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DIOPHANTINE INEQUALITY WITH FOUR SQUARES AND ONE kTH POWER OF PRIMES

Li Zhu

Abstract. Let k be an integer with $k \geq 3$. Define $h(k) = \left[\frac{k+1}{2}\right]$, $\sigma(k) = \min\left(2^{h(k)-1}, \frac{1}{2}h(k)(h(k)+1)\right)$. Suppose that $\lambda_1, \ldots, \lambda_5$ are non-zero real numbers, not all of the same sign, satisfying that $\frac{\lambda_1}{\lambda_2}$ is irrational. Then for any given real number η and $\varepsilon > 0$, the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^k + \eta| < (\max_{1\leq j\leq 5} p_j)^{-\frac{3}{20\sigma(k)}+\varepsilon}$$

has infinitely many solutions in prime variables $p_1, \ldots, p_5$. This gives an improvement of the recent results.

1. Introduction

In 1938, Hua [5] proved that for sufficiently large integers $N \equiv 5 \pmod{24}$, the equation

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 \tag{1.1}$$

is solvable in primes $p_1, \ldots, p_5$. In 2004, Harman [3] considered an analogous problem for Diophantine inequality. He showed that there are infinitely many solutions in primes p_j to the inequality

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^2 + \eta\right| < (\max_{1\leq j\leq 5} p_j)^{-\frac{1}{8}+\varepsilon}, \tag{1.2}$$

where λ_j $(1 \leq j \leq 5)$ are non-zero real numbers, not all of the same sign, with $\frac{\lambda_1}{\lambda_2}$ irrational, η real and $\varepsilon > 0$.

In [7], Li and Wang considered a transformation of (1.2). Suppose that λ_i $(1 \leq i \leq 5)$ satisfied the above conditions. They proved that there are infinitely many solutions in primes p_j to the inequality

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^k + \eta\right| < (\max_{1\leq j\leq 5} p_j)^{-\theta(k)+\varepsilon}, \tag{1.3}$$

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where $k \geq 3$ is an integer and $\theta(k) = (3k2^k)^{-1}$. By employing the techniques in Languasco and Zaccagnini [6] and Harman [3], Mu [9] improved the value of the exponent $\theta(k)$ to $\frac{1}{16}$ for $k = 3$, $\frac{5}{3k2^k}$ for $k = 4, 5$ and $\frac{40}{21k2^k}$ for $k \geq 6$. Let $[x]$ be the greatest integer not exceeding x and

$$h(k) = \left[\frac{k+1}{2}\right],\ \sigma(k) = \min\left(2^{h(k)-1},\ \frac{1}{2}h(k)(h(k)+1)\right). \tag{1.4}$$

Based on a more detailed analysis on the minor arcs, Ge and Wang [2] showed that they can replace $\theta(k)$ in (1.3) with $\frac{1}{8\sigma(k)}$.

Motivated by [12, Theorem 2], we can make a further improvement on the exponent in (1.3). The improvement derives from the use of the function $\rho(m)$ constructed in Harman and Kumchev [4]. However, in the proof of [12], the function $\rho(m)$ can only be available under the assumption that $\frac{\lambda_1}{\lambda_2}$ and $\frac{\lambda_1}{\lambda_3}$ are both irrational. In this paper, we introduced Brüdern [1]'s technique into the investigation of (1.3). By employing Brüdern's technique, we can use the function $\rho(m)$ without further assuming that $\frac{\lambda_1}{\lambda_3}$ is irrational and obtain the following sharper result.

Theorem. *Let k be an integer with $k \geq 3$. Suppose that $\lambda_1, \ldots, \lambda_5$ are non-zero real numbers, not all of the same sign with $\frac{\lambda_1}{\lambda_2}$ irrational. Then for any given real number η and $\varepsilon > 0$, the inequality*

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^k + \eta| < (\max_{1\leq j\leq 5} p_j)^{-\frac{3}{20\sigma(k)}+\varepsilon} \tag{1.5}$$

has infinitely many solutions in prime variables $p_1, \ldots, p_5$, where $\sigma(k)$ is defined by (1.4).

2. Notation and outline of the method

The proof of the Theorem is dependent on the Davenport-Heilbronn circle method (see [11, Chap. 11]). Throughout the paper, λ is a non-zero real number. The letter p, with or without subscript, is reserved for a prime number and k is a positive integer with $k \geq 3$. ε is a sufficiently small positive number and $\delta > 0$ is a small constant depending on the coefficients $\lambda_1, \ldots, \lambda_5$. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$. $[x]$ denotes the greatest integer not exceeding x. Since $\frac{\lambda_1}{\lambda_2}$ is irrational, we let $\frac{a}{q}$ be a convergent to $\frac{\lambda_1}{\lambda_2}$, with the denominator q sufficiently large. Write

$$X = q^{\frac{4}{3}},\quad h(k) = \left[\frac{k+1}{2}\right],\ \sigma(k) = \min\left(2^{h(k)-1},\ \frac{1}{2}h(k)(h(k)+1)\right),$$
$$\tau = X^{-\frac{3}{40\sigma(k)}+3\varepsilon},\qquad L = \log X.$$

We make use of the non-negative functions $\rho_1(m)$, $\rho_2(m)$, $\rho_3(m)$, $\rho_4(m)$ constructed in [4, Section 5] (or see [12, Section 6]). It follows from [12, Section 6] that

$$\rho_1(m) = \begin{cases} 1, & \text{if } m \text{ is a prime,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho_1(m)\rho_1(k) \geq \rho_1(m)\rho_3(k) - \rho_2(m)\rho_4(k), \tag{2.1}$$

$$\sum_{u<m\leq v} \rho_j(m) = C_j(v-u)L^{-1} + O(X^{\frac{1}{2}}L^{-2}) \tag{2.2}$$

for any u and v with $(\delta X)^{\frac{1}{2}} \leq u < v \leq X^{\frac{1}{2}}$ and the constants C_j (j=1,2,3,4) satisfy

$$C_1C_3 - C_2C_4 > 0, \quad C_1 = 1. \tag{2.3}$$

Let

$$I_2 = [(\delta X)^{\frac{1}{2}}, X^{\frac{1}{2}}],\ I_k = [(\delta X)^{\frac{1}{k}}, X^{\frac{1}{k}}]$$

and

$$\begin{aligned} S_i^*(\alpha) &= \sum_{m\in I_2} \rho_i(m)e(m^2\alpha), \\ S_j(\alpha) &= \sum_{p\in I_j} e(p^j\alpha)\log p, \\ T_j(\alpha) &= \int_{I_j} e(t^j\alpha)dt. \end{aligned} \tag{2.4}$$

