J. Korean Math. Soc. **56** (2019), No. 4, pp. 947–984
https://doi.org/10.4134/JKMS.j180488
pISSN: 0304-9914 / eISSN: 2234-3008

SHIFTED TABLEAU SWITCHINGS AND SHIFTED LITTLEWOOD–RICHARDSON COEFFICIENTS

Seung-Il Choi, Sun-Young Nam, and Young-Tak Oh

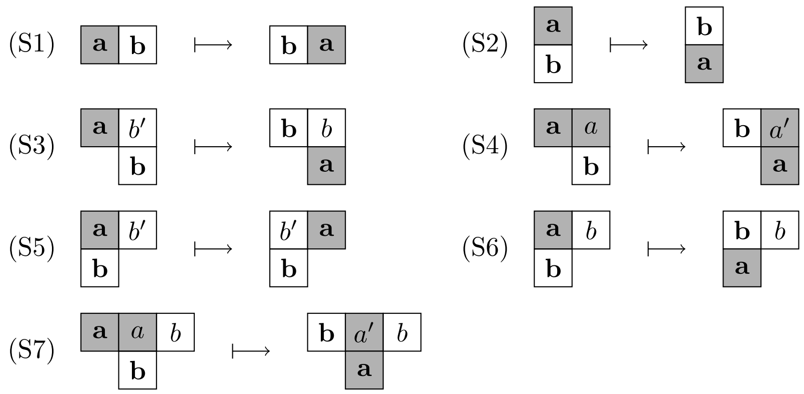
Abstract. We provide two shifted analogues of the tableau switching process due to Benkart, Sottile, and Stroomer; the shifted tableau switching process and the modified shifted tableau switching process. They are performed by applying a sequence of elementary transformations called *switches* and shares many nice properties with the tableau switching process. For instance, the maps induced from these algorithms are involutive and behave very nicely with respect to the lattice property. We also introduce shifted generalized evacuation which exactly agrees with the shifted J-operation due to Worley when applied to shifted Young tableaux of normal shape. Finally, as an application, we give combinatorial interpretations of Schur P- and Schur Q-function related identities.

1. Introduction

The (tableau) switching process was introduced by Benkart–Sottile–Stroomer [1] in 1996 as a unification of the previously defined switching algorithms on Young tableaux due to Haiman [7], James–Kerber [9], etc. It is an algorithm which takes a pair (S,T) of Young tableaux such that T extends S, and moves them through each other, yielding a pair $({}^{S}T, S_T)$ of Young tableaux such that S_T extends ${}^{S}T$. Since the switching process is built upon a mixture of Schützenberger's jeu de taquin [12] and its reverse process, it shares many nice properties with jeu de taquin process. For instance, it preserves Knuth equivalence relations on words, so ${}^{S}T$ equals the rectification of T when $S\cup T$ is of normal shape. Besides, the map induced from the switching process is involutive and provides nice combinatorial interpretations of Schur function related identities.

Received July 20, 2018; Revised October 30, 2018; Accepted November 29, 2018.
2010 *Mathematics Subject Classification.* 05E05.
Key words and phrases. shifted tableau switchings, shifted jeu de taquin, Schur P- and Schur Q-functions, shifted Littlewood–Richardson coefficients.
The first was supported by NRF Grant ♯2012R1A1A2001635, NRF Grant ♯2015R1D1-A1A01056670 and Samsung Science and Technology Foundation under Project Number SSTF-BA1501-01. The second author was supported by NRF Grant ♯2012R1A1A2001635, NRF Grant ♯2015R1D1-A1A01056670 and NRF Grant ♯2017R1D1A1B03030945. The third author was supported by NRF Grant ♯2012R1A1A2001635 and NRF Grant ♯2015R1D1-A1A01056670.

The purpose of the present paper is to provide two shifted analogues of the switching process, called the *shifted* (*tableau*) *switching process* and *modified shifted* (*tableau*) *switching process*, by mimicking the strategy of [1]. To begin with, we establish the notion of shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs and then contrive the following seven elementary transformations called *switches* to interchange two letters in a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair:



The above switches are so designed that they recover the shifted slides used in the shifted jeu de taquin introduced by Worley [19] and Sagan [11], when $\mathbf{a}$-boxes are viewed as empty boxes. Applying appropriate switches among (S1) through (S7) in succession, we obtain the shifted switching process on the set of shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs (A, B) such that B extends A. The most crucial step here is to show that the shifted perforateness has being maintained throughout this process (Proposition 3.3). Keeping this process, we finally obtain an algorithm which takes a pair (S, T) of shifted Young tableaux such that T extends S, and moves them through each other, yielding a pair $({}^{S}T, S_T)$ of shifted Young tableaux such that S_T extends ${}^{S}T$ (Theorem 3.6).

The shifted switching process so obtained has many nice properties as the switching process does. The most significant property, which is one of the main results of this paper, is that the map induced from the shifted switching process, called *the shifted tableau switching*, is involutive and preserves the lattice property (Theorem 4.3). Also, it provides combinatorial interpretations of various identities related with Schur P- and Q-functions, and Schur S-functions. For instance, we can explain the following well-known identity

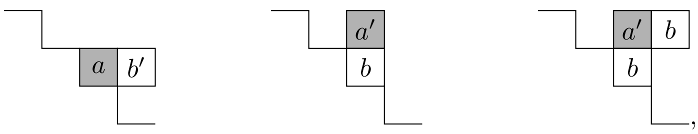
$$Q_{\nu/\lambda}(x) = \sum_{\mu} f^{\nu}_{\mu\lambda} Q_{\mu}(x)$$

in a combinatorial way by affording the shifted tableau switching to certain pairs of shifted Young tableaux (see Section 8(a)). The other examples can be found in Section 8.

It should be noted that there have already been two operations similar to our shifted switching process. The one is due to Buch–Kresch–Tamvakis [2], who exploited sliding unmarked and marked holes in any shifted Littlewood–Richardson tableau. Their slides may be viewed as switches by identifying o-holes with a-boxes and o′-holes with a'-boxes. Indeed the switches obtained in this way agree with our switches with the exception that two horizontally neighboring holes and **a**-boxes lie on the main diagonal. In the exceptional case, we specially contrive the switch (S4) and (S7) to assign the involutiveness to the shifted tableau switching (Remark 3.8(a)). The other is the infusion introduced by Thomas–Yong [18] to provide a root-theoretic uniform generalization of the classical Littlewood–Richardson rule for the Schubert calculus of (co)minuscule variety. Particularly, in type C case, the infusion coincides with the shifted tableau switching on pairs of standard shifted Young tableaux and their generalization with the shifted Littlewood–Richardson rule [17, Lemma 8.4]. Moreover, the authors in [18] showed the symmetry of shifted Littlewood–Richardson coefficients by using the involutivity of the infusion. Our shifted switching process can be viewed as a semistandardization of the infusion process (Remark 8.1). We emphasize that the shifted tableau switching gives not only a weight-preserving bijective proof of the symmetry of shifted Littlewood–Richardson coefficients but also combinatorial interpretations of various identities related with Schur P-, Q-, and S-functions.

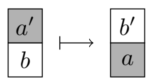
The shifted J-operation, introduced by Worley [19], was utilized intensively to show the symmetric identity $f^{\nu}_{\lambda\mu} = f^{\nu}_{\mu\lambda}$ and properties about the coefficients of expansions involving $Q_{\lambda/\mu}(x)$. In Section 5, we introduce shifted generalized evacuation using the shifted switching and show that it produces the same result with the shifted J-operation when applied to shifted Young tableaux of normal shape (Theorem 5.8). It should be remarked that our shifted generalized evacuation is similar, but not same as the shifted evacuation given in [8, Theorem 14.12], which led us to introduce a new shifted evacuation (Remark 5.7).

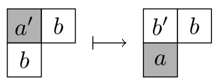
Despite of the above good properties, the shifted switching process is rather unsatisfactory in that the shifted tableau switching cannot be reduced to a bijection on the set of pairs (S, T) of semistandard shifted Young tableaux such that T extends S since the shifted jeu de taquin does not necessarily take a semistandard shifted Young tableau to another semistandard shifted Young tableau. In more detail, if a switch is applied to the following **a**-boxes



then the output so obtained have a primed letter on the main diagonal. To avoid this phenomenon, we introduce the following three *modified switches*:

(S1′) 

(S2′) 

(S6′) 

We construct the modified shifted switching process by applying (S1′), (S2′)

and (S6′) in the above three cases and (S1) through (S7) in the other cases exactly in the same way as the shifted switching process has been defined. We first verify that the shifted perforateness is being maintained throughout the modified shifted switching process when it is applied to any shifted perforated $(\mathbf{a},\mathbf{b})$-pair (A,B) such that B extends A and no primed letters are on the main diagonal of $A\cup B$ (Proposition 6.1). Once the modified shifted switching process is established, we show that the induced map, called the *modified shifted tableau switching*, is involutive (Theorem 6.2). It would be very nice to develop a modified version of shifted jeu de taquin which plays the same role as that of the shifted jeu de taquin in the shifted switching process.

Many applications of the modified shifted tableau switchings also concern Schur P- and Q-function identities. Letting $Q_\lambda(x)Q_\mu(x)=\sum_\nu g^\nu_{\lambda\mu}Q_\nu(x)$, it can be easily seen that $g^\nu_{\lambda\mu}$ counts semistandard shifted Young tableaux T of shape ν/λ and weight μ satisfying the lattice property (see Theorem 4.1). We call these tableaux *modified Littlewood–Richardson–Stembridge* (*LRS*) *tableaux*. Compared with LRS tableaux, the study of modified LRS tableaux is not yet well established. In the best knowledge of the authors, even the combinatorial interpretation of $g^\nu_{\lambda\mu}=g^\nu_{\mu\lambda}$ has not been known. Using the modified shifted tableau switching we here give its combinatorial interpretation and also that of $P_{\nu/\lambda}(x)=\sum_\mu g^\nu_{\lambda\mu}P_\mu(x)$ (see Section 7 and Section 8).

The present paper is organized as follows: In Section 2, we collect the definitions and notation necessary to develop our arguments. In Section 3, we introduce the shifted tableau switching and its properties. To do this we first define the shifted switching process on shifted perforated pairs (Algorithm 1). Using this we define the shifted switching process on pairs of shifted Young tableaux (Algorithm 2). In Section 4, we investigate the properties of the shifted tableau switching and in Section 5, we provide a shifted analogue of the generalized evacuation in [2] and redescribe the shifted switching process in the spirit of the shifted J-operation. In Section 6 and Section 7 we introduce another switching algorithm: the modified shifted tableau switching, and also investigate the properties of this algorithm. Section 8 deals with applications of the shifted tableau switching and modified shifted tableau switching. Finally Appendix A, B, C and D are devoted to the proof of Proposition 3.3, Lemma 4.4, Proposition 6.1 and Theorem 6.2, which are placed here for their length and complexity.

2. Definitions, terminologies, and notation

In this section, we provide definitions and notation which are required to develop our arguments. More details can be found in [2, 11, 17, 19].

