substituting (2.14) into (2.13), for all $t \in (0, t^\star)$, we get

$$\begin{aligned}\phi_1'(t) \le M_1&\left(\int_\Omega u^{\overline{q}+1}dx\right)^{\frac{q+1}{\overline{q}+1}}\\ &\left(|\Omega|^{\frac{\overline{q}-q}{\overline{q}+1}} - \frac{M_2(t)}{M_1}(\phi_1(t))^{\frac{\overline{q}-q}{2}}\left(\int_\Omega (\varrho'(u))^{\frac{\overline{q}+1}{\overline{q}-1}}dx\right)^{\frac{(\overline{q}-q)(1-\overline{q})}{2(\overline{q}+1)}}\right).\end{aligned}\tag{2.15}$$

From the inequality (2.15), we declare that $\phi_1(t)$ remains bounded for all time under the conditions in Theorem 2.1 since $M_2(t)$ is a positive $C^1(\mathbb{R}^+)$ function. In fact, if $u(x,t)$ blows up at finite time $t^\star$, then

$$\lim_{t\to t^{\star-}} M_2(t) = M(t^\star), \ \lim_{t\to t^{\star-}} \phi_1(t) = +\infty,$$

and thus,

$$\lim_{t\to t^{\star -}} \frac{M_2(t)}{M_1}(\phi_1(t))^{\frac{\overline{q}-q}{2}}\left(\int_\Omega (\varrho'(u))^{\frac{\overline{q}+1}{\overline{q}-1}}dx\right)^{\frac{(\overline{q}-q)(1-\overline{q})}{2(\overline{q}+1)}} = +\infty.$$

Therefore $\phi_1(t)$ is unbounded near $t^\star$ which forces $\phi_1'(t) \le 0$ in some interval $[t_0, t^\star)$. So we have $\phi_1(t) \le \phi_1(t_0)$ in $[t_0, t^\star)$, which implies that $\phi_1(t)$ is bounded in $[t_0, t^\star)$, which is a contradiction. The proof of Theorem 2.1 is completed. □

3. Upper bound estimation of $t^\star$ for blow-up time

In this section, we do not need the assumption that $\Omega \subset \mathbb{R}^N$ is a star-sharped domain, and we will establish the conditions on the nonlinearities to get the upper bound of the blow-up solution.

Theorem 3.1. *Let $u(x,t)$ be the nonnegative solution of problem* (1.1)*, and assume that $\mu \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $p \in C(\mathbb{R}^+, \mathbb{R}^+)$ and the nonnegative and integrable function $\varrho \in C^2(\mathbb{R}^+, \mathbb{R})$ satisfy conditions*

$$\begin{aligned} &xh(x) \ge 2(1+\alpha)H(x) > 0,\ xp(x) \ge 2(1+\beta)P(x),\ x \in \mathbb{R}^+, \\ &\mu'(t) \ge 0,\ t \in (0, t^\star),\ \mu(0) > 0, \\ &\varrho''(s) \le 0,\ s \in \mathbb{R}^+, \end{aligned} \tag{3.1}$$

where $0 \le \alpha \le \beta$, and

$$H(x) = \int_0^x h(s)ds,\ P(x) = \int_0^x p(s)ds.$$

Moreover, assume that $\Lambda(0) > 0$ with

$$\Lambda(t) = 2\int_{\partial\Omega} P(u)dS - \int_\Omega \sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx + 2\mu(t)\int_\Omega F(u)dx. \tag{3.2}$$

Then $u(x,t)$ blows up at time $t^\star < T$ with

$$T = \frac{1}{2\alpha(\alpha+1)A}(\phi_1(0))^{-\alpha},$$

where $\phi_1(t)$ is defined by (2.1) *and $A = \Lambda(0)(\phi_1(0))^{-(1+\alpha)}$ is a constant.*