Denote

$$K_\tau(\alpha) = \begin{cases} \left(\frac{\sin(\pi\tau\alpha)}{\pi\alpha}\right)^2, & \text{if } \alpha \neq 0, \\ \tau^2, & \text{if } \alpha = 0. \end{cases}$$

Then

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \tag{2.5}$$

$$\int_{-\infty}^{+\infty} e(xy)K_\tau(x)dx = \max(0, \tau - |y|). \tag{2.6}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, let

$$I(\tau,\eta,\mathfrak{X}) = I_1(\tau,\eta,\mathfrak{X}) - I_2(\tau,\eta,\mathfrak{X}), \tag{2.7}$$

where

$$I_1(\tau,\eta,\mathfrak{X}) = \int_{\mathfrak{X}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_1^*(\lambda_3\alpha)S_3^*(\lambda_4\alpha)S_k(\lambda_5\alpha)K_\tau(\alpha)e(\alpha\eta)d\alpha \tag{2.8}$$

and

$$I_2(\tau,\eta,\mathfrak{X}) = \int_{\mathfrak{X}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2^*(\lambda_3\alpha)S_4^*(\lambda_4\alpha)S_k(\lambda_5\alpha)K_\tau(\alpha)e(\alpha\eta)d\alpha. \tag{2.9}$$

From (2.1) and (2.6)-(2.9), we have

$$\begin{aligned}
I(\tau,\eta,\mathbb{R}) = &\sum_{\substack{m_3,m_4,p_1,p_2\in I_2\\ p_5\in I_k}} \log p_1 \log p_2 \log p_5 (\rho_1(m_3)\rho_3(m_4)-\rho_2(m_3)\rho_4(m_4))\\
&\times \int_{\mathbb{R}} e((\lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 m_3^2+\lambda_4 m_4^2+\lambda_5 p_5^k+\eta)\alpha)K_\tau(\alpha)d\alpha\\
= &\sum_{\substack{m_3,m_4,p_1,p_2\in I_2\\ p_5\in I_k}} \log p_1 \log p_2 \log p_5 (\rho_1(m_3)\rho_3(m_4)-\rho_2(m_3)\rho_4(m_4))\\
&\times \max(0,\tau-|\lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 m_3^2+\lambda_4 m_4^2+\lambda_5 p_5^k+\eta|)\\
\le &\sum_{\substack{m_3,m_4,p_1,p_2\in I_2\\ p_5\in I_k}} \log p_1 \log p_2 \log p_5 \rho_1(m_3)\rho_1(m_4)\\
&\times \max(0,\tau-|\lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 m_3^2+\lambda_4 m_4^2+\lambda_5 p_5^k+\eta|)\\
\le &\,\tau N_\tau(X)L^3, \qquad (2.10)
\end{aligned}$$

where $N_\tau(X)$ denotes the number of solutions to the inequality

$$|\lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 p_3^2+\lambda_4 p_4^2+\lambda_5 p_5^k+\eta| < \tau \qquad (2.11)$$

with $p_1,p_2,p_3,p_4 \in I_2$ and $p_5 \in I_k$. Let $\psi = X^{-\frac{1}{2k}}$, $\xi = \tau^{-2}X^{\varepsilon}$. We now divide the real line into three parts

$$\mathfrak{M} = \{\alpha : |\alpha| \le \psi\},\ \mathfrak{m} = \{\alpha : \psi < |\alpha| \le \xi\},\ \mathfrak{t} = \{\alpha : |\alpha| > \xi\}. \qquad (2.12)$$

These sets are called the major arc, the minor arcs and the trivial arcs, respectively. Thus

$$I(\tau,\eta,\mathbb{R}) = I(\tau,\eta,\mathfrak{M}) + I(\tau,\eta,\mathfrak{m}) + I(\tau,\eta,\mathfrak{t}). \qquad (2.13)$$

In Sections 3, 4 and 5, we will prove

$$\begin{aligned}
&I(\tau,\eta,\mathfrak{M}) \gg \tau^2 X^{1+\frac{1}{k}}L^{-2},\\
&|I(\tau,\eta,\mathfrak{m})| = O(\tau^2 X^{1+\frac{1}{k}-\varepsilon}),\\
&|I(\tau,\eta,\mathfrak{t})| = O(\tau^2 X^{1+\frac{1}{k}-\varepsilon}).
\end{aligned} \qquad (2.14)$$

Then we can deduce from (2.10), (2.13) and (2.14) that

$$N_\tau(X) \gg \tau X^{1+\frac{1}{k}}L^{-5}. \qquad (2.15)$$

3. The major arc $\mathfrak{M}$

In this section, we give the lower bound for $I(\tau,\eta,\mathfrak{M})$. We first subdivide the major arc into three parts:

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3, \qquad (3.1)$$

where

$$\begin{aligned}\mathfrak{M}_1 &= \{\alpha : |\alpha| \le X^{-1+\frac{5}{12k}-\varepsilon}\},\\ \mathfrak{M}_2 &= \{\alpha : X^{-1+\frac{5}{12k}-\varepsilon} < |\alpha| \le X^{-1+\frac{1}{2k}}\},\\ \mathfrak{M}_3 &= \{\alpha : X^{-1+\frac{1}{2k}} < |\alpha| \le X^{-\frac{1}{2k}}\}.\end{aligned} \tag{3.2}$$

Thus

$$I(\tau,\eta,\mathfrak{M}) = I(\tau,\eta,\mathfrak{M}_1) + I(\tau,\eta,\mathfrak{M}_2) + I(\tau,\eta,\mathfrak{M}_3). \tag{3.3}$$

For brevity, let

$$H = \int_{-\infty}^{+\infty} \prod_{1\le j\le 4} T_2(\lambda_j\alpha)T_k(\lambda_5\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha,$$

$$V_1(\alpha) = S_2^*(\lambda_3\alpha)S_4^*(\lambda_4\alpha),\ \ V_2(\alpha) = T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_k(\lambda_5\alpha).$$