For a strict partition $\lambda=(\lambda_1>\lambda_2>\cdots>\lambda_\ell>0)$, we define the *length* $\ell(\lambda)$ of λ to be the number of positive parts of λ with $\lambda_i=0$ for all $i>\ell(\lambda)$. And we denote the set of all strict partitions by Λ^+. For $\lambda,\mu\in\Lambda^+$ with $\lambda_i\geq\mu_i$ for all $i\geq 1$, we write $\lambda\supseteq\mu$, and define the *shifted diagram* $S(\lambda/\mu)$ of λ/μ to be the set

$$\left\{(i,j) \in \mathbb{Z}^2 : 1 \le i \le \ell(\lambda),\ \mu_i + i \le j \le \lambda_i + i - 1\right\}.$$

The intersection of $\{(i,i) : 1 \le i \le \ell(\lambda)\}$ and $S(\lambda/\mu)$ is called the *main diagonal* of $S(\lambda/\mu)$. Particularly, when $\mu = \emptyset$, we briefly write $S(\lambda)$. It should be mentioned that, throughout this paper, the elements of $S(\lambda/\mu)$ are visualized as boxes in a plane with matrix-style coordinates and $S(\lambda/\mu)$ is frequently denoted by λ/μ.

For given two boxes (i,j) and (k,l) in a shifted diagram, (i,j) is said to be *north* of (k,l) if $i \le k$, and *northwest* of (k,l) if $i \le k$ and $j \le l$. When $i = k-1$ and $j = l$, (i,j) is said to be the *neighbor to the north* of (k,l). The other directions and neighbors can be defined in the same manner.

Definition 2.1. Let $\lambda, \mu \in \Lambda^+$ with $\lambda \supseteq \mu$.

(a) A *shifted Young tableau* (ShYT) T of shape λ/μ is a filling of λ/μ with letters in $X = \{1' < 1 < 2' < 2 < \cdots\}$ such that

(1) the entries are weakly increasing along each row and down each column,
(2) each column contains at most one k for each $k \ge 1$, and
(3) each row contains at most one k' for each $k \ge 1$.

(b) A *semistandard shifted Young tableau* (SShYT) T of shape λ/μ is a shifted Young tableau satisfying the following extra condition:

(4) there is no primed entry on the main diagonal.

For a given ShYT T of shape λ/μ, we define the *shape* $\mathrm{sh}(T)$ *of* T to be the shifted diagram $S(\lambda/\mu)$ and the *weight* $\mathrm{wt}(T)$ *of* T to be the sequence $(\mathrm{wt}_1(T), \mathrm{wt}_2(T), \ldots)$, where $\mathrm{wt}_i(T)$ denotes the number of occurrences of i and i' in T. Especially, T is called *standard* if its weight is $(1,1,\ldots,1)$ and it has no primed letter.

For any filling T of skew shape, the *reading word of* T, denoted by $\mathrm{w}(T)$, is defined to be the word obtained by reading entries of T from right to left, starting with the topmost row in T.

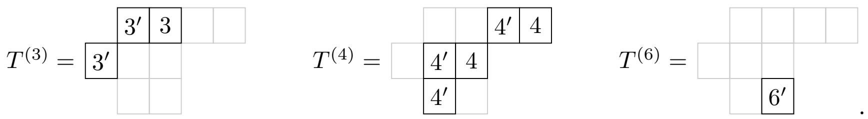
Example 2.2.

$$S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad T = \begin{array}{ccccc} & 3' & 3 & 4' & 4 \\ 3' & 4' & 4 & & \\ & 4' & 6' & & \end{array} \quad \begin{cases} \mathrm{sh}(S) = (2), & \mathrm{sh}(T) = (6,3,2)/(2) \\ \mathrm{wt}(S) = (1,1), & \mathrm{wt}(T) = (0,0,3,5,0,1) \\ \mathrm{w}(S) = 21, & \mathrm{w}(T) = 44'33'44'3'6'4' \\ S \text{ is standard.} & \end{cases}$$

When $\nu \subseteq \mu \subseteq \lambda$, we say that λ/μ *extends* μ/ν or μ/ν is *extended* by λ/μ. For two ShYTs S and T, we say that T extends S if $\mathrm{sh}(T)$ extends $\mathrm{sh}(S)$. In Example 2.2 we see that T extends S. If a single box B is extended by λ/μ, B is called an *inner corner* of λ/μ. It should be distinguished from a *removable corner* of λ/μ, which is a box in λ/μ that is neither a neighbor to the south nor a neighbor to the east in λ/μ.

For given two fillings F_1 of shape λ/μ and F_2 of shape γ/δ, if $\mathrm{sh}(F_1) \cap \mathrm{sh}(F_2) = \emptyset$, we can naturally consider $F_1 \cup F_2$, which means the filling obtained by drawing F_1 and F_2 in a plane simultaneously. For any filling F, let $F^{(i)}$ be the subset of F consisting of all i', i in F. Especially, if T is an ShYT, then $T^{(i)}$ is an ShYT of border strip shape consisting of all i', i in T. We call this the *i-border strip ShYT*. By the *maximum* $\max(T)$ of an ShYT T we mean the largest integer i such that $\mathrm{wt}_i(T)$ is positive and the *minimum* $\min(T)$ of T be defined similarly. If $\max(T) = m$ and $\min(T) = n$, then we have $T = T^{(n)} \cup T^{(n+1)} \cup \cdots \cup T^{(m)}$.

For example, continuing Example 2.2,

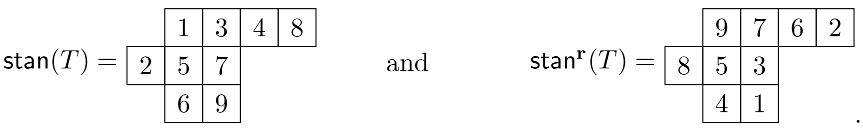


Definition 2.3. Let T be an ShYT.

(a) The *standardization* of T, denoted by $\mathsf{stan}(T)$, is defined to be the standard ShYT obtained from T by replacing all $1'$'s with $1, 2, \ldots, k_1$ from top to bottom, and next all 1's with $k_1+1, k_1+2, \ldots, k_1+k_2$ from left to right, and so on. Here k_1 and k_2 denote the number of occurrences of $1'$ and 1, respectively.

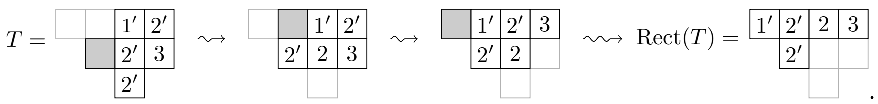
(b) The *reverse standardization* of T, denoted by $\mathsf{stan}^{\mathsf{r}}(T)$, is defined to be the filling of $\mathrm{sh}(T)$ obtained from $\mathsf{stan}(T)$ by replacing i with $|\mathrm{sh}(T)| - i + 1$.

For example, from Example 2.2 we have



Worley [19] and Sagan [11] independently introduced the shifted analogue of Schützenberger's jeu de taquin, which is a combinatorial algorithm taking an ShYT to another ShYT with the same weight, but different shape. This can be carried out by applying the shifted jeu de taquin slides described in Figure 1 (for short, shifted slides) in succession. Here $\mathbf{k}$ means a letter $k \in X$ without specifying whether it is primed or not, and let $|\mathbf{k}|$ denote the unprimed letter k.

Given an ShYT T of skew shape, one can apply shifted slides to T iteratively to get an ShYT of normal shape (which means no more shifted slides are possible). This can be done in many different ways, but the resulting ShYT of normal shape is known to be same for all possible choices. This ShYT is called the *rectification* of T and denoted by $\mathrm{Rect}(T)$. For example,



Here the shaded boxes denote inner corners.

We say that two ShYTs are *shifted jeu de taquin equivalent* if one can be changed into the other by a sequence of shifted slides. Equivalently, two ShYTs

S and T are shifted jeu de taquin equivalent if and only if $\mathrm{Rect}(S)$ and $\mathrm{Rect}(T)$ are the same. As in the ordinary case, shifted jeu de taquin equivalence can be described in terms of shifted Knuth equivalence. Given any word $u = u_1 u_2 \cdots u_n$ in X, the possible shifted Knuth relations that can be applied to u in order to form another word are

- replace yzx by yxz or vice versa if $x \le y < z$,
- replace xzy by zxy or vice versa if $x < y \le z$,
- for the last two elements of u we may
  replace yx by xy or vice versa if $|x| \neq |y|$,
  replace $x'x$ by xx or vice versa.

Here, $(x')'$ denotes x for an unprimed letter x. Shifted Knuth equivalence of words, denoted by $\equiv_{\mathrm{SK}}$, can induce the equivalence relation on the set of words in X in such a way that u and v are *shifted Knuth equivalent* if one can be transformed into the other under the relations above. For more information, refer to [11].

Proposition 2.4. ([11, 19]) *Given two ShYTs, they are shifted jeu de taquin equivalent if and only if their reading words are shifted Knuth equivalent.*

3. Shifted tableau switching

We here introduce the shifted version of the tableau switching given in [1]. For this purpose we first introduce the notion of shifted perforated tableaux.

Definition 3.1. For a skew shape θ, a *shifted perforated* $\mathbf{a}$*-tableau* A in θ is a filling of some of the boxes in θ with $a', a \in X$ satisfying the following conditions:

(1) no a''s are placed to southeast of any a,
(2) each column of A contains at most one a,
(3) each row of A contains at most one a', and
(4) the main diagonal of A contains at most one $\mathbf{a}$.

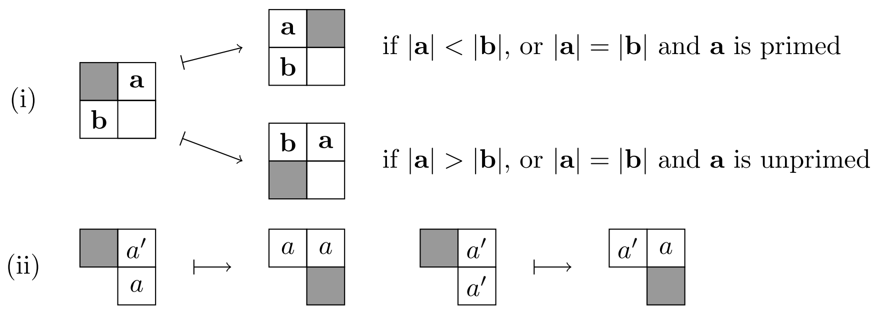
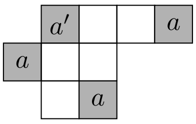
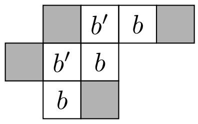
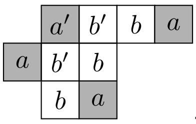


FIGURE 1. The shifted slides

Define the *shape* sh(A) of A by the subset of θ consisting of all boxes filled with a' and a. For a shifted perforated $\mathbf{a}$-tableau A and $\mathbf{b}$-tableau B in θ, (A, B) is called a *shifted perforated* $(\mathbf{a}, \mathbf{b})$*-pair of shape* θ if θ is the disjoint union of sh(A) and sh(B). For a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair (A, B) of shape θ we can consider $A \cup B$ in the obvious way. For instance,

$A =$  and $B =$ 

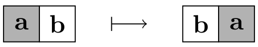
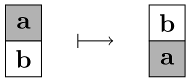
form a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair of shape $(6, 3, 2)/(2)$ and

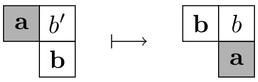
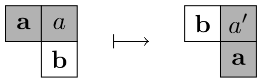
$A \cup B =$  .

For any filling F, let $F_{i,j}$ be the entry of (i, j) in F. And we call (i, j) an $\mathbf{a}$-*box* in case where $F_{i,j} = \mathbf{a}$.