Proof. Differentiating (2.1), and by the Green formula, for all $t \in (0, t^\star)$, following the computations in (2.4), we have

$$\begin{aligned} \phi_1'(t) &\ge 4(1+\beta)\int_{\partial\Omega} P(u)dS - 2\int_\Omega \sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx \\ &\quad + 4\mu(t)(1+\alpha)\int_\Omega H(u)dx \\ &\ge 2\left[2(1+\beta)\int_{\partial\Omega} P(u)dS - \int_\Omega \sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx\right. \end{aligned} \tag{3.3}$$

$$+2\mu(t)(1+\alpha)\int_\Omega H(u)dx\Big]$$
$$\geq 2(1+\alpha)\Lambda(t),$$

where $\Lambda(t)$ is as (3.2). Differentiating $\Lambda(t)$, for all $t \in (0, t^\star)$, we have

$$\begin{aligned}
\Lambda'(t) &= 2\int_{\partial\Omega} p(u)u_t dS - \int_\Omega \left(\sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}\right)_t dx + 2\mu'(t)\int_\Omega H(u)dx \\
&\quad + 2\mu(t)\int_\Omega h(u)u_t dx \\
&= 2\int_{\partial\Omega} p(u)u_t dS + 2\int_\Omega u_t \sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} dx - 2\int_{\partial\Omega} p(u)u_t dS \\
&\quad + 2\mu'(t)\int_\Omega H(u)dx + 2\mu(t)\int_\Omega h(u)u_t dx \\
&\geq 2\int_\Omega u_t \sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} dx + 2\mu(t)\int_\Omega h(u)u_t dx \\
&= 2\int_\Omega u_t^2 \varrho'(u) dx > 0,
\end{aligned} \tag{3.4}$$

which with $\Lambda(0) > 0$ implies $\Lambda(t) > 0$ for all $t \in (0, t^\star)$. Using (3.3), (3.4) and Hölder inequality, we have

$$\begin{aligned}
2\int_\Omega \varrho'(u)u_t^2 dx \int_\Omega \varrho'(u)u^2 dx &\geq 2\left(\int_\Omega u\varrho'(u)u_t dx\right)^2 \\
&= \frac{1}{2}(\phi_1(t))^2 \geq (1+\alpha)\Lambda(t)\phi_1'(t),\ t \in (0, t^\star).
\end{aligned} \tag{3.5}$$

Using integration by parts, we have

$$\begin{aligned}
\Re(u) &= 2\int_0^u y\varrho'(y)dy = \int_0^u \varrho'(y)dy^2 \\
&= \varrho'(u)u^2 - \int_0^u \varrho''(y)y^2 dy \\
&\geq \varrho'(u)u^2.
\end{aligned} \tag{3.6}$$

Hence, by (3.4)-(3.6), for all $t \in (0, t^\star)$, we have

$$(1+\alpha)\Lambda(t)\phi_1'(t) \leq 2\int_\Omega \varrho'(u)u_t^2 dx \int_\Omega \Re(u)dx \leq \Lambda'(t)\phi_1(t), \tag{3.7}$$

i.e.,

$$\left(\Lambda(t)\phi_1^{-(1+\alpha)}\right)' \geq 0. \tag{3.8}$$

Integrating (3.8) and making use of (3.3), we have

$$\begin{aligned}\frac{1}{2(1+\alpha)}\phi_1'(t)(\phi_1(t))^{-(1+\alpha)} &\geq \Lambda(t)(\phi_1(t))^{-(1+\alpha)}\\ &\geq \Lambda(0)(\phi_1(0))^{-(1+\alpha)} = A.\end{aligned} \tag{3.9}$$

Integrating (3.9) form 0 to t, we have

$$(\phi_1(t))^{-\alpha} \leq (\phi_1(0))^{-\alpha} - 2\alpha(\alpha+1)At.$$

Clearly, this inequality cannot hold for all time, which implies

$$\phi_1(t) \to +\infty \text{ as } t \to T = \frac{1}{2\alpha(\alpha+1)A}\left(\phi_1(0)\right)^{-\alpha},$$

so we conclude that

$$t^\star \leq \frac{1}{2\alpha(\alpha+1)A}\left(\phi_1(0)\right)^{-\alpha}.$$

The proof of Theorem 3.1 is completed. □

Remark 3.1. The nonnegative solution $u(x,t)$ of problem (1.1) will blow-up when $\alpha > 0$, but the solution will exists globally when $\alpha = 0$.