It is easy to see that

$$\begin{aligned}&I_2(\tau,\eta,\mathfrak{M}_1)\\ &= \frac{C_2C_4}{L^2}\int_{\mathfrak{M}_1}\prod_{1\le j\le 4} T_2(\lambda_j\alpha)T_k(\lambda_5\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &\quad + \int_{\mathfrak{M}_1}(S_2(\lambda_1\alpha)-T_2(\lambda_1\alpha))S_2(\lambda_2\alpha)S_k(\lambda_5\alpha)V_1(\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &\quad + \int_{\mathfrak{M}_1}(S_2(\lambda_2\alpha)-T_2(\lambda_2\alpha))T_2(\lambda_1\alpha)S_k(\lambda_5\alpha)V_1(\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &\quad + \int_{\mathfrak{M}_1}(S_k(\lambda_5\alpha)-T_k(\lambda_5\alpha))T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)V_1(\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &\quad + \int_{\mathfrak{M}_1}(S_2^*(\lambda_3\alpha)-\frac{C_2}{L}T_2(\lambda_3\alpha))S_4^*(\lambda_4\alpha)V_2(\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &\quad + \frac{C_2}{L}\int_{\mathfrak{M}_1}(S_4^*(\lambda_4\alpha)-\frac{C_4}{L}T_2(\lambda_4\alpha))T_2(\lambda_3\alpha)V_2(\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\\ &= J_1+J_2+J_3+J_4+J_5+J_6.\end{aligned} \tag{3.4}$$

By the first derivative estimate for trigonometric integrals (see [10, (2.8)]), we can obtain

$$T_j(\alpha) \ll X^{\frac{1}{j}-1}\min(X,|\alpha|^{-1}). \tag{3.5}$$

Then we can deduce from (2.5) and (3.5) that

$$\begin{aligned}&\frac{C_2C_4}{L^2}H - J_1\\ &\ll \left|\frac{C_2C_4}{L^2}\int_{X^{-1+\frac{5}{12k}-\varepsilon}}^{+\infty}\prod_{1\le j\le 4} T_2(\lambda_j\alpha)T_k(\lambda_5\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha\right|\end{aligned}$$

$$\ll \tau^2 L^{-2} X^{-3+\frac{1}{k}} \int_{X^{-1+\frac{5}{12k}-\varepsilon}}^{+\infty} \frac{1}{\alpha^5} d\alpha = O(\tau^2 L^{-2} X^{1+\frac{1}{k}-\frac{5}{3k}+4\varepsilon}). \tag{3.6}$$

In order to estimate J_i $(2 \le i \le 6)$, we need the following lemma.

Lemma 3.1. *Suppose that j is a positive integer with $j \le k$. Then we have*
(i) $\int_{\mathfrak{M}_1} |S_j(\lambda\alpha)|^2 d\alpha \ll X^{\frac{2}{j}-1}$,
(ii) $\int_{\mathfrak{M}_1} |T_j(\lambda\alpha)|^2 d\alpha \ll X^{\frac{2}{j}-1}$,
(iii) $\int_{\mathfrak{M}_1} |S_j(\lambda\alpha) - T_j(\lambda\alpha)|^2 d\alpha \ll X^{\frac{2}{j}-1}\exp(-\log^{\frac{1}{4}} X)$.

Proof. This is [10, Lemma 3.3]. □

By Cauchy's inequality, Lemma 3.1(i), (iii) and (2.5), we have

$$\begin{aligned} J_2 + J_3 &\ll \tau^2 X^{\frac{3}{2}} \left(\int_{\mathfrak{M}_1} |S_k(\lambda_5\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}_1} |\big(S_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)\big)|^2 d\alpha \right)^{\frac{1}{2}} \\ &\quad + \tau^2 X^{\frac{3}{2}} \left(\int_{\mathfrak{M}_1} |S_k(\lambda_5\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}_1} |\big(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha)\big)|^2 d\alpha \right)^{\frac{1}{2}} \\ &\ll \tau^2 X^{\frac{3}{2}+\frac{1}{k}-\frac{1}{2}} \exp\left(-\frac{1}{2}\log^{\frac{1}{4}} X\right) = \tau^2 X^{1+\frac{1}{k}} \exp\left(-\frac{1}{2}\log^{\frac{1}{4}} X\right). \end{aligned} \tag{3.7}$$

In a similar manner, we can obtain

$$\begin{aligned} J_4 &\ll \tau^2 X^{\frac{3}{2}} \left(\int_{\mathfrak{M}_1} |\big(S_k(\lambda_5\alpha) - T_k(\lambda_5\alpha)\big)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}_1} |T_2(\lambda_1\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\ &\ll \tau^2 X^{1+\frac{1}{k}} \exp\left(-\frac{1}{2}\log^{\frac{1}{4}} X\right). \end{aligned} \tag{3.8}$$

Note that $\rho_j(m) \ge 0$. Hence from partial summation and (2.2), we have

$$|S_j^*(\alpha)| \le |S_j^*(0)| \ll X^{\frac{1}{2}} L^{-1}, \quad \left| S_j^*(\alpha) - \frac{C_j}{L} T_2(\alpha) \right| \ll X^{\frac{1}{2}} L^{-2}(1 + |\alpha| X). \tag{3.9}$$

Then we can deduce from (2.5), (3.5) and (3.9) that

$$\begin{aligned} J_5 + J_6 &\ll \tau^2 X^{\frac{1}{2}} L^{-1} \left(\int_0^1 X^{\frac{1}{2}} L^{-2}(1 + |\alpha| X) \frac{X^{1+\frac{1}{k}}}{(1 + X|\alpha|)^3} d\alpha \right) \\ &\ll \tau^2 X^{\frac{1}{2}} L^{-1} \left(\int_0^{\frac{1}{X}} X^{\frac{3}{2}+\frac{1}{k}} L^{-2} d\alpha + \int_{\frac{1}{X}}^1 \frac{X^{-\frac{1}{2}+\frac{1}{k}} L^{-2}}{\alpha^2} d\alpha \right) \\ &\ll \tau^2 X^{1+\frac{1}{k}} L^{-3}. \end{aligned} \tag{3.10}$$

Now combining (3.4), (3.6), (3.7)-(3.8) and (3.10), we have

$$I_2(\tau, \eta, \mathfrak{M}_1) = \frac{C_2 C_4}{L^2} H + O(\tau^2 X^{1+\frac{1}{k}} L^{-3}). \tag{3.11}$$

In the same manner, we can get

$$I_1(\tau,\eta,\mathfrak{M}_1)=\frac{C_1C_3}{L^2}H+O(\tau^2X^{1+\frac{1}{k}}L^{-3}). \tag{3.12}$$

Furthermore, from [9, Section 3.1], we can get

$$H\gg \tau^2X^{1+\frac{1}{k}}. \tag{3.13}$$

Since $C_1C_3-C_2C_4>0$, we have

$$\begin{aligned} &I(\tau,\eta,\mathfrak{M}_1)\\ &=I_1(\tau,\eta,\mathfrak{M}_1)-I_2(\tau,\eta,\mathfrak{M}_1)\\ &=\frac{C_1C_3-C_2C_4}{L^2}H+O(\tau^2X^{1+\frac{1}{k}}L^{-3})\gg \tau^2X^{1+\frac{1}{k}}L^{-2}. \end{aligned} \tag{3.14}$$

We next give the nontrivial upper bound for $I(\tau,\eta,\mathfrak{M}_2)$ and $I(\tau,\eta,\mathfrak{M}_3)$.