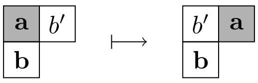
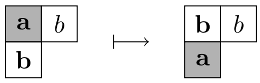
As the tableau switching in [1], our shifted tableau switching can be performed by applying a sequence of elementary transformations called switches. Let (A, B) be a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair. Interchanging an $\mathbf{a}$-box and a $\mathbf{b}$-box in $A \cup B$ is called a *switch* if it is subject to the following rules.

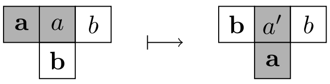
(1) The case where an $\mathbf{a}$-box is adjacent to a unique $\mathbf{b}$-box:

(S1)  (S2) 

(S3)  (S4) 

(2) The case where an $\mathbf{a}$-box is adjacent to two $\mathbf{b}$-boxes:

(S5)  (S6) 

(S7) 

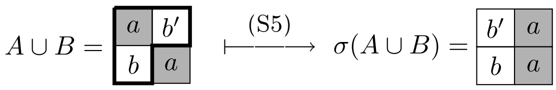
An $\mathbf{a}$-box in $A \cup B$ is said to be *fully switched* if it can be switched with no $\mathbf{b}$-boxes. If every $\mathbf{a}$-box in $A \cup B$ is fully switched, then $A \cup B$ is said to be *fully switched*.

Remark 3.2. (a) The shifted slides in Figure 1 can be recovered from the above switches by viewing $\mathbf{a}$-boxes as empty boxes.

(b) The switches cannot be expressed as a composition of two other switches. For instance, (S7) is different from the composition of (S6) and (S1).

Let $F = A \cup B$ be a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair which is not fully switched. Choose the easternmost a-box in $F^{(a)}$ which is the neighbor to the north or west of a $\mathbf{b}$-box if such an a-box exists. Otherwise, choose the southernmost a'-box which is the neighbor to the north or west of a $\mathbf{b}$-box. Denote this box by $\square(F)$. Then we apply the adequate switch among (S1) through (S7) to $\square(F)$ and denote the resulting filling by $\sigma(F)$. Let us apply this process in succession until $\sigma^n(F)$ gets fully switched, where $\sigma^i(F)$ is inductively defined by $\sigma(\sigma^{i-1}(F))$ for $i = 2, 3, \ldots, n$. We call the process from F to $\sigma^n(F)$ the *shifted (tableau) switching process*, and write A_B for $\sigma^n(F)^{(a)}$ and ${}^{A}B$ for $\sigma^n(F)^{(b)}$ to respect the notation in [1].

As the following example shows, even though $A \cup B$ is a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair, $\sigma(A \cup B)$ is not necessarily a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair:



This problem, however, does not arise in practice in case where B extends A. Going further, we can state the following proposition whose proof will be given in Appendix A.

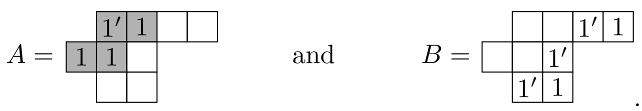
Proposition 3.3. *Let* (A, B) *be a shifted perforated* $(\mathbf{a}, \mathbf{b})$*-pair such that* B *extends* A*. Suppose that* $\sigma^n(A \cup B)$ *is fully switched for some positive integer* n*. Then* $\sigma^m(A \cup B)$ *is still a shifted perforated* $(\mathbf{a}, \mathbf{b})$*-pair for all* $m = 1, 2, \ldots, n$*.*

Let A and B be fillings consisting of $\mathbf{a}$ and $\mathbf{b}$, respectively. Notice that, when B extends A, (A, B) is a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair if and only if A, B are pairs of a- and b-border strip ShYTs, respectively. Hence, Proposition 3.3 yields the following (well-defined) algorithm on shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs (A, B) such that B extends A.

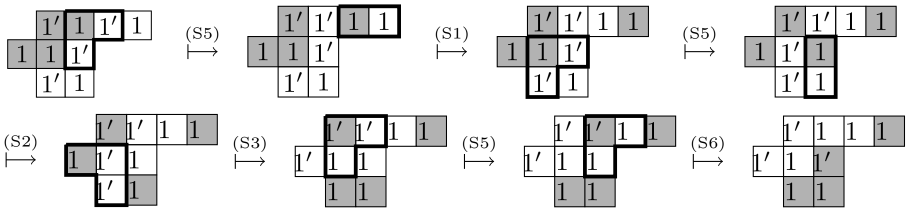
define SP(A, B) **do**
  set $F \leftarrow A \cup B$
  while F is not fully switched **do**
    $F \leftarrow \sigma(F)$
  set $A \leftarrow F^{(a)}$ and $B \leftarrow F^{(b)}$
  return A, B

ALGORITHM 1. The shifted switching process on pairs of border strip ShYTs

Example 3.4. Let



We graphically illustrate how ALGORITHM 1 acts on (A, B) step by step:



Theorem 3.5. *Let (A, B) be a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair such that B extends A. Then we have*

(a) ${}^{A}B \cup A_B$ *has the same shape as* $A \cup B$.

(b) A_B *extends* ${}^{A}B$. *In addition,* A_B *and* ${}^{A}B$ *are* a- *and* b-*border strip ShYTs respectively.*

Proof. (a) The assertion follows from the fact that no switches affect on the shape.

(b) By Remark 3.2(a), ${}^{A}B$ can be obtained by applying a sequence of shifted slides to B following the order in which we choose inner corners given by $\mathsf{stan}^{\mathsf{r}}(A)$. This tells us that ${}^{A}B$ is a b-border strip ShYT extended by A_B. Since A_B is a shifted perforated **a**-tableau due to Proposition 3.3, it follows that A_B is an a-border strip ShYT. □

We now are ready to present the shifted switching process on pairs (S, T) of ShYTs such that T extends S.

define SW(S, T) **do**
    set $m \leftarrow \max(S)$ and $n \leftarrow \max(T)$
    for i from m to 1
        for j from 1 to n
            SP$(S^{(i)}, T^{(j)})$
    set ${}^{S}T \leftarrow T^{(1)} \cup \cdots \cup T^{(n)}$ and $S_T \leftarrow S^{(1)} \cup \cdots \cup S^{(m)}$
    return ${}^{S}T, S_T$

ALGORITHM 2. The shifted switching process on pairs of ShYTs

Following the notation of the above algorithm, we write S_T and ${}^{S}T$ for the fillings that S and T respectively yield when the shifted switching process is applied to (S, T). We define the *shifted* (*tableau*) *switching* Σ to be the map sending (S, T) to $({}^{S}T, S_T)$. The next theorem is the main result of this section.

Theorem 3.6. *Let (S, T) be a pair of ShYTs such that T extends S. The shifted tableau switching $(S, T) \mapsto ({}^{S}T, S_T)$ satisfies the following.*

(a) ${}^{S}T \cup S_T$ *has the same shape as* $S \cup T$. *In addition,* S_T *and* ${}^{S}T$ *has the same weight as* S *and* T, *respectively.*

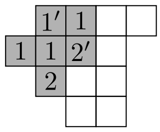
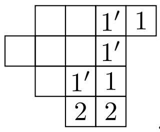
(b) $({}^{S}T, S_T)$ *is a pair of ShYTs such that* S_T *extends* ${}^{S}T$.

Proof. Let $m := \max(S)$ and $n := \max(T)$. As we can see in ALGORITHM 2, S_T and ST are obtained by performing only $\mathrm{SP}(S^{(i)}, T^{(j)})_{m \le i \le 1, 1 \le j \le n}$ in the order given in ALGORITHM 2. But, every resulting pair obtained from $\mathrm{SP}(S^{(i)}, T^{(j)})$ does, due to Theorem 3.5, satisfy the assertions (a) and (b) in Theorem 3.6. Hence, using the induction on the number of border strip ShYTs together with the property

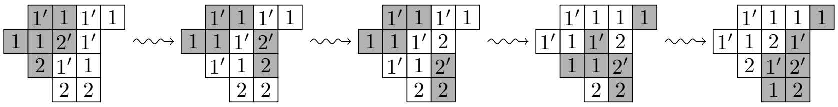
$$\Sigma(S,T) = \Sigma(S^{(1)} \cup S', T) = \Sigma(S^{(1)}, {}^{S'}T) \cup S'_T\,, \qquad (S' := S^{(2)} \cup \cdots \cup S^{(m)})$$

we can easily see that S_T and ST satisfy the desired properties. □

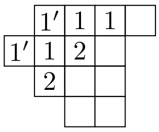
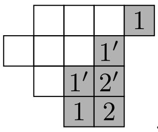
Example 3.7. Let

$S =$  and $T =$  .

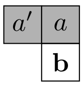
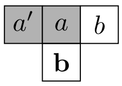
We graphically illustrate how ALGORITHM 2 acts on (S,T) step by step:



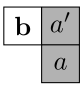
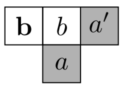
As a consequence, we have

${}^ST =$  and $S_T =$  .

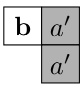
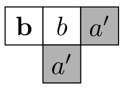
Remark 3.8. (a) There is a slight difference between the slides in [2] and our switches. For clarity, we denote the slides in [2] by $\mapsto$. Let us consider the following shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs:

$U =$  $V =$ 

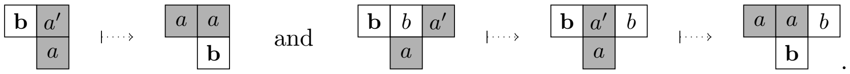
Under the identification of o-holes with a-boxes and a'-boxes with o$'$-holes, these pairs are transformed by the sliding process given in [2] as follows:

$U' =$  $V' =$ 

On the contrary, U and V are transformed by our switching process in the following form:

$U'' =$  $V'' =$ 

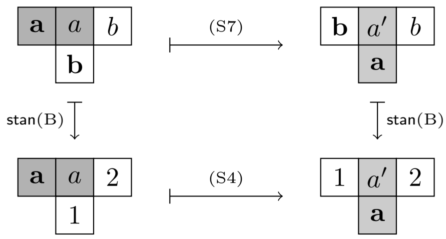
It should be pointed out that U and V cannot be recovered from U' and V', respectively, by applying $\mapsto$ since



But, by applying our switches, they can be recovered from U'' and V'', respectively. In fact we will prove in Section 4 that our shifted tableau switching is an involution (see Theorem 4.3).

(b) Given a shifted perforated $(\mathbf{a},\mathbf{b})$-pair (A,B) such that B extends A, let us identify a'-boxes and a-boxes with o$'$-holes and o-holes, respectively. Let $(\overline{{}^A B}, \overline{A_B})$ be the shifted perforated $(\mathbf{b},\mathbf{a})$-pair obtained through the sliding process given in [2]. In case where neither U nor V presented in (a) does not occur in the middle of the sliding process, one can easily verify that $({}^A B, A_B)$ is exactly equal to $(\overline{{}^A B}, \overline{A_B})$. However, if either U or V occurs, then A_B is different from $\overline{A_B}$ although ${}^A B$ is still equal to $\overline{{}^A B}$. The difference between A_B and $\overline{A_B}$ lies in that the former has a primed letter in the southernmost box on the main diagonal, whereas the latter has an unprimed letter therein. It implies that $\mathsf{stan}(A_B) = \mathsf{stan}(\overline{A_B})$.