4. Blow-up and lower bound estimation of $t^\star$

In this section, let Ω be a bounded domain in $\mathbb{R}^3$ and also be star-shaped and convex in two orthogonal directions, we will establish the conditions on the nonlinearities to get the lower bound of the blow-up solution.

Theorem 4.1. *Assume that nonnegative functions $h \in C(\mathbb{R}^+, \mathbb{R}^+)$, $p \in (\mathbb{R}^+, \mathbb{R}^+)$, $\mu \in C^1((0,t^\star), \mathbb{R})$ and $\varrho \in C^2(\mathbb{R}^+, \mathbb{R})$ satisfy*

$$\begin{aligned}&h(x) \leq x^2,\ p(x) \leq ax,\ x \in \mathbb{R}^+,\\ &\frac{\mu'(t)}{\mu(t)} \leq b,\ t \in (0,t^\star),\\ &\varrho'(s) \geq m,\ s \in (0,+\infty),\end{aligned} \tag{4.1}$$

where $a > 0, 0 \leq b < \infty, m > 0$. And for fixed u, we define

$$\phi_2(t) = \mu^2(t)\int_\Omega \Re(u(x,t))dx, \tag{4.2}$$

where $\Re$ is defined by (2.1)*. Then the solution $u(x,t)$ blows up at $t^\star$, and*

$$\int_{\varphi_2(0)}^{\varphi_2(t)} \frac{d\eta}{c_1\eta + c_2\eta^{\frac{3}{2}} + c_3\eta^3} \leq t^\star,$$

where c_1, c_2 are as in (3.15)*,* (3.16)*.*

Proof. Since $(a^{ij}(x))_{3\times 3}$ is a positive definite matrix, there exists a real number $\theta > 0$ such that for all $\eta \in \mathbb{R}^n$, $x \in \Omega$,

$$\sum_{i,j=1}^{3} a^{ij}(x)\eta_i\eta_j \geq \theta|\eta|^2. \tag{4.3}$$

Differentiating (4.2), and using (4.1), for all $t \in (0, t^\star)$, we have

$$\begin{aligned}
\phi_2'(t) &= 2\mu(t)\mu'(t)\int_\Omega \Re(u)dx + \mu^2(t)\int_\Omega \varrho'(u)uu_t dx \\
&= 2\frac{\mu'(t)}{\mu(t)}\phi_2(t) + \mu^2(t)\int_\Omega u\left(\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})_{x_j} + \mu(t)h(u)\right)dx \\
&\leq 2b\phi_2(t) + \mu^2(t)\int_\Omega u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})_{x_j}dx + \mu^3(t)\int_\Omega uh(u)dx \qquad (4.4)\\
&\leq 2b\phi_2(t) + \mu^2(t)\int_\Omega u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})_{x_j}dx + \mu^3(t)\int_\Omega u^3dx.
\end{aligned}$$

Using the Green formula and (4.1), (4.2), we get

$$\begin{aligned}
\int_\Omega u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})_{x_j}dx &= \int_{\partial\Omega} u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})n_j dS \\
&\quad - \int_\Omega \sum_{i,j=1}^{3} a^{ij}(x)u_{x_i}u_{x_j}dx \\
&= \int_{\partial\Omega} ug(u)dS - \int_\Omega \sum_{i,j=1}^{3} a^{ij}(x)u_{x_i}u_{x_j}dx \\
&\leq a\int_{\partial\Omega} u^2dS - \int_\Omega \sum_{i,j=1}^{3} a^{ij}(x)u_{x_i}u_{x_j}dx \qquad (4.5)\\
&\leq a\int_{\partial\Omega} u^2dS - \theta\int_\Omega |\nabla u|^2dx.
\end{aligned}$$