Lemma 3.2. *Suppose that $k\geq 3$ is a positive integer. Then we have*

$$S_k(\lambda_5\alpha)\ll\begin{cases} X^{\frac{1}{k}(1-k4^{1-k})+\varepsilon}|\alpha|^{-4^{1-k}}, & X^{-1}\leq|\alpha|\leq X^{-1+\frac{1}{2k}},\\ X^{\frac{1}{k}(1-\frac{1}{2}4^{1-k})+\varepsilon}, & X^{-1+\frac{1}{2k}}\leq|\alpha|\leq X^{-\frac{1}{2k}}. \end{cases} \tag{3.15}$$

Proof. This is [10, Corollary 3.5]. □

On recalling the definition of $\mathfrak{M}_2$ and $\mathfrak{M}_3$, we can deduce from Lemma 3.2 that

$$\max_{\alpha\in\mathfrak{M}_2}|S_k(\lambda_5\alpha)|\ll X^{\frac{1}{k}-\frac{5}{12k}4^{1-k}+2\varepsilon},\quad \max_{\alpha\in\mathfrak{M}_3}|S_k(\lambda_5\alpha)|\ll X^{\frac{1}{k}-\frac{1}{2k}4^{1-k}+\varepsilon}. \tag{3.16}$$

Moreover, it follows easily from [4, (iv) p. 1975] that $\max_{m\in I_2}|\rho_i(m)|\ll X^{\varepsilon}$. So, we can deduce from Hua's inequality that

$$\int_0^1|S_2(\alpha)^4|d\alpha\ll X^{1+\varepsilon},\quad \int_0^1|S_i^*(\alpha)^4|d\alpha\ll X^{1+\varepsilon}\quad (1\leq i\leq 4). \tag{3.17}$$

By Hölder's inequality, (2.5) and (3.16)-(3.17), we have

$$\begin{aligned} &|I_1(\tau,\eta,\mathfrak{M}_2)|\\ &\ll \tau^2\max_{\alpha\in\mathfrak{M}_2}|S_k(\lambda_5\alpha)|\left(\int_0^1|S_2(\lambda_1\alpha)|^4d\alpha\right)^{\frac14}\left(\int_0^1|S_2(\lambda_2\alpha)|^4d\alpha\right)^{\frac14}\\ &\quad\times\left(\int_0^1|S_1^*(\lambda_3\alpha)|^4d\alpha\right)^{\frac14}\left(\int_0^1|S_3^*(\lambda_4\alpha)|^4d\alpha\right)^{\frac14}\\ &\ll \tau^2X^{1+\frac{1}{k}-\frac{5}{12k}4^{1-k}+3\varepsilon}. \end{aligned} \tag{3.18}$$

Similarly

$$|I_2(\tau,\eta,\mathfrak{M}_2)|\ll \tau^2X^{1+\frac{1}{k}-\frac{5}{12k}4^{1-k}+3\varepsilon}, \tag{3.19}$$

$$|I_1(\tau,\eta,\mathfrak{M}_3)|+|I_2(\tau,\eta,\mathfrak{M}_3)|\ll \tau^2X^{1+\frac{1}{k}-\frac{1}{2k}4^{1-k}+3\varepsilon}. \tag{3.20}$$

From (3.18)-(3.20), we have

$$|I(\tau,\eta,\mathfrak{M}_2)|+|I(\tau,\eta,\mathfrak{M}_3)|\ll O(\tau^2X^{1+\frac{1}{k}-\varepsilon}). \tag{3.21}$$

Now combining (3.3), (3.14) and (3.21), we obtain

$$I(\tau,\eta,\mathfrak{M})\gg\tau^2L^{-2}X^{1+\frac{1}{k}}. \tag{3.22}$$

4. The minor arcs $\mathfrak{m}$

In this section, we show that

$$|I(\tau,\eta,\mathfrak{m})|\ll|I_1(\tau,\eta,\mathfrak{m})|+|I_2(\tau,\eta,\mathfrak{m})|\ll\tau^2X^{1+\frac{1}{k}-\varepsilon}. \tag{4.1}$$

In order to obtain (4.1), we need the following Lemmas.

Lemma 4.1. *For $1\le i\le 4$, we have*

$$\int_{-\infty}^{+\infty}|S_2(\lambda\alpha)|^4K_\tau(\alpha)d\alpha\ll\tau X^{1+\varepsilon},\int_{-\infty}^{+\infty}|S_i^*(\lambda\alpha)|^4K_\tau(\alpha)d\alpha\ll\tau X^{1+\varepsilon}. \tag{4.2}$$

Proof. From [4, (iv) p. 1975], we can find that $\max\limits_{m\in I_2}|\rho_i(m)|\ll X^{\varepsilon}$. Then Lemma 4.1 follows easily from Hua's inequality or [12, (2.5)]. □

Lemma 4.2. *For $1\le i\le 4$, we have*

$$\int_{-\infty}^{+\infty}|S_i^*(\lambda\alpha)|^2|S_k(\lambda_5\alpha)|^{2\sigma(k)}K_\tau(\alpha)d\alpha\ll\tau X^{\frac{2\sigma(k)}{k}+\varepsilon}. \tag{4.3}$$

Proof. It follows from the fact $\max\limits_{m\in I_2}|\rho_i(m)|\ll X^{\varepsilon}$ and [2, Lemma 5.1]. □

Lemma 4.3. *Suppose that $X^{\frac{1}{2}}\ge Z\ge X^{\frac{7}{16}+2\varepsilon}$ and $|S_2(\lambda\alpha)|>Z$. Then there are integers a,q satisfying*

$$(a,q)=1,\ \ q\le\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2,\ \ |q\lambda\alpha-a|\ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2. \tag{4.4}$$

Proof. This is [8, Lemma 2.2]. □

Lemma 4.4. *Let $i=2$ or 3. Suppose that $X^{\frac{1}{2}}\ge Z_2\ge X^{\frac{17}{40}+2\varepsilon}$ and $|S_i^*(\lambda\alpha)|>Z_2$. Then there are integers a_2 and q_2 satisfying*

$$(a_2,q_2)=1,\ \ q_2\le\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2,\ \ |q_2\lambda\alpha-a_2|\ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2. \tag{4.5}$$