(c) The definition of the sliding process presented in [2] tells us that $\overline{{}^A B} = \overline{{}^{\mathsf{stan}(A)} B}$ and $\mathsf{stan}(A)_B = \overline{\mathsf{stan}(A)_B}$. From this it follows that ${}^A B = {}^{\mathsf{stan}(A)} B$ and $\mathsf{stan}(A_B) = \mathsf{stan}(A)_B$. Similarly, one can observe that the standardization of $\mathbf{b}$ letters is compatible with our switches (S1) through (S7). The below example shows that (S7) followed by the standardization of $\mathbf{b}$ letters and the standardization of $\mathbf{b}$ letters followed by (S4) produce the same result:



As a consequence, we can derive that $A_B = A_{\mathsf{stan}(B)}$ and $\mathsf{stan}({}^A B) = {}^A\mathsf{stan}(B)$.

(d) Suppose that (A', B) is a shifted perforated $(\mathbf{a},\mathbf{b})$-pair such that B extends A'. If $\mathsf{stan}(A) = \mathsf{stan}(A')$, then $\mathsf{stan}(A)_B = \mathsf{stan}(A')_B$ and therefore $\mathsf{stan}(A_B) = \mathsf{stan}(A'_B)$ by (c).

4. Properties of the shifted tableau switching

The primary purpose of this section is to show that our shifted tableau switching is an involution and has a very nice behavior with respect to tableaux whose reading words satisfy the lattice property. As a byproduct, we give a combinatorial interpretation of various identities related to Schur P- and Q-functions. To do this let us review Schur P- and Q-functions very briefly.

For $\lambda, \mu \in \Lambda^+$ with $\lambda \supseteq \mu$, let $\widetilde{\mathcal{Y}}(\lambda/\mu)$ and $\mathcal{Y}(\lambda/\mu)$ denote the set of all ShYTs of shape λ/μ and the set of all SShYTs of shape λ/μ respectively. Then (skew) Schur P- and Q-functions are defined as follows:

$$P_{\lambda/\mu}(x) = \sum_{T \in \mathcal{Y}(\lambda/\mu)} x^T \qquad \text{and} \qquad Q_{\lambda/\mu}(x) = \sum_{T \in \widetilde{\mathcal{Y}}(\lambda/\mu)} x^T,$$

where x^T is $\prod_{i \geq 1} x_i^{\mathrm{wt}_i(T)}$. When $\mu = \emptyset$, we simply write them as $P_\lambda(x)$ and $Q_\lambda(x)$. For $\lambda, \mu, \nu \in \Lambda^+$, the *shifted Littlewood–Richardson* (*LR*) *coefficients* $f^\nu_{\lambda\mu}$ are the structure constants appearing in the Schur P-expansion of the product of two P_λ and P_μ, that is,

$$P_\lambda(x) \cdot P_\mu(x) = \sum_\nu f^\nu_{\lambda\mu} P_\nu(x).$$

To prove the positivity of $f^\nu_{\lambda\mu}$ in a combinatorial way has been an important and interesting problem in the study of combinatorics of symmetric functions. For instance, [11, 17, 19] approach this problem with ShYTs and very recently [3, 4, 6, 14] with another combinatorial objects called 'semistandard decomposition tableaux'. We here focus on the former approach.

It is due to Sagan [11] and Worley [19] that $f^\nu_{\lambda\mu}$ counts the standard shifted Young tableaux S of shape ν/λ with $\mathrm{Rect}(S) = T$ for an arbitrarily chosen standard shifted Young tableau T of shape μ. Moreover, they proved that $\mathrm{Rect}(S)$ depends not on the shape of S but on its reading word $\mathrm{w}(S)$. By virtue of their seminal works one can interpret the identity

$$Q_{\nu/\lambda}(x) = \sum_\mu f^\nu_{\lambda\mu} Q_\mu(x) \tag{1}$$

in a combinatorial way by using the shifted jeu de taquin. Later, we will give another combinatorial interpretation of Eq. (1) by using our shifted tableau switching. To do this, we need the Stembridge's combinatorial interpretation of $f^\nu_{\lambda\mu}$, which gives a characterization for the ShYTs that rectify to the special ShYT, denoted by R_μ, whose ith row consists of only i's for all $i = 1, \ldots, \ell(\mu)$.

Theorem 4.1 ([17, Theorem 8.3])**.** *The shifted Littlewood–Richardson coefficient $f^\nu_{\lambda\mu}$ is the number of ShYTs T of shape* $\mathrm{sh}(T) = \nu/\lambda$ *and* $\mathrm{wt}(T) = \mu$ *such that*

(a) $\mathrm{w}(T)$ *satisfies the lattice property, and*
(b) *the rightmost i in $|\mathrm{w}(T)|$ is unprimed for each $1 \leq i \leq \ell(\mu)$.*

(*For the definition of the lattice property, see* [17, Section 8]).

We call these ShYTs *shifted Littlewood–Richardson* (*LR*) *tableaux*, and a word satisfying both the lattice property and Theorem 4.1(b) a *shifted Littlewood–Richardson* (*LR*) *word*. Then the following lemma tells us that for each shifted Knuth equivalence class on words, either all or none of its elements are shifted LR words.

Lemma 4.2 ([2, Lemma 1])**.** *Let u, v be shifted Knuth equivalent words. Then u is a shifted LR word if and only if v is a shifted LR word.*

Now we are ready to state our main theorem.

Theorem 4.3. *Let (S, T) be a pair of ShYTs such that T extends S. If the shifted tableau switching transforms (S, T) into $({}^{S}T, S_T)$, then we have the following.*

(a) *The shifted tableau switching transforms $({}^{S}T, S_T)$ into (S, T).*
(b) *When T is a shifted LR tableau, so is ${}^{S}T$.*
(c) *When S is a shifted LR tableau, so is S_T.*

To prove this theorem, we first claim that the shifted tableau switching is involutive on pairs (A, B) of a- and b-border strip ShYTs (see Lemma 4.4). The proof of this claim is straightforward but quite complicated, so we will prove it in Appendix B.

Lemma 4.4. *Let A and B be an a- and a b-border strip ShYT, respectively. If B extends A, then $\Sigma({}^{A}B, A_B) = (A, B)$.*

Proof of Theorem 4.3. (a) Notice that, in the middle of the shifted switching process on (S, T), two border strips in S which are not adjacent do not effect each other. So the assertion follows from Lemma 4.4.

(b) Let T be a shifted LR tableau and T' any filling obtained by a sequence of shifted slides to T. By applying Lemma 4.2 in iteration, we can deduce that T is a shifted LR tableau if and only if $\mathrm{w}(T')$ is a shifted LR word. Since ${}^{S}T = \mathrm{Rect}(T)$ due to Remark 3.2(a), $\mathrm{w}({}^{S}T)$ is a shifted LR word and thus ${}^{S}T$ is a shifted LR tableau.

(c) Let S be a shifted LR tableau. By (b), S_T is a shifted LR tableau if and only if ${}^{{}^{S}T}S_T$ is a shifted LR tableau. Now our assertion is straightforward since ${}^{{}^{S}T}S_T = S$ (by (a)). □

5. The shifted generalized evacuation

Worley [19] introduced a shifted analogue of Schützenberger's J-operation [13] in order to explain the identities related with shifted Littlewood–Richardson coefficients in a combinatorial way. The purpose of this section is to demonstrate the relationship between Worley's shifted J-operation and our shifted tableau switching.

Let $-X := \{\cdots < -2' < -2 < -1' < -1\}$ and $\overline{X} := \{\cdots < -1' < -1 < 1' < 1 < \cdots\}$. It should be particularly noticed that, in the present section, an ShYT always denotes an ShYT with letters in $\overline{X}$. Recall that an ShYT T of shape λ/μ is of normal shape if $\mu = \emptyset$. For an unprimed letter $a \in \overline{X}$, set $a^* := -a'$ and $(a')^* := -a$. Conventionally, $-(-\mathbf{a})$ is set to be $\mathbf{a}$.

Definition 5.1 ([19])**.** Given $\lambda, \mu \in \Lambda^+$ with $\lambda \supseteq \mu$, let T be an ShYT of shape λ/μ. Then T^* is defined to be the ShYT obtained by the following steps:

(a) Each box (i, j) is transported into $(-j, -i)$ and then moved on the original partial plane.
(b) Each entry a is replaced by $-a'$, where $(a')' := a$.

The shifted J-operation for ShYTs is the mapping $T \mapsto T^J$ defined by

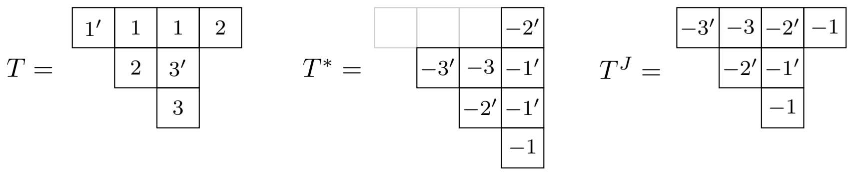
$$T^J := \mathrm{Rect}(T^*)\,.$$

Then the following property of the shifted J-operation is well-known.

Lemma 5.2 ([19, Lemma 7.1.4])**.** *Let T be an ShYT. Then we have the following.*

(a) *If T is of normal shape, then* $\mathrm{sh}(T^J) = \mathrm{sh}(T)$ *and* $(T^J)^J = T$.
(b) *If T is of skew shape, then* $\mathrm{w}(\mathrm{Rect}(T^*)) \equiv_{SK} \mathrm{w}(\mathrm{Rect}(T)^*)$, *that is,* $T^J = \mathrm{Rect}(T)^J$.

Example 5.3. We give an example for T^* and T^J:



In the previous section, we showed $f^{\nu}_{\lambda\mu} = f^{\nu}_{\mu\lambda}$ by using the shifted tableau switching. Worley had, however, obtained this result much earlier by using the shifted J-operation. His argument can be restated in the following manner.

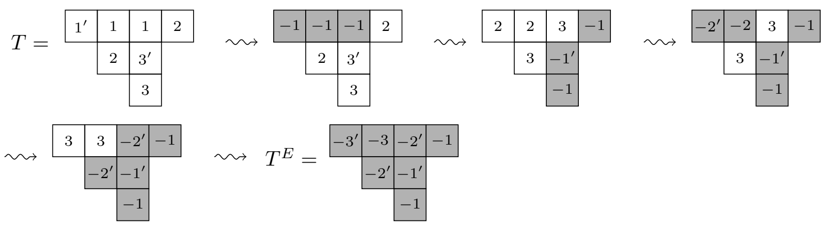
Theorem 5.4 ([19, Theorem 7.2.2])**.** *For each $T \in \mathcal{F}^{\nu}_{\lambda\mu}$ define $\rho(T)$ to be the ShYT uniquely determined by the condition that $((R_\lambda)^J \cup T)^J = (R_\mu)^J \cup \rho(T)$. Then the function*

$$\rho : \mathcal{F}^{\nu}_{\lambda\mu} \to \mathcal{F}^{\nu}_{\mu\lambda},\ T \mapsto \rho(T)$$

is a bijection.

Hence it would be very natural to search for the relationship between Worley's shifted J-operation and our shifted tableau switching. For this purpose, we first provide a shifted analogue of [1, Algorithm 5.2], which is called the *generalized evacuation* there (See Algorithm 4). Let T be an ShYT of normal shape.

Example 5.5. Continuing Example 5.3, we show how to obtain T^E step by step:



define SGE(T) **do**
 set $m \leftarrow \max(T)$ and $T^E \leftarrow \emptyset$
 for a from 1 to m
  if $T_{1,1} = a$ **then**
   set $T^{(a)} \leftarrow$ $\boxed{-a' \mid -a \mid \cdots \mid -a}$
  else
   set $T^{(a)} \leftarrow$ $\boxed{-a \mid -a \mid \cdots \mid -a}$
  if $a = m$ **then**
   return $T^E \leftarrow T^{(a)} \cup T^E$
  else
   set $T \leftarrow {}^{T^{(a)}}(T^{(a+1)} \cup \cdots \cup T^{(m)})$,
    $T^E \leftarrow (T^{(a)})_{T^{(a+1)} \cup \cdots \cup T^{(m)}} \cup T^E$

ALGORITHM 4. The shifted generalized evacuation

Notice that it is obvious from Theorem 3.6 that T^E is an ShYT. The following theorem shows the relationship between the shifted J-operation and the shifted generalized evacuation for ShYTs of normal shape.