By Lemma 2.1, we have

$$\int_{\partial\Omega} u^2dS \leq \frac{3}{\rho_0}\int_\Omega u^2dx + \frac{2d}{\rho_0}\int_\Omega u|\nabla u|dx. \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\begin{aligned}
\int_\Omega u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})_{x_j}dx &\leq \int_{\partial\Omega} u\sum_{i,j=1}^{3}(a^{ij}(x)u_{x_i})n_j dS - \theta\int_\Omega |\nabla u|^2dx \\
&\leq a\int_{\partial\Omega} u^2dS - \theta\int_\Omega |\nabla u|^2dx \qquad (4.7)
\end{aligned}$$

$$\leq \frac{3a}{\rho_0}\int_\Omega u^2dx + \frac{2da}{\rho_0}\int_\Omega u|\nabla u|dx - \theta\int_\Omega |\nabla u|^2dx.$$

Substituting (4.7) into (4.4) and using Cauchy inequality, for all $t \in (0, t^\star)$, we have

$$\begin{aligned}\phi_2'(t) &\leq 2b\phi_2(t) + \frac{3a}{\rho_0}\mu^2(t)\int_\Omega u^2dx + \frac{2da}{\rho_0}\mu^2(t)\int_\Omega u|\nabla u|dx\\ &\quad - \mu^2(t)\theta\int_\Omega |\nabla u|^2dx + \mu^3(t)\int_\Omega u^3dx.\\ &\leq 2b\phi_2(t) + \frac{3a}{\rho_0}\mu^2(t)\int_\Omega u^2dx + \frac{a_1da}{\rho_0}\mu^2(t)\int_\Omega u^2dx \qquad (4.8)\\ &\quad + \frac{da}{\rho_0a_1}\mu^2(t)\int_\Omega |\nabla u|^2dx - \mu^2(t)\theta\int_\Omega |\nabla u|^2dx + \mu^3(t)\int_\Omega u^3dx,\end{aligned}$$

where a_1 is to be determined later. By (2.16) of [10], we get a upper bound of $\int_\Omega u^3dx$, that is

$$\int_\Omega u^3dx \leq 3^{-\frac{3}{4}}\left\{\frac{3}{\rho_0}\int_\Omega u^2dx + \left(\frac{d}{\rho}+1\right)\left(\int_\Omega u^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2dx\right)\right\}^{\frac{3}{2}}. \tag{4.9}$$

Making use of the inequalities

$$(a+b)^{\frac{3}{2}} \leq 2^{\frac{1}{2}}(a^{\frac{3}{2}} + b^{\frac{3}{2}}),\ a>0, b>0,$$
$$a^{r_1}b^{r_2} \leq r_1a + r_2b,\ r_1 + r_2 = 1,$$

we have

$$\int_\Omega u^3dx \leq 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left[\left(\frac{3}{\rho_0}\int_\Omega u^2dx\right)^{\frac{3}{2}} + \left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}\left(\int_\Omega u^2dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla u|^2dx\right)^{\frac{3}{4}}\right]. \tag{4.10}$$

By (4.2), we have

$$\Re(u) = \int_0^u \varrho'(y)ydy \geq m\int_0^u ydy = \frac{m}{2}u^2. \tag{4.11}$$

Hence, by (4.2), (4.10), (4.11), for all $t \in (0, t^\star)$, we have

$$\begin{aligned}&\mu^3(t)\int_\Omega u^3dx \qquad (4.12)\\ &\leq 3^{-\frac{3}{4}}2^{\frac{1}{2}}\mu^3(t)\left(\frac{3}{\rho_0}\int_\Omega u^2dx\right)^{\frac{3}{2}}\\ &\quad + 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}\mu^3(t)\left(\int_\Omega u^2dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla u|^2dx\right)^{\frac{3}{4}}\\ &\leq 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{3}{\rho_0}\mu^2(t)\frac{2}{m}\int_\Omega \Re(u)dx\right)^{\frac{3}{2}}\\ &\quad + 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}\left(\int_\Omega \mu^2(t)\frac{2}{m}\Re(u)dx\right)^{\frac{3}{4}}\left(\int_\Omega \mu^2(t)|\nabla u|^2dx\right)^{\frac{3}{4}}\end{aligned}$$