Proof. By Dirichlet's Theorem, there exist co-prime integers $a_2,\ q_2$, such that

$$1\le q_2\le X\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^{-4},\ \ |q_2\lambda\alpha-a_2|\ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4. \tag{4.6}$$

For $i = 2$ or 3, it follows from [12, (6.2)] that

$$|S_i^*(\lambda\alpha)| \ll X^{\frac{17}{40}+\frac{1}{2}\varepsilon} + X^{\frac{1}{2}+\frac{1}{2}\varepsilon}\left(\frac{1}{q_2}+\frac{q_2}{X}\right)^{\frac{1}{4}}. \tag{4.7}$$

Since $|S_i^*(\lambda\alpha)| > Z_2 \geq X^{\frac{17}{40}+2\varepsilon}$ and $X^{\frac{1}{4}+\frac{1}{2}\varepsilon}q_2^{\frac{1}{4}} \ll Z_2X^{-\frac{1}{2}\varepsilon}$, we have

$$Z_2 < |S_i^*(\lambda\alpha)| \ll X^{\frac{1}{2}+\frac{1}{2}\varepsilon}q_2^{-\frac{1}{4}}.$$

Hence

$$q_2 \leq \left(\frac{X^{\frac{1}{2}+\frac{1}{2}\varepsilon}}{Z_2}\right)^4, \ \ |q_2\lambda\alpha - a_2| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4. \tag{4.8}$$

Combining $|S_i^*(\lambda\alpha)| > Z_2 \geq X^{\frac{17}{40}+2\varepsilon}$ and (4.8), we can deduce from [4, Lemma 8] (or see the proof of [4, Lemma 9]) that

$$Z_2 < |S_i^*(\lambda\alpha)| \ll \frac{X^{\frac{1}{2}+\varepsilon}}{(q_2 + X|q_2\lambda\alpha - a_2|)^{\frac{1}{2}}}. \tag{4.9}$$

Therefore, we obtain

$$q_2 \leq \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2, \ \ |q_2\lambda\alpha - a_2| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2.$$

□

Lemma 4.5. *Let $i = 2$ or 3 and*

$$\mathfrak{N}_1(Z) = \{\alpha : \alpha \in \mathfrak{m},\ Z \leq |S_i^*(\lambda\alpha)| \leq X^{\frac{1}{2}}\}.$$

Then for $Z \gg X^{\frac{17}{40}+2\varepsilon}$, we have

(i) $\int_{\mathfrak{N}_1(Z)} |S_i^*(\lambda\alpha)|^2|S_k(\lambda_5\alpha)|^4K_\tau(\alpha)d\alpha \ll \tau\left(X^{\frac{4}{k}+\varepsilon} + \frac{X^{1+\frac{2}{k}+\varepsilon}}{Z^2}\right)$,

(ii) $\int_{\mathfrak{N}_1(Z)} |S_i^*(\lambda\alpha)|^2|S_k(\lambda_5\alpha)|^2K_\tau(\alpha)d\alpha \ll \tau\left(X^{\frac{2}{k}+\varepsilon} + \frac{X^{1+\frac{1}{k}+\varepsilon}}{Z^2}\right)$.

Proof. It is easy to see that $K_\tau(\frac{\alpha}{\lambda}) = \lambda^2K_{\frac{\tau}{\lambda}}(\alpha)$. Hence

$$\begin{aligned}&\int_{\mathfrak{N}_1(Z)} |S_i^*(\lambda\alpha)|^2|S_k(\lambda_5\alpha)|^4K_\tau(\alpha)d\alpha \\ &= \lambda\int_{\mathfrak{N}_2(Z)}\left|S_i^*(\alpha)^2S_k\left(\frac{\lambda_5}{\lambda}\alpha\right)^4\right|K_{\frac{\tau}{\lambda}}(\alpha)d\alpha,\end{aligned} \tag{4.10}$$

where

$$\mathfrak{N}_2(Z) = \{\alpha : \frac{\alpha}{\lambda} \in \mathfrak{m},\ Z \leq |S_i^*(\alpha)| \leq X^{\frac{1}{2}}\}.$$

Denote

$$\mathfrak{N}^*(n,Z)=\bigcup_{1\le q\le \frac{X^{1+\frac16\varepsilon}}{Z^2}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\left(n+\frac{a}{q}-\frac{X^{\frac16\varepsilon}}{qZ^2},n+\frac{a}{q}+\frac{X^{\frac16\varepsilon}}{qZ^2}\right],$$
(4.11)
$$\mathfrak{N}^*(Z)=\bigcup_{n=-\infty}^{+\infty}\mathfrak{N}^*(n,Z).$$

Let $G(\alpha)$ be the function of period 1, and defined for $\alpha\in[0,1)$ by

$$G(\alpha)=\begin{cases}(q+X|q\alpha-a|)^{-1},\ \alpha\in\mathfrak{N}^*(Z)\bigcap[0,1),\\ 0,\ \ \alpha\in[0,1)\setminus\mathfrak{N}^*(Z).\end{cases}$$

We can find from Lemma 4.4 and (4.9) that

$$\mathfrak{N}_2(Z)\subseteq\mathfrak{N}^*(Z) \tag{4.12}$$

and

$$|S_i^*(\alpha)|\ll X^{\frac12+\frac16\varepsilon}G^{\frac12}(\alpha)\ \text{for}\ \alpha\in\mathfrak{N}_2(Z). \tag{4.13}$$

Write

$$\psi(v)=\sum_{\substack{p_1^k-p_2^k+p_3^k-p_4^k=v\\(\delta X)^{\frac1k}\le p_1,p_2,p_3,p_4\le X^{\frac1k}}}\ \prod_{1\le j\le4}\log p_j,$$
(4.14)
$$\Psi(\alpha)=\left|S_k\left(\frac{\lambda_5}{\lambda}\alpha\right)^4\right|=\sum_v\psi(v)e\left(\frac{\lambda_5}{\lambda}v\alpha\right).$$

Then by (4.12)-(4.14), we have

$$\begin{aligned}&\int_{\mathfrak{N}_2(Z)}\left|S_i^*(\alpha)^2S_k\left(\frac{\lambda_5}{\lambda}\alpha\right)^4\right|K_{\frac{\tau}{\lambda}}(\alpha)d\alpha\\ &\ll\int_{\mathfrak{N}_2(Z)}X^{1+\frac13\varepsilon}G(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda}}(\alpha)d\alpha\\ &\ll X^{1+\frac13\varepsilon}\int_{\mathfrak{N}^*(Z)}G(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda}}(\alpha)d\alpha.\end{aligned} \tag{4.15}$$