Theorem 5.6. *Let T be an ShYT of normal shape. Then $T^E = T^J$.*

Proof. Recall that T is uniquely decomposed as $T^{(1)} \cup T^{(2)} \cup \cdots \cup T^{(\max(T))}$. For simplicity, let us call this decomposition *the elementary decomposition of* T. We will prove our assertion by induction on the number of border strip ShYTs, denoted by $\mathbf{b}(T)$, appearing in the elementary decomposition of T.

If $\mathbf{b}(T) = 1$, then $T = T^{(a)}$ for some (unprimed) $a \in X$, that is,

$$T = \boxed{a' \mid a \mid \cdots \mid a} \quad \text{or} \quad \boxed{a \mid a \mid \cdots \mid a}\ .$$

In the former case, $T^E = \boxed{-a \mid -a \mid \cdots \mid -a}$ and $\mathrm{w}(T^*) = -a' \cdots - a' - a$. In the latter case, $T^E = \boxed{-a' \mid -a \mid \cdots \mid -a}$ and $\mathrm{w}(T^*) = -a' \cdots - a'$. Lemma 5.2 says that $T^J = \mathrm{Rect}(T^*)$ is an ShYT of normal shape $\mathrm{sh}(T)$ consisting of $-a', -a$. Moreover, it is not difficult to see that if T' is any ShYT obtained by applying shifted slides to T^*, then the rightmost letter in $\mathrm{w}(T')$ equals that in $\mathrm{w}(T^*)$. Therefore we can conclude that $T^E = T^J$.

Now assume that the assertion holds for all ShYTs R of normal shape with $\mathbf{b}(R) < n$. Suppose that T is an ShYT of normal shape with $\mathbf{b}(T) = n$. By shifting all the entries in T if necessary, we may assume that $T = T^{(1)} \cup T^{(2)} \cup \cdots \cup T^{(n)}$. Let $U := T^{(1)}$ and $V := T^{(2)} \cup \cdots \cup T^{(n)}$.

To prove the assertion, we first show that $(U \cup V)^J = V^J \cup (U^J)_V$, equivalently, $(U^J \cup V)^J = V^J \cup U_V$. Let δ be the smallest staircase partition containing $\mathrm{sh}(T)$. Choose any ShYT W such that W extends T and $\mathrm{sh}(T \cup W) = \delta$. Let $U' = ((U^J \cup V)^*)^{(1)}$. Then we see that $(U^J \cup V)^* = V^* \cup U'$. It is clear that U' is of connected vertical strip shape contained in δ. On the other hand, by applying the shifted tableau switching to (U_V, W) we get the 1-border strip

ShYT $(U_V)_W$. Since $\mathrm{sh}({}^U V \cup {}^{U_V} W \cup (U_V)_W) = \delta$ due to Theorem 3.6, this 1-border strip ShYT is also of connected vertical strip shape in δ. From the case where $\mathbf{b}(T) = 1$ it follows that the southernmost entry in U' and $(U_V)_W$ are identical and hence U' and $(U_V)_W$ are the same ShYT.

Notice that $\mathrm{Rect}(W)$ extends V^* since W^* and $\mathrm{Rect}(W)$ are of the same normal shape. So we have

$$\Sigma(\mathrm{Rect}(W), V^*) = (\mathrm{Rect}(V^*), \mathrm{Rect}(W)_{V^*}) = (V^J, \mathrm{Rect}(W)_{V^*})$$

and $\mathrm{sh}(V^J) = \mathrm{sh}(\mathrm{Rect}(V)) = \mathrm{sh}({}^U V)$. This shows that $\mathrm{sh}(\mathrm{Rect}(W)_{V^*}) = \mathrm{sh}({}^{U_V} W)$ and thus $(U_V)_W$ extends $\mathrm{Rect}(W)_{V^*}$.

Let S be an ShYT of arbitrary shape such that $\mathrm{sh}(S) \subseteq \delta$. Let $S^\vee$ be an ShYT obtained by applying reverse shifted slides to S in succession until the shape of the resulting ShYT gets δ/κ for some $\kappa \subseteq \delta$. Indeed $S^\vee$ is uniquely determined because $S^\vee$ and $(S^J)^*$ produce the same resulting tableau. In view of this observation, we can deduce that $(\mathrm{Rect}(W)_{V^*}, (U_V)_W)$ goes to (U_V, W) by the shifted tableau switching. Hence $\mathrm{Rect}(W)_{V^*}$ and ${}^{U_V} W$ are the same ShYT due to Theorem 4.3. Furthermore, we have

$$(U^J \cup V)^J = \mathrm{Rect}(V^* \cup U') = V^J \cup^{\mathrm{Rect}(W)_{V^*}} U' = V^J \cup^{{}^{U_V} W} (U_V)_W = V^J \cup U_V \, .$$

Here the second equality follows from the fact that the rectification does not depend on the choice of inner corners.

Now we are ready to prove our assertion. By the definition of ALGORITHM 4 and induction hypothesis, T^E can be expressed as

$$T^E = (U \cup V)^E = \mathrm{Rect}(V)^E \cup (U^E)_V = \mathrm{Rect}(V)^J \cup (U^J)_V \, .$$

On the other hand, from $(U \cup V)^J = V^J \cup (U^J)_V$ along with Lemma 5.2 it follows that

$$T^J = (U \cup V)^J = V^J \cup (U^J)_V = \mathrm{Rect}(V)^J \cup (U^J)_V \, .$$

Therefore $T^E = T^J$. □

Remark 5.7. We define the shifted generalized evacuation on the set of ShYTs. The similar operation defined on the set of words is already described in [8, Theorem 14.12] as follows: for any ShYT T,

$$\mathscr{E}(T) := \mathsf{ins}(\mathsf{opp}(\mathrm{w}(T))).$$

Here ins is the Sagan–Worley insertion in [11,19] and opp is $*$ on words described at the beginning of this section. But it does not produce the same result with our shifted generalized evacuation. With T in Example 5.3 we have

$$\mathscr{E}(T) = \begin{array}{|c|c|c|c|} \hline -3' & -3 & -2' & -1 \\ \hline \multicolumn{1}{c|}{} & -2' & -1' & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & -1' & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \, .$$

Let us consider the operation on the set of pairs of ShYTs defined by

$$\varphi(S \cup T) := S^J \cup T.$$

Then the relationship between the shifted J-operation and the shifted tableau switching can be summarized as follows:

Theorem 5.8. *For $\lambda, \nu \in \Lambda^+$ with $\lambda \subseteq \nu$, let (S,T) be a pair of ShYTs with entries X such that S and T are of shape λ and ν/λ respectively. Then $(S^J \cup T)^J = T^J \cup S_T$ and $J \circ \varphi = \varphi \circ \Sigma$.*

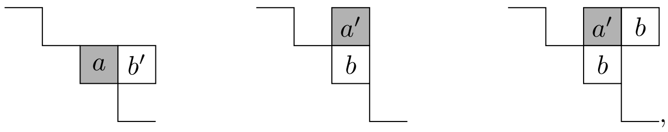
Proof. Since $S^J \cup T$ is an ShYT of shape ν with letters in $\overline{X}$, so is $(S^J \cup T)^J$. In addition, we may write $(S^J \cup T)^J$ as $\mathrm{Rect}(T)^J \cup Y$ for some ShYT Y of shape $\nu/\mathrm{sh}(\mathrm{Rect}(T))$ and with entries $-X$. On the other hand, from Theorem 5.6 and Algorithm 4 we have

$$(S^J \cup T)^J = (S^J \cup T)^E = \mathrm{Rect}(T)^E \cup ((S^J)^E)_T\,.$$

Since J is an involution on ShYTs of normal shape and S^J is of normal shape, $(S^J)^E = (S^J)^J = S$. This proves the first assertion. The second one is immediate from the first. □

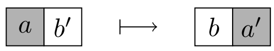
6. The modified shifted tableau switching

Throughout this section, we will consider only shifted perforated $(\mathbf{a}, \mathbf{b})$-pairs with no primed letters on the main diagonal. Unfortunately the set of these pairs is not closed under the shifted tableau switching. In more detail, if a switch defined in Section 3 is applied to the following $\mathbf{a}$-boxes

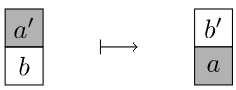


then the resulting pairs have a primed letter on the main diagonal. To avoid this phenomenon, we introduce three new switches labelled by (S1′), (S2′) and (S6′).

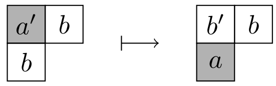
- The case where an a-box on the main diagonal is adjacent to a unique b'-box:

(S1′) 

- The case where an a'-box is adjacent to a unique b-box on the main diagonal:

(S2′) 

- The case where an a'-box is horizontally adjacent to a b-box and vertically adjacent to a b-box on the main diagonal:

(S6′) 

For the clarity we call these the *modified switches*. The original (S1), (S2) and (S6) will be used only in the cases other than the above three ones. Using the switches (S1) through (S7) together with (S1′), (S2′) and (S6′), let us define the *modified shifted (tableau) switching process* on shifted perforated $(\mathbf{a},\mathbf{b})$-pairs in the same way as the shifted switching process has been defined. It should be remarked that the modified shifted switching process in which no modified switches are used is identical to the shifted switching process.

Suppose that T is a filling of a double border strip shape with letters $\mathbf{a}$ and $\mathbf{b}$ and is not fully switched. Let $\widetilde{\sigma}(T)$ denote the resulting filling obtained from T by applying a modified switch.

We will prove the following proposition in Appendix C.

Proposition 6.1. *Let* (A,B) *be a shifted perforated* $(\mathbf{a},\mathbf{b})$*-pair such that* B *extends* A*. Then,*

(a) *modified switches in total can appear at most once in the modified shifted switching process on* (A,B)*,*
(b) *each* $(\mathbf{a},\mathbf{b})$*-pair appearing in the modified shifted switching process on* (A,B) *is still a shifted perforated* $(\mathbf{a},\mathbf{b})$*-pair.*

From now on, we will use the notation $\widetilde{\Sigma}$ to distinguish the modified shifted switching process from the shifted switching process. For a shifted perforated $(\mathbf{a},\mathbf{b})$-pair (A,B) such that B extends A we will write $\widetilde{A_B}$ for $\widetilde{\Sigma}(A,B)^{(a)}$ and $\widetilde{{}^{A}B}$ for $\widetilde{\Sigma}(A,B)^{(b)}$. We also call the map $\widetilde{\Sigma}$ obtained via this process *the modified shifted tableau switching.*

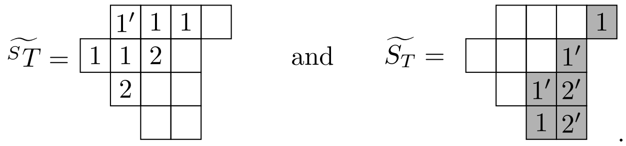
The subsequent theorem shows that the set of pairs of SShYTs is invariant under the modified shifted tableau switching. The proof of this theorem will be given in Appendix D.