$$
\begin{aligned}
&\leq 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{6}{\rho_0 m}\right)^{\frac{3}{2}}(\phi_2(t))^{\frac{3}{2}} \\
&\quad + 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}\left(\frac{2}{m}\right)^{\frac{3}{4}}(\varphi_2^3(t))^{\frac{1}{4}}\left(\int_\Omega \mu^2(t)|\nabla u|^2dx\right)^{\frac{3}{4}} \\
&\leq 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{6}{\rho_0 m}\right)^{\frac{3}{2}}(\phi_2(t))^{\frac{3}{2}} \\
&\quad + 3^{-\frac{3}{4}}2^{\frac{1}{2}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}\left(\frac{2}{m}\right)^{\frac{3}{4}}\left(\frac{1}{4a_2^3}(\varphi_2(t))^3+\frac{3a_2}{4}\mu^2(t)\int_\Omega|\nabla u|^2dx\right).
\end{aligned}
$$

Substituting (4.11), (4.12) into (4.8), for all $t \in (0, t^\star)$, we have

$$
\begin{aligned}
\phi_2'(t) \leq\ & 2b\phi_2(t) + \left(\frac{3a}{\rho_0}+\frac{aa_1d}{\rho_0}\right)\mu^2(t)\int_\Omega \frac{2}{m}\Re(u)dx \\
& + \left(\frac{ad}{\rho_0 a_1} - \theta + \frac{3^{\frac{1}{4}}2^{-\frac{3}{4}}(\frac{d}{\rho_0}+1)^{\frac{3}{2}}a_2}{m^{\frac{3}{4}}}\right)\mu^2(t)\int_\Omega|\nabla u|^2dx \\
& + \frac{4\cdot 3^{\frac{3}{4}}}{(\rho_0 m)^{\frac{3}{2}}}\phi_2^{\frac{3}{2}}(t) + \frac{3^{-\frac{3}{4}}2^{\frac{13}{4}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}}{a_2^3}\phi_2^3(t).
\end{aligned}
\tag{4.13}
$$

Let $a_1 = \frac{2ad}{\rho_0\theta}$, $a_2 = \frac{2\theta m^{\frac{3}{4}}}{3^{\frac{1}{4}}2^{\frac{5}{4}}(\frac{d}{\rho_0}+1)^{\frac{3}{2}}}$, then

$$
\phi_2'(t) \leq c_1\phi_2(t) + c_2(\phi_2(t))^{\frac{3}{2}} + c_3(\phi_2(t))^3, \tag{4.14}
$$

where

$$
c_1 = 2b + \frac{6a}{\rho_0 m} + \frac{2a_1ad}{m\rho_0}, c_2 = \frac{4\cdot 3^{\frac{3}{4}}}{(\rho_0 m)^{\frac{3}{2}}}, c_3 = \frac{3^{-\frac{3}{4}}2^{\frac{13}{4}}\left(\frac{d}{\rho_0}+1\right)^{\frac{3}{2}}}{a_2^3}. \tag{4.15}
$$

Integrating (4.14) from 0 to t, we have

$$
\int_0^t \frac{d\phi_2(t)}{c_1\phi_2(t)+c_2\phi_2^{\frac{3}{2}}(t)+c_3\phi_2^3(t)} \leq t^\star,
$$

i.e.

$$
\int_{\phi_2(0)}^{\phi_2(t)} \frac{d\eta}{c_1\eta + c_2\eta^{\frac{3}{2}} + c_3\eta^3} \leq t^\star. \tag{4.16}
$$

The proof of Theorem 4.1 is completed. □

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