From [1, Lemma 3] with $Q=\frac{X^{1+\frac16\varepsilon}}{Z^2}$, we deduce that

$$\begin{aligned}&\int_{\mathfrak{N}^*(Z)}G(\alpha)\Psi(\alpha)K_{\frac{\tau}{\lambda}}(\alpha)d\alpha\\ &\ll\tau X^{\frac13\varepsilon}(1+\tau)^{1+\varepsilon}X^{-1}\left(\sum_v|\psi(v)|+\frac{X^{1+\frac16\varepsilon}}{Z^2}\sum_{|\frac{\lambda_5}{\lambda}v|\le\frac{\tau}{|\lambda|}}|\psi(v)|\right)\end{aligned}$$

$$\ll \tau X^{-1+\frac{1}{3}\varepsilon}\left(\sum_{v}|\psi(v)|+\frac{X^{1+\frac{1}{6}\varepsilon}}{Z^2}\sum_{|v|\leq\frac{\tau}{|\lambda_5|}}|\psi(v)|\right). \tag{4.16}$$

It is easy to find that

$$\sum_{v}|\psi(v)|=\Psi(0)\ll X^{\frac{4}{k}}L^4. \tag{4.17}$$

Since $\tau=X^{-\frac{3}{40\sigma(k)}+3\varepsilon}$, by Hua's inequality, we have

$$\sum_{|v|\leq\frac{\tau}{|\lambda_5|}}|\psi(v)|\leq\sum_{\substack{|p_1^k-p_2^k+p_3^k-p_4^k|\leq\frac{\tau}{|\lambda_5|}\\ (\delta X)^{\frac{1}{k}}\leq p_1,p_2,p_3,p_4\leq X^{\frac{1}{k}}}}L^4\leq L^4\int_0^1|S_k(\alpha)|^4d\alpha\leq X^{\frac{2}{k}+\frac{1}{6}\varepsilon}. \tag{4.18}$$

Combining (4.10) and (4.15)-(4.18), we have

$$\int_{\mathfrak{N}_1(Z)}|S_i^*(\lambda\alpha)^2S_k(\lambda_5\alpha)^4|K_\tau(\alpha)d\alpha\ll\tau\left(X^{\frac{4}{k}+\varepsilon}+\frac{X^{1+\frac{2}{k}+\varepsilon}}{Z^2}\right). \tag{4.19}$$

Now (i) is proved. On replacing $\Psi(\alpha)=\left|S_k\left(\frac{\lambda_5}{\lambda}\alpha\right)^4\right|$ in (4.14) with $\Psi(\alpha)=\left|S_k\left(\frac{\lambda_5}{\lambda}\alpha\right)^2\right|$, we can obtain (ii) by the similar method leading to (4.19). □

Let

$$\begin{aligned}
\mathfrak{m}_1&=\{\alpha\in\mathfrak{m}:\ |S_3^*(\lambda_4\alpha)|\leq X^{\frac{17}{40}+2\varepsilon}\},\\
\mathfrak{m}_2&=\{\alpha\in\mathfrak{m}:\ |S_2(\lambda_1\alpha)|\leq X^{\frac{7}{16}+2\varepsilon},\ |S_3^*(\lambda_4\alpha)|>X^{\frac{17}{40}+2\varepsilon}\},\\
\mathfrak{m}_3&=\{\alpha\in\mathfrak{m}:\ |S_2(\lambda_2\alpha)|\leq X^{\frac{7}{16}+2\varepsilon},\ |S_3^*(\lambda_4\alpha)|>X^{\frac{17}{40}+2\varepsilon}\},\\
\mathfrak{m}_4&=\mathfrak{m}\setminus(\mathfrak{m}_1\bigcup\mathfrak{m}_2\bigcup\mathfrak{m}_3).
\end{aligned}$$

Thus

$$|I_1(\tau,\eta,\mathfrak{m})|\leq\sum_{1\leq i\leq 4}|I_1(\tau,\eta,\mathfrak{m}_i)|. \tag{4.20}$$

Applying Hölder's inequality, Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
&|I_1(\tau,\eta,\mathfrak{m}_1)|\\
\ll\ &\max_{\alpha\in\mathfrak{m}_1}|S_3^*(\lambda_4\alpha)|^{\frac{1}{\sigma(k)}}\left(\int_{-\infty}^{+\infty}|S_3^*(\lambda_4\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac{1}{4}-\frac{1}{2\sigma(k)}}\\
&\times\left(\int_{-\infty}^{+\infty}|S_2(\lambda_1\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac{1}{4}}\left(\int_{-\infty}^{+\infty}|S_2(\lambda_2\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac{1}{4}}\\
&\times\left(\int_{-\infty}^{+\infty}|S_1^*(\lambda_3\alpha)^4|K_\tau(\alpha)d\alpha\right)^{\frac{1}{4}}
\end{aligned}$$

$$\times\left(\int_{-\infty}^{+\infty}|S_3^*(\lambda_4\alpha)^2S_k(\lambda_5\alpha)^{2\sigma(k)}|K_\tau(\alpha)d\alpha\right)^{\frac{1}{2\sigma(k)}}$$

$$\ll \tau X^{\frac{17}{40\sigma(k)}+\frac{1}{4}-\frac{1}{2\sigma(k)}+\frac{3}{4}+\frac{1}{k}+2\varepsilon}\ll \tau X^{1+\frac{1}{k}-\frac{3}{40\sigma(k)}+2\varepsilon}\ll \tau^2X^{1+\frac{1}{k}-\varepsilon}. \tag{4.21}$$

Let $\mathfrak{N}_1(Z)$ be defined as in Lemma 4.5. When $3\le k\le 13$, we can find from Hölder's inequality, Lemma 4.1 and Lemma 4.5(ii) that

$$|I_1(\tau,\eta,\mathfrak{m}_2)|$$

$$\ll \max_{\alpha\in\mathfrak{m}_2}|S_2(\lambda_1\alpha)|\left(\int_{\mathfrak{N}_1(X^{\frac{17}{40}+2\varepsilon})}|S_3^*(\lambda_4\alpha)|^2|S_k(\lambda_5\alpha)|^2K_\tau(\alpha)d\alpha\right)^{\frac12}$$

$$\times\left(\int_{-\infty}^{+\infty}|S_1^*(\lambda_3\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{-\infty}^{+\infty}|S_2(\lambda_2\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}$$

$$\ll \tau X^{\frac{7}{16}+\frac12+3\varepsilon}(X^{\frac1k+\varepsilon}+X^{\frac{3}{40}+\frac{1}{2k}+\varepsilon})$$

$$=\tau(X^{1+\frac1k-\frac{1}{16}+4\varepsilon}+X^{1+\frac1k-\frac{1}{2k}+\frac{1}{80}+4\varepsilon})\ll\tau^2X^{1+\frac1k-\varepsilon}. \tag{4.22}$$