Theorem 6.2. *Let* (S,T) *be a pair of SShYTs such that* T *extends* S*. The modified shifted tableau switching* $(S,T)\mapsto(\widetilde{{}^{S}T},\widetilde{S_T})$ *satisfies the following.*

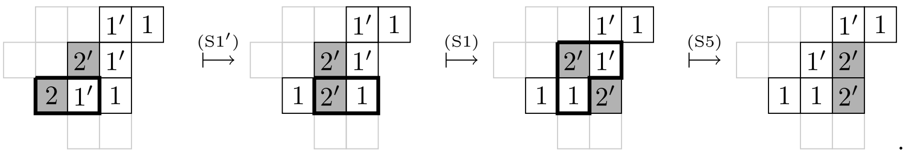
(a) $\widetilde{S_T}\cup\widetilde{{}^{S}T}$ *has the same shape as* $S\cup T$*. In addition,* $\widetilde{S_T}$ *(resp.,* $\widetilde{{}^{S}T}$*) has the same weight as* S *(resp.,* T*).*
(b) $\widetilde{S_T}$ *extends* $\widetilde{{}^{S}T}$*. In addition,* $\widetilde{S_T}$ *and* $\widetilde{{}^{S}T}$ *are SShYTs.*

The following example illustrates how the modified shifted switching process works.

Let S,T be the SShYTs in Example 3.7. Keeping the modified shifted switching process, we finally obtain:



In particular, the modified shifted switching process to $(S^{(2)}, T^{(1)})$ yields



7. Properties of the modified shifted tableau switching

The aim of this section is to show that the map via the modified shifted tableau switching defined in Section 6 is an involution and also behaves nicely with respect to whose reading words satisfied the modified lattice property (see below). To do this we first recall Schur Q-expansion of the product of two Schur Q-functions, and Schur P-expansion of skew Schur P-functions.

For $\lambda, \mu, \nu \in \Lambda^+$, the shifted Littlewood–Richardson coefficients $\widetilde{f}^{\nu}_{\lambda\mu}$ are given by

$$P_{\nu/\lambda}(x) = \sum_{\mu \in \Lambda^+} \widetilde{f}^{\nu}_{\lambda\mu} P_\mu(x) \qquad \text{or} \qquad Q_\lambda(x) Q_\mu(x) = \sum_{\nu \in \Lambda^+} \widetilde{f}^{\nu}_{\lambda\mu} Q_\nu(x).$$

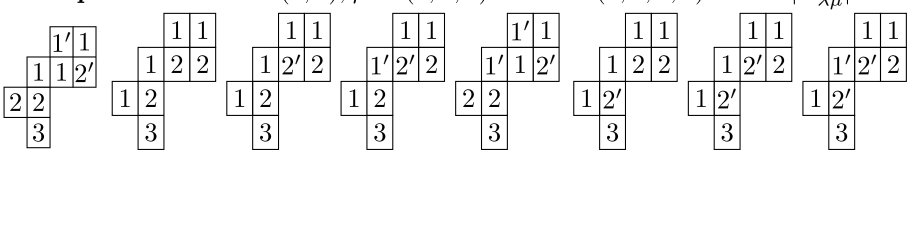
It follows from the facts $\widetilde{f}^{\nu}_{\lambda\mu} = 2^{\ell(\lambda)+\ell(\mu)-\ell(\nu)} f^{\nu}_{\lambda\mu}$ and $\ell(\lambda) + \ell(\mu) - \ell(\nu) \geq 0$ (whenever $f^{\nu}_{\lambda\mu} \geq 0$) that these coefficients are nonnegative integers.

To find an appropriate tableau model for these coefficients, we note that, for each i, the rightmost entry i of $|\mathrm{w}(T)|$ in Theorem 4.1(b) is unprimed. Indeed it was pointed out in [17, page 126] that the lattice property is unaffected when the primes of these extremal entries are changed arbitrarily. This shows that the number of ShYTs of shape ν/λ and weight μ satisfying the lattice property is given by $2^{\ell(\mu)} f^{\nu}_{\lambda\mu}$. Since every entry on the main diagonal of T is an extremal entry, we have the following lemma.

Lemma 7.1. *The modified Littlewood–Richardson coefficient $\widetilde{f}^{\nu}_{\lambda/\mu}$ is the number of SShYTs of T of shape $\mathrm{sh}(T) = \nu/\lambda$ and $\mathrm{wt}(T) = \mu$ such that $w = \mathrm{w}(T)$ satisfies the lattice property.*

We denote by $\widetilde{\mathcal{F}}^{\nu}_{\lambda\mu}$ the set of these tableaux and call an SShYT in $\widetilde{\mathcal{F}}^{\nu}_{\lambda\mu}$ a *modified Littlewood–Richardson–Stembridge* (*LRS*) *tabelau.*

Example 7.2. Let $\lambda = (4,2)$, $\mu = (4,3,1)$ and $\nu = (6,5,2,1)$. Then $|\widetilde{\mathcal{F}}^{\nu}_{\lambda\mu}| = 8$:



The main result of this section is the following theorem which is an analogue of Theorem 4.3 for the modified shifted tableau switching.

Theorem 7.3. *Suppose that S and T are SShYTs such that T extends S. If the modified shifted tableau switching transforms (S,T) into $(\widetilde{{}^ST}, \widetilde{S_T})$, then we have the following.*

(a) *The modified shifted tableau switching transforms $(\widetilde{{}^ST}, \widetilde{S_T})$ into (S,T).*
(b) *When T is a modified LRS tableau, so is $\widetilde{{}^ST}$.*
(c) *When S is a modified LRS tableau, so is $\widetilde{S_T}$.*

Proof. (a) It is enough to show that $\widetilde{\Sigma} \circ \widetilde{\Sigma}(A \cup B) = A \cup B$ for any shifted perforated $(\mathbf{a}, \mathbf{b})$-pair (A,B) such that B extends A. If a modified switch is not used in the modified shifted switching process, the assertion follows from Lemma 4.4. Otherwise, we have

$$\begin{aligned}
\widetilde{\Sigma}(\widetilde{{}^AB}, \widetilde{A_B}) &= \widetilde{\Sigma}(\widetilde{\Sigma}(A,B)) \\
&= \Sigma\left(\Omega\left(\Sigma\left(\Omega(A,B)\right)\right)\right) && \text{(by Lemma D.1(a))} \\
&= \Sigma\left(\Omega\left(\Omega\left(\Sigma(A,B)\right)\right)\right) && \text{(by Lemma D.1(b))} \\
&= \Sigma\left(\Sigma(A,B)\right) && \text{(since } \Omega \circ \Omega = \mathrm{id}\text{)} \\
&= (A,B) && \text{(by Lemma 4.4)}
\end{aligned}$$

as desired. Here Ω is defined in Appendix D.

(b) Suppose that (S,T) is a pair of SShYTs such that T extends S. Denote by (S',T') any pair obtained by applying a sequence of shifted and modified switches to (S,T). To prove our assertion, it is suffices to show that $\mathrm{w}(T)$ is a lattice word if and only if $\mathrm{w}(T')$ is a lattice word. Notice that every word shifted Knuth equivalent to an LRS word is an LRS word by Lemma 4.2 and every word obtained by changing the primes of any rightmost letters is also a lattice word by [17, page 126], and hence one can immediately derive that every word shifted Knuth equivalent to a lattice word is a lattice word. Moreover, by Lemma D.1(a), $\mathrm{w}(T')$ is obtained from $\mathrm{w}(T)$ by applying a sequence of shifted Knuth transformations and by priming some rightmost letters if necessary. This shows that $\mathrm{w}(T)$ is a lattice word if and only if $\mathrm{w}(T')$ is a lattice word.

(c) Let S be a modified LRS tableau. From (b) it follows that $\widetilde{S_T}$ is a modified LRS tableau if and only if $\widetilde{{}^{\widetilde{S_T}}\widetilde{S_T}}$ is a modified LRS tableau. Now the assertion is straightforward by (a). □

Remark 7.4. Let S and T be shifted tableaux such that T extends S. When the shifted tableau switching is applied to (S,T), the image of (S,T) can also be obtained by destandardizing the image of $(\mathsf{stan}(S), \mathsf{stan}(T))$ since each shifted switch is compatible with the standardization of pairs of ShYTs (see Remark 3.8(c)). Thus we can consider the shifted tableau switching as the tableau switching which is defined on pairs of ShYTs with special slide (see Figure 1(ii)). However, in the case of the modified shifted tableau switching, we cannot recover the image of (S,T) by destandardizing the image of

$(\mathsf{stan}(S), \mathsf{stan}(T))$. That is because in case of modified switches (S1′), (S2′) and (S6′) is used during the modified shifted tableau switching is being applied to (S,T), the southwesternmost i' or i-box in $\widetilde{\Sigma}(S,T)$ do not necessarily come from those in $S \cup T$ respectively. At the moment we have no idea how the modified shifted tableau switching can be regarded as the tableau switching on pairs of SShYTs.

Finally, it should be pointed out that, unlike the switching process in [1], our switching processes depend on an order of inner corners to which (modified) switches are applied. It would be interesting to find a 'nice' order of inner corners to make the modified shifted tableau switching most efficient.

8. Applications of the shifted tableau switchings

We here explain various identities related to Schur P- and Q-functions in a combinatorial manner by using our (modified) shifted tableau switching.

To begin with, recall that R_λ is the ShYT whose ith row consists of only i's for all $i = 1, \ldots, \ell(\lambda)$, and Λ^+ is the set of strict partitions. For $\lambda, \mu, \nu \in \Lambda^+$, let $\mathcal{F}^{\nu}_{\lambda\mu}$ be the set of all shifted LR tableaux of shape ν/λ and weight μ. It is obvious that R_λ is a unique ShYT in $\mathcal{F}^{\lambda}_{\emptyset\lambda}$. Likewise, we denote Λ by the set of partitions and R_α, for $\alpha \in \Lambda$, by the semistandard Young tableau (SST) whose ith row consists of only i's for all $i = 1, \ldots, \ell(\alpha)$. For α, β and $\gamma \in \Lambda$, let $\mathcal{C}^{\gamma}_{\alpha\beta}$ be the set of all (ordinary) Littlewood–Richardson tableaux of shape γ/α and weight β. It is also obvious that R_α is a unique SST in $\mathcal{C}^{\alpha}_{\emptyset\alpha}$. In this section we define the addition and subtraction of two (ordinary or strict) partitions componentwise.

(a) (The (modified) shifted Littlewood–Richardson rule) Let $\lambda, \nu \in \Lambda^+$ with $\lambda \subseteq \nu$. By Theorem 4.3, The shifted tableau switching affords a bijection

$$\Sigma : \mathcal{F}^{\lambda}_{\emptyset\lambda} \times \widetilde{\mathcal{Y}}(\nu/\lambda) \longrightarrow \bigcup_{\mu \subseteq \nu} \widetilde{\mathcal{Y}}(\mu) \times \mathcal{F}^{\nu}_{\mu\lambda}, \quad (R_\lambda, T) \mapsto \left({}^{R_\lambda}T, (R_\lambda)_T\right).$$

This immediately gives a combinatorial interpretation of

$$Q_{\nu/\lambda}(x) = \sum_{\mu} f^{\nu}_{\mu\lambda} Q_\mu(x).$$

Similarly, by Theorem 7.3 the modified shifted tableau switching affords a bijection

$$\Sigma : \widetilde{\mathcal{F}}^{\lambda}_{\emptyset\lambda} \times \mathcal{Y}(\nu/\lambda) \to \bigcup_{\mu \subseteq \nu} \mathcal{Y}(\mu) \times \widetilde{\mathcal{F}}^{\nu}_{\mu\lambda}, \quad (R_\lambda, T) \mapsto \left(\widetilde{{}^{R_\lambda}T}, \widetilde{(R_\lambda)_T}\right).$$

It immediately yields a combinatorial interpretation of $P_{\nu/\lambda}(x) = \sum_\mu \widetilde{f}^{\nu}_{\mu\lambda} P_\mu(x)$.