When $k\ge 14$, let

$$\mathfrak{m}_5=\{\alpha\in\mathfrak{m}_2:\ |S_2(\lambda_1\alpha)|\le X^{\frac{7}{16}+2\varepsilon},\ X^{\frac{17}{40}+2\varepsilon}<|S_3^*(\lambda_4\alpha)|<X^{\frac12-\frac1k+2\varepsilon}\},$$

$$\mathfrak{m}_6=\{\alpha\in\mathfrak{m}_2:\ |S_2(\lambda_1\alpha)|\le X^{\frac{7}{16}+2\varepsilon},\ X^{\frac12-\frac1k+2\varepsilon}\le|S_3^*(\lambda_4\alpha)|<X^{\frac12}\}.$$

Since $\mathfrak{m}_2=\mathfrak{m}_5\bigcup\mathfrak{m}_6$, we have

$$I_1(\tau,\eta,\mathfrak{m}_2)=I_1(\tau,\eta,\mathfrak{m}_5)+I_1(\tau,\eta,\mathfrak{m}_6). \tag{4.23}$$

By the similar method leading to (4.22), we can get

$$|I_1(\tau,\eta,\mathfrak{m}_6)|$$

$$\ll \max_{\alpha\in\mathfrak{m}_6}|S_2(\lambda_1\alpha)|\left(\int_{\mathfrak{N}_1(X^{\frac12-\frac1k+2\varepsilon})}|S_3^*(\lambda_4\alpha)|^2|S_k(\lambda_5\alpha)|^2K_\tau(\alpha)d\alpha\right)^{\frac12}$$

$$\times\left(\int_{-\infty}^{+\infty}|S_1^*(\lambda_3\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{-\infty}^{+\infty}|S_2(\lambda_2\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}$$

$$\ll \tau X^{\frac{7}{16}+\frac12+3\varepsilon}(X^{\frac2k+\varepsilon}+X^{\frac3k+\varepsilon})^{\frac12}$$

$$\ll \tau X^{1+\frac1k-\frac{1}{16}+\frac{1}{2k}+4\varepsilon}\ll\tau^2X^{1+\frac1k-\varepsilon}. \tag{4.24}$$

In order to estimate $I_1(\tau,\eta,\mathfrak{m}_5)$, we subdivide $\mathfrak{m}_5$ into disjoint sets $S(Z)$ such that for $\alpha\in S(Z)$, we have

$$Z<|S_3^*(\lambda_4\alpha)|\le 2Z,$$

where $Z=2^{k_1}X^{\frac{17}{40}+2\varepsilon}$ for some positive integers k_1. Then we can deduce from Lemma 4.5(i) and the dyadic arguments that

$$\int_{\mathfrak{m}_5}|S_3^*(\lambda_4\alpha)^4S_k(\lambda_5\alpha)^4|K_\tau(\alpha)d\alpha$$

$$\ll \max_{X^{\frac{17}{40}+2\varepsilon}<Z\le X^{\frac{1}{2}-\frac{1}{k}+2\varepsilon}} Z^2 L\int_{S(Z)}|S_3^*(\lambda_4\alpha)^2S_k(\lambda_5\alpha)^4|K_\tau(\alpha)d\alpha$$

$$\ll \max_{X^{\frac{17}{40}+2\varepsilon}<Z\le X^{\frac{1}{2}-\frac{1}{k}+2\varepsilon}} Z^2 L\int_{\mathfrak{N}_1(Z)}|S_3^*(\lambda_4\alpha)^2S_k(\lambda_5\alpha)^4|K_\tau(\alpha)d\alpha$$

$$\ll \max_{X^{\frac{17}{40}+2\varepsilon}<Z\le X^{\frac{1}{2}-\frac{1}{k}+2\varepsilon}} \tau Z^2 L\left(\frac{X^{1+\frac{2}{k}+\varepsilon}}{Z^2}+X^{\frac{4}{k}+\varepsilon}\right)\ll \tau X^{1+\frac{2}{k}+2\varepsilon}. \tag{4.25}$$

So by Hölder's inequality, Lemma 4.1 and (4.25), we have

$$\begin{aligned}&|I_1(\tau,\eta,\mathfrak{m}_5)|\\ \ll\ &\left(\int_{\mathfrak{m}_5}|S_3^*(\lambda_4\alpha)^4S_k(\lambda_5\alpha)^4|K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{-\infty}^{+\infty}|S_1^*(\lambda_3\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}\\ &\times\left(\int_{-\infty}^{+\infty}|S_2(\lambda_1\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}\left(\int_{-\infty}^{+\infty}|S_2(\lambda_2\alpha)|^4K_\tau(\alpha)d\alpha\right)^{\frac14}\\ \ll\ &\tau X^{\frac14+\frac{1}{2k}+\frac34+2\varepsilon}=\tau X^{1+\frac1k-\frac{1}{2k}+2\varepsilon}\ll\tau^2X^{1+\frac1k-\varepsilon}.\end{aligned}\tag{4.26}$$

Combining (4.22)-(4.24) and (4.26), we get

$$|I_1(\tau,\eta,\mathfrak{m}_2)|\ll\tau^2X^{1+\frac1k-\varepsilon}. \tag{4.27}$$

By the same argument leading to (4.27), we can get

$$|I_1(\tau,\eta,\mathfrak{m}_3)|\ll\tau^2X^{1+\frac1k-\varepsilon}. \tag{4.28}$$

Now we consider the range $\mathfrak{m}_4=\mathfrak{m}\setminus\left(\mathfrak{m}_1\bigcup\mathfrak{m}_2\bigcup\mathfrak{m}_3\right)$. Note that for $\alpha\in\mathfrak{m}_4$, we have

$$|S_2(\lambda_1\alpha)|>X^{\frac{7}{16}+2\varepsilon},\quad |S_2(\lambda_2\alpha)|>X^{\frac{7}{16}+2\varepsilon}.$$