(b) (Symmetry of the (modified) shifted Littlewood–Richardson coefficients) By Theorem 4.3(b) and (c), the map Σ in (a) restricts to a bijection

$$\Sigma : \mathcal{F}^{\lambda}_{\emptyset\lambda} \times \mathcal{F}^{\nu}_{\lambda\mu} \longrightarrow \mathcal{F}^{\mu}_{\emptyset\mu} \times \mathcal{F}^{\nu}_{\mu\lambda}, \quad (R_\lambda, T) \mapsto (R_\mu, (R_\lambda)_T).$$

Hence, not only have we proven $f^{\nu}_{\lambda\mu} = f^{\nu}_{\mu\lambda}$, but also we have constructed an explicit involution that interchanges the inner shape with the weight of a shifted LR tableau. This result is already introduced in [18, Corollary 4.7].

Similarly, by Theorem 7.3(b) and (c), the map $\widetilde{\Sigma}$ in (a) restricts to a bijection

$$\widetilde{\Sigma} : \widetilde{\mathcal{F}}^{\lambda}_{\emptyset\lambda} \times \widetilde{\mathcal{F}}^{\nu}_{\lambda\mu} \longrightarrow \widetilde{\mathcal{F}}^{\mu}_{\emptyset\mu} \times \widetilde{\mathcal{F}}^{\nu}_{\mu\lambda}, \quad (R_\lambda, T) \mapsto (R_\mu, (R_\lambda)_T)\,.$$

It gives an explicit involution of the symmetry $\widetilde{f}^{\nu}_{\lambda\mu} = \widetilde{f}^{\nu}_{\mu\lambda}$.

(c) (The generalized skew (modified) shifted Littlewood–Richardson rule) Let $\lambda, \mu, \nu \in \Lambda^+$ with $\lambda \supseteq \mu \supseteq \nu$. As above, Theorem 4.3 and Theorem 7.3 afford bijections

$$\Sigma : \bigcup_{\substack{\zeta\in\Lambda^+ \\ |\zeta|=|\mu|-|\nu|}} \mathcal{F}^{\mu}_{\nu\zeta} \times \widetilde{\mathcal{Y}}(\lambda/\mu) \longrightarrow \bigcup_{\substack{(\nu\subseteq)\eta\in\Lambda^+ \\ |\eta|=|\lambda|-|\zeta|}} \widetilde{\mathcal{Y}}(\eta/\nu) \times \mathcal{F}^{\lambda}_{\eta\zeta}, \quad (S,T) \mapsto \left({}^{S}T, S_T\right).$$

$$\widetilde{\Sigma} : \bigcup_{\substack{\zeta\in\Lambda^+ \\ |\zeta|=|\mu|-|\nu|}} \mathcal{G}^{\mu}_{\nu\zeta} \times \mathcal{Y}(\lambda/\mu) \longrightarrow \bigcup_{\substack{(\nu\subseteq)\eta\in\Lambda^+ \\ |\eta|=|\lambda|-|\zeta|}} \mathcal{Y}(\eta/\nu) \times \mathcal{G}^{\lambda}_{\eta\zeta}, \quad (S,T) \mapsto \left({}^{S}T, S_T\right),$$

respectively. These give combinatorial interpretations of

$$\sum_{\zeta} f^{\mu}_{\nu\zeta} Q_{\lambda/\mu}(x) = \sum_{\eta} f^{\lambda}_{\eta\zeta} Q_{\eta/\nu}(x) \text{ and } \sum_{\zeta} g^{\mu}_{\nu\zeta} P_{\lambda/\mu}(x) = \sum_{\eta} g^{\lambda}_{\eta\zeta} P_{\eta/\nu}(x),$$

which is a shifted analogue of [1, Example 3.4].

(d) (A bijection between ShYTs of a given shape and ShYTs of its flip) The *flip* of $S(\lambda/\mu)$ is defined to be the skew shifted diagram obtained by flipping $S(\lambda/\mu)$ over its anti-diagonal, that is, $\{(i, \lambda_1 - i + 1) : 1 \le i \le \ell(\lambda)\} \cap S(\lambda)$. Denote it by $(\lambda/\mu)^{\mathsf{fl}}$. Given an ShYT T of shape λ/μ, let $T = T^{(1)} \cup \cdots \cup T^{(m)}$, where $m = \max(T)$. By applying ALGORITHM 1 to each pair $(T^{(i)}, T^{(j)})$ for $(i, j) = (1, 2), \ldots, (1, m), (2, 3), \ldots, (2, m), \ldots, (m-1, m)$ and using the reversed order, that is, $\cdots < 2' < 2 < 1' < 1$, we obtain a unique tableau of the same shape. On the other hand, it can also be viewed as an ShYT of shape $(\lambda/\mu)^{\mathsf{fl}}$ by changing primed letters to unprimed letters and vice versa and then reading the columns (in this tableau) from right to left. It induces the identity $Q_{(\lambda/\mu)^{\mathsf{fl}}}(x) = Q_{\lambda/\mu}(x)$. In fact, this identity first appeared in [5, Proposition IV.13].

(e) (A connection between unshifted and shifted standard Young tableaux) For $r \ge 2$, let $\delta_r := (r-1, r-2, \ldots, 1)$ and $\delta_1 = \emptyset$. Given $\nu \in \Lambda^+$, let ν° denote the ordinary partition $\nu - \delta_{\ell(\nu)}$. By setting $\lambda = \delta_{\ell(\nu)}$ in (a), we have

$$S_{\nu^\circ}(x) = \sum_{\mu} f^{\nu}_{\mu\,\delta_{\ell(\nu)}} Q_\mu(x),$$

where $S_{\nu^\circ}(x)$ is the Schur S-function which is defined by the weight generating function of tableaux of unshifted shape ν° satisfying three conditions in Definition 2.1(a). We call these tableaux *marked tableaux of shape* ν°. Then the symmetry of shifted LR coefficients in (b) implies that

$$S_{\nu^\circ}(x) = \sum_{\mu\in\Lambda^+} g_{\mu\nu^\circ} Q_\mu(x), \tag{2}$$

where $g_{\mu\nu^\circ}$ is the number of marked tableaux of unshifted shape ν° and weight μ, and whose reading word is a shifted LR word (see [17, p. 132]). For $\mu \in \Lambda^+$ let g^μ denote the number of standard shifted Young tableaux of shape μ. The identity Eq. (2), when we only consider the contribution of standard shifted Young tableaux, immediately induces

$$f^{\nu^\circ} = \sum_{\mu\in\Lambda^+} g_{\mu\nu^\circ}\, g^\mu,$$

where f^{ν° is the number of standard Young tableaux of unshifted shape ν°. This identity can be obtained in a direct way by using our shifted tableau switching. We denote $\delta_{\ell(\nu)}$ briefly by δ. Since $g_{\mu\nu^\circ} = f^\nu_{\delta\mu}$, Theorem 4.3 affords a bijection

$$\Sigma : \mathcal{F}^\delta_{\emptyset\delta} \times \mathsf{S}(\nu^\circ) \longrightarrow \bigcup_{\mu(\subseteq\nu)\in\Lambda^+} \mathsf{S}^+(\mu) \times \mathcal{F}^\nu_{\mu\delta}, \quad (R_\delta, T) \mapsto \left({}^{R_\delta}T, (R_\delta)_T\right),$$

where $\mathsf{S}(\nu^\circ)$ denotes the set of standard Young tableaux of unshifted shape ν° and $\mathsf{S}^+(\mu)$ denotes the set of standard shifted Young tableaux of shape μ.

(f) (The Littlewood–Richardson rule) Let $\alpha, \beta \in \Lambda$ with $\alpha \supseteq \beta$, and $\delta = \delta_{\ell(\alpha)}$. Then (c) gives rise to a bijection

$$\Sigma : \bigcup_{\lambda\in\Lambda^+} \mathcal{F}^{\beta+\delta}_{\delta\,\lambda} \times \widetilde{\mathcal{Y}}((\alpha+\delta)/(\beta+\delta)) \longrightarrow \bigcup_{\delta\subseteq\mu\in\Lambda^+} \widetilde{\mathcal{Y}}(\mu/\delta) \times \mathcal{F}^{\alpha+\delta}_{\mu\,\lambda},\ (S,T) \mapsto \left({}^{S}T, S_T\right).$$

It gives a combinatorial interpretation of

$$\sum_{\lambda\in\Lambda^+} g_{\lambda\beta} S_{\alpha/\beta}(x) = \sum_{\mu\in\Lambda^+} f^{\alpha+\delta}_{\mu\,\lambda} S_{\mu-\delta}(x).$$

In particular, if $\lambda = \beta$, then no switches (S3), (S4), (S7) occur in the shifted switching process, so we can say that the above bijection is afforded by the ordinary(=unshifted) switching process from (R_μ, T) to $({}^{R_\mu}T, (R_\mu)_T)$. Hence, by [1, Theorem 3.1], it also gives a combinatorial interpretation of

$$S_{\alpha/\beta} = \sum_{\gamma\in\Lambda} c^\alpha_{\gamma\beta} S_\gamma, \tag{3}$$

where $c^\alpha_{\gamma\beta}$ are the (ordinary) Littlewood–Richardson coefficients.

(g) (Schur expansions of Schur Q-functions) Let $\lambda \in \Lambda^+$. Recall that $P_\lambda = \sum_{\gamma\in\Lambda} g_{\lambda\gamma} s_\gamma$. So we have

$$\begin{aligned} Q_\lambda(x) &= 2^{\ell(\lambda)} P_\lambda(x) = 2^{\ell(\lambda)} \sum_{\alpha\in\Lambda} g_{\lambda\,\alpha} s_\alpha(x) = 2^{\ell(\lambda)} \sum_{\alpha\in\Lambda} f^{\alpha+\delta}_{\delta\lambda} s_\alpha(x) \\ &= \sum_{\alpha\in\Lambda} 2^{\ell(\lambda)+\ell(\delta)-\ell(\delta+\alpha)} f^{\alpha+\delta}_{\delta\lambda} s_\alpha(x) = \sum_{\alpha\in\Lambda} \widetilde{f}^{\alpha+\delta}_{\delta\lambda} s_\alpha(x). \end{aligned}$$

Here δ's are staircase partitions for α's with $\ell(\alpha) \geq \ell(\alpha)$. We denote by $\widetilde{g}_{\lambda\alpha}$ the coefficient $\widetilde{f}^{\alpha+\delta}_{\delta\lambda}$ of $s_\alpha(x)$.