So we can divide $\mathfrak{m}_4$ into disjoint sets $S(Z_1,Z_2,y)$ such that for $\alpha\in S(Z_1,Z_2,y)$, we have

$$Z_1<|S_2(\lambda_1\alpha)|\le 2Z_1,\ \ Z_2<|S_2(\lambda_2\alpha)|\le 2Z_2,\ \ y<|\alpha|\le 2y, \tag{4.29}$$

where $Z_1=2^{t_1}X^{\frac{7}{16}+2\varepsilon}$, $Z_2=2^{t_2}X^{\frac{7}{16}+2\varepsilon}$ and $y=2^rX^{-\frac{1}{2k}}$ for some positive integers t_1, t_2 and r. Thus by Lemma 4.3, there are co-prime integers (a_1,q_1), (a_2,q_2) with

$$\begin{aligned}q_1&\le\left(\frac{X^{\frac12+\varepsilon}}{Z_1}\right)^2,\ \ |q_1\lambda_1\alpha-a_1|\ll X^{-1}\left(\frac{X^{\frac12+\varepsilon}}{Z_1}\right)^2,\\ q_2&\le\left(\frac{X^{\frac12+\varepsilon}}{Z_2}\right)^2,\ \ |q_2\lambda_2\alpha-a_2|\ll X^{-1}\left(\frac{X^{\frac12+\varepsilon}}{Z_2}\right)^2.\end{aligned}\tag{4.30}$$

We remark that $a_1a_2 \neq 0$, since otherwise we have $\alpha \in \mathfrak{M}$. Furthermore, we subdivide $S(Z_1, Z_2, y)$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j < q_j \leq 2Q_j$ on each set. Then

$$\begin{aligned} \left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| &= \left|\frac{a_2(q_1\lambda_1\alpha - a_1) + a_1(a_2 - q_2\lambda_2\alpha)}{\lambda_2\alpha}\right| \\ &\ll Q_2X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2 + Q_1X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2 \\ &\ll \frac{X^{1+4\varepsilon}}{Z_1^2Z_2^2} \ll X^{-\frac{3}{4}-\varepsilon}. \end{aligned} \tag{4.31}$$

Note that $q = X^{\frac{3}{4}}$. Thus

$$|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2| = o(q^{-1}). \tag{4.32}$$

We also have

$$|a_2q_1| \ll yQ_1Q_2. \tag{4.33}$$

Hence, if $|a_2q_1|$ takes R distinct values, we could deduce the existence of n satisfying

$$\|n\frac{\lambda_1}{\lambda_2}\| \ll X^{-\frac{3}{4}-\varepsilon}, \ n \ll \frac{yQ_1Q_2}{R}. \tag{4.34}$$

This would contradict $\frac{a}{q}$ being a convergent to $\frac{\lambda_1}{\lambda_2}$ if q is sufficiently large, unless

$$R \ll \frac{yQ_1Q_2}{q}. \tag{4.35}$$

By (4.31) and the well known bound on the divisor function, we find that each value of a_2q_1 corresponds to $O(X^\varepsilon)$ values of a_2, q_1 and a_1, q_2. Then we obtain that each set of $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $O(RX^\varepsilon)$ intervals of length

$$\min\left(\frac{1}{Q_1X}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2, \frac{1}{Q_2X}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2\right). \tag{4.36}$$

Let $\mathfrak{L}$ denote such a set $S(Z_1, Z_2, y, Q_1, Q_2)$. Note that

$$\begin{aligned} \int_{\mathfrak{L}} 1d\alpha &\ll yQ_1Q_2q^{-1}\min\left(\frac{1}{Q_1X}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2, \frac{1}{Q_2X}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2\right) \\ &\ll \frac{yX^{1+4\varepsilon}}{qZ_1^2Z_2^2}. \end{aligned} \tag{4.37}$$

Recall that $\tau = X^{-\frac{3}{40\sigma(k)}+3\varepsilon}$, $y \ll \xi = \tau^{-2}X^\varepsilon$, $q = X^{\frac{3}{4}}$ and $Z_2 \gg X^{\frac{7}{16}+2\varepsilon}$, $Z_1 \gg X^{\frac{7}{16}+2\varepsilon}$ and $K_\tau(\alpha) \ll \tau^2$. Hence

$$\int_{\mathfrak{L}} 1d\alpha \ll X^{\frac{3}{20\sigma(k)}-\frac{3}{2}} \tag{4.38}$$

and

$$|I_1(\tau,\eta,\mathfrak{L})| \ll \tau^2 X^{2+\frac{1}{k}}\left(\int_{\mathfrak{L}} 1 d\alpha\right) \ll \tau^2 X^{\frac{3}{20\sigma(k)}+\frac{1}{k}+\frac{1}{2}}. \tag{4.39}$$

Summing over all possible values of y, Q_1, Q_2, Z_1, Z_2, we get

$$|I_1(\tau,\eta,\mathfrak{m}_4)| \ll |I_1(\tau,\eta,\mathfrak{L})|L^5 \ll \tau^2 X^{\frac{3}{20\sigma(k)}+\frac{1}{k}+\frac{1}{2}+\varepsilon} \ll \tau^2 X^{1+\frac{1}{k}-\varepsilon}. \tag{4.40}$$

Combining (4.20)-(4.21), (4.27)-(4.28) and (4.40), we have

$$|I_1(\tau,\eta,\mathfrak{m})| \ll \tau^2 X^{1+\frac{1}{k}-\varepsilon}. \tag{4.41}$$

By the similar method leading to (4.41), we can get

$$|I_2(\tau,\eta,\mathfrak{m})| \ll \tau^2 X^{1+\frac{1}{k}-\varepsilon}. \tag{4.42}$$

Hence, we have

$$|I(\tau,\eta,\mathfrak{m})| \ll \tau^2 X^{1+\frac{1}{k}-\varepsilon}. \tag{4.43}$$

5. The trivial arcs $\mathfrak{t}$

By the same argument as in [9, Section 4], we can get

$$|I(\tau,\eta,\mathfrak{t})| \leq |I_1(\tau,\eta,\mathfrak{t})| + |I_2(\tau,\eta,\mathfrak{t})| \ll \tau^2 X^{1+\frac{1}{k}-\varepsilon}. \tag{5.1}$$

Combining (2.10), (2.13), (3.22), (4.43) and (5.1), we get

$$I(\tau,\eta,\mathbb{R}) \gg \tau^2 X^{1+\frac{1}{k}} L^{-2}, \quad N_\tau(X) \gg \tau X^{1+\frac{1}{k}} L^{-5}. \tag{5.2}$$

Since $\frac{\lambda_1}{\lambda_2}$ is irrational, there are infinitely many pairs of co-prime integers q and a such that $\frac{a}{q}$ is a convergent to $\frac{\lambda_1}{\lambda_2}$. Then we have $X = q^{\frac{4}{3}} \to +\infty$, as $q \to +\infty$. This imply that (5.2) holds for infinite sequence of values of X. Thus the proof of the Theorem is completed.

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Li Zhu
School of Mathematical Sciences
Tongji University
Shanghai, 200092, P. R. China
Email address: zhuli15@tongji.edu.cn