(h) (Schur expansions of the products of Schur and Schur P-(or Q-)functions) With the hypothesis in (f) it follows that Eq. (3) yields a bijection

$$\Sigma : \mathcal{C}^{\beta}_{\emptyset\beta} \times \mathcal{F}^{\alpha+\delta}_{\beta+\delta\,\lambda} \longrightarrow \bigcup_{\gamma\in\Lambda} \mathcal{F}^{\gamma+\delta}_{\delta\,\lambda} \times \mathcal{C}^{\alpha}_{\gamma\beta}, \qquad (R_\beta, T) \mapsto ({}^{R_\beta}T, (R_\beta)_T),$$

which gives

$$f^{\alpha+\delta}_{\beta+\delta\,\lambda} = \sum_{\gamma\in\Lambda} g_{\lambda\gamma}\, c^{\alpha}_{\gamma\beta}. \tag{4}$$

Recall that $s_\gamma s_\beta = \sum_{\alpha\in\Lambda} c^{\alpha}_{\gamma\beta} s_\alpha$. So we have

$$\begin{aligned} s_\beta P_\lambda(x) &= s_\beta \sum_{\gamma\in\Lambda} g_{\lambda\gamma} s_\gamma(x) = \sum_{\gamma\in\Lambda} g_{\lambda\gamma} \left(\sum_{\alpha\in\Lambda} c^{\alpha}_{\beta\gamma} s_\alpha(x) \right) \\ &= \sum_{\alpha\in\Lambda} \left(\sum_{\gamma\in\Lambda} g_{\lambda\gamma} c^{\alpha}_{\beta\gamma} \right) s_\alpha(x) = \sum_{\alpha\in\Lambda} f^{\alpha+\delta}_{\beta+\delta\,\lambda}\, s_\alpha(x). \end{aligned}$$

The last equality is followed by Eq. (4) and the symmetry of Littlewood–Richardson coefficients. By using shifted tableau switching it gives a combinatorial proof of [15, Theorem 4.1].

Similarly, we have that from (g) and (b)

$$\begin{aligned} s_\beta Q_\lambda(x) &= s_\beta \sum_{\gamma\in\Lambda} \widetilde{g}_{\lambda\gamma} s_\gamma(x) = \sum_{\gamma\in\Lambda} \widetilde{g}_{\lambda\gamma} \left(\sum_{\alpha\in\Lambda} c^{\alpha}_{\beta\gamma} s_\alpha(x) \right) \\ &= \sum_{\alpha\in\Lambda} \left(\sum_{\gamma\in\Lambda} \widetilde{g}_{\lambda\gamma} c^{\alpha}_{\beta\gamma} \right) s_\alpha(x) = \sum_{\alpha\in\Lambda} \widetilde{f}^{\alpha+\delta}_{\beta+\delta\,\lambda}\, s_\alpha. \end{aligned}$$

(i) (The $\mathbb{Z}_2$-symmetries of shifted Littlewood–Richardson coefficients) Let $\alpha, \beta \in \Lambda$ with $\alpha \supseteq \beta$ and $\delta = \delta_{\max\{\alpha_1, \ell(\alpha)\}}$. We write $g^{\alpha/\beta}_{\lambda}$ instead of $f^{\alpha+\delta}_{\beta+\delta\,\lambda}$ in (f) briefly. It follows from (g) that

$$S_{\alpha/\beta} = \sum_{\lambda\in\Lambda^+} g^{\alpha/\beta}_{\lambda} Q_\lambda.$$

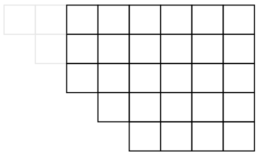
Let γ' be the conjugate partition of γ and $(\alpha/\beta)^\pi$ be the skew diagram obtained from α/β by rotating 180 degree. From the result $S_{\alpha/\beta} = S_{\alpha'/\beta'} = S_{(\alpha/\beta)^\pi}$ (see [5, Proposition IV.15]), we have

$$g^{\alpha/\beta}_{\lambda} = g^{(\alpha/\beta)'}_{\lambda} = g^{(\alpha/\beta)^\pi}_{\lambda}.$$

(j) (The Schur Q-expansion of Schur S-functions of truncated shapes) For positive integers r, t with $r > t$, let $\delta_{r,t}$ be the partition $(r-1, r-2, \ldots, t)$.

In particular, $\delta_{r,1} = \delta_r$. For $1 \le s \le r - t$, let $\Delta^r_{s,t}$ be the shifted diagram $S(\delta_{r,t}/\delta_s)$. For example,

$$\Delta^9_{3,4} =$$



.

Similar to (a), the shifted tableau switching affords a bijection

$$\Sigma : \mathcal{F}^{\delta_s}_{\emptyset\delta_s} \times \widetilde{\mathcal{Y}}(\Delta^r_{s,t}) \longrightarrow \bigcup_{\substack{\lambda\in\Lambda^+ \\ |\lambda|=|\Delta^r_{s,t}|}} \widetilde{\mathcal{Y}}(\lambda) \times \mathcal{F}^{\delta_{r,t}}_{\lambda\,\delta_s}, \quad (R_{\delta_s}, T) \mapsto \left({}^{R_{\delta_s}}T, (R_{\delta_s})_T\right).$$

For the same reason of (b) we have $f^{\delta_{r,t}}_{\lambda\,\delta_s} = f^{\delta_{r,t}}_{\delta_s\,\lambda}$. By Theorem 4.1, for each $i \ge 1$ there is only one way to fill in the boxes of hook shape

$$(\{(i, s-1+i+y) : y \ge 0\} \cup \{(i+x, s-1+i) : x \ge 1\}) \cap S(\Delta^r_{s,t})$$

by entries i, i'. So we have

$$f^{\delta_{r,t}}_{\delta_s\,\lambda} = \begin{cases} 1 & \text{if } \lambda = (r-1, r-2, \ldots, s+t-1, s+t-3, \ldots, s+t-2\min\{s,t\}+1), \\ 0 & \text{otherwise.} \end{cases}$$

It gives a combinatorial interpretation of

$$Q_{\Delta^r_{s,t}}(x) = Q_{(r-1,\ldots,s+t-1,s+t-3,\ldots,s+t-2\min\{s,t\})+1)}.$$

Specially, if $r = s + t$, then $\Delta^r_{s,t}$ is of rectangular shape (t^s), so we have $Q_{\Delta^r_{s,t}}(x) = S_{(t^s)}(x) = Q_{(r-1,r-3,\ldots,r-2\min\{s,t\}+1)}(x)$. (cf. Dewitt [5, Eq. (4.5)], Worley [19, Eq. (7.19)])

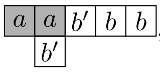
(k) (Content-reversing involutions on ShYTs and SShYTs) In [16, Chapter 6] Stembridge gave a combinatorial proof that Schur Q-functions are symmetric. To do this he showed that there is a content-reversing involution on the set of ShYTs. We remark that our shifted tableau switching also give a combinatorial proof of the symmetry of Schur Q-functions. More precisely, for an ShYT T we can define a content-reversing involution by performing the shifted switching process on $(T^{(i)}, T^{(i+1)})$ and then interchanging $i \leftrightarrow i+1, i' \leftrightarrow (i+1)'$. It should be emphasized that the involution coming from our shifted tableau switching is different to some extent from Stembridge's involution.

Moreover, the set of SShYTs is not closed under this involution, thus Stembridge's method does not induce a combinatorial proof that Schur P-functions are symmetric. So the modified shifted tableau switching (Theorem 6.2) give a combinatorial proof of the symmetry of Schur P-functions. Also we remark that Cho [3] gave another combinatorial proof of the symmetry of Schur P-functions by giving a Bender-Knuth type involution on the set of semistandard decomposition tableaux.

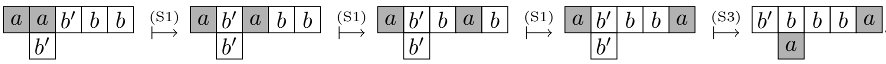
Remark 8.1. In [18], Thomas-Yong devised a combinatorial tool called the infusion to provide a root-theoretic generalization of the Littlewood–Richardson rule for the Schubert calculus of (co)minuscule variety. More precisely, for a minuscule flag variety G/P, they introduced notion of the rectification of a standard tableau that depends on the root system, and then showed that the Schubert intersection number $c^{\nu}_{\lambda,\mu}(G/P)$ equals the number of standard tableaux of shape ν/λ whose rectification is a fixed standard tableau of shape μ. They also showed symmetric properties that $c^{\nu}_{\lambda,\mu}(G/P)$ satisfies (see [18, Corollary 4.7]). In the case of isotropic Grassmannian, their generalization coincides with the shifted Littlewood–Richardson rule [17, Lemma 8.4] and the symmetry of $c^{\nu}_{\lambda,\mu}(G/P)$ with the symmetry of shifted LR coefficients described in (b). Although the shifted tableau switching can be recovered from the infusion map of type C with the semistandardization procedure for ShYTs introduced in [10, Section 9] by Pechenik and Yong, the modified shifted tableau switching cannot be recovered, which is an extended result about it.

Appendix A. Proof of Proposition 3.3

To begin with, we provide two lemmas which are required to prove Proposition 3.3. Suppose that (A, B) is a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair such that B extends A. For each $\mathbf{a}$-box in $A \cup B$, define *its switching path* by the set of boxes it passes through during the shifted switching process. For example, if

$A \cup B =$  ,

then the switching path of the $(1, 2)$ box is $\{(1, 2), (1, 3), (1, 4), (1, 5)\}$ and that of the $(1, 1)$ box is $\{(1, 1), (1, 2), (2, 2)\}$ since



Lemma A.1. *Let (A, B) be a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair such that B extends A. Then the switching path α of the (i, j) box in A has one of the following forms*:

T1 $\{(i, j), \ldots, (i, j + y)\}$ *for some* $y \geq 0$ (*horizontal shape*)
T2 $\{(i, j), \ldots, (i + x, j)\}$ *for some* $x \geq 1$ (*vertical shape*)
T3 $\{(i, j), (i, j + 1), (i + 1, j + 1)\}$ (*hook shape*)

Proof. Since $A \cup B$ is the union of two border strip ShYTs, it suffices to show that no paths contain the following type subsets:

E1 $\{(s, t), (s + 1, t), (s + 1, t + 1)\}$
E2 $\{(s, t), (s, t + 1), (s, t + 2), (s + 1, t + 2)\}$
E3 $\{(s, t), (s, t + 1), (s + 1, t + 1), (s + 2, t + 1)\}$

We first deal with the case where the (i, j) box in A is $\square(A \cup B)$. When α contains **E1**, the $(s, t + 1)$ box is filled with b, which contradicts Definition 3.1 since the $(s + 1, t + 1)$ box is in B. When α contains **E2**, the $(s, t + 1), (s, t +$

$2), (s+1,t), (s+1,t+1)$ and $(s+1,t+2)$ boxes should be contained in B. This cannot occur since B is a b-border strip ShYT. The case where α contains **E3** can be treated in the same manner.

Next we deal with the case where the (i,j) box in A is not $\square(A\cup B)$. When the (i,j) box in A is $\square(\sigma^k(A\cup B))$ for some k, our assertion follows from the above argument. The case where this fails occurs only when $j=i$ and the (i,i) box in A is moved to the $(i,i+1)$ position by either (S4) or (S7). But, in this case, the $(i,i+2)$ box is empty or filled with b. It implies that the $(i+1)$st row consists of only one $\mathbf{b}$-box, hence α is horizontal. $\square$

Given a shifted perforated pair (A,B) such that B extends A, let us identify a'-boxes and a-boxes with o$'$-holes and o-holes, respectively. We remark that the switching path of a given $\mathbf{a}$-box is exactly same to its sliding path defined in [1, p.82] except that the final position of the switching path and the sliding path may be different. More precisely, if either (S4) or (S7) occurs in the shifted switching process of a given unprimed a-box, then the final position of its switching path is filled with a'. On the contrary, the final position of the corresponding o-hole is still an o-hole (see Remark 3.8(a)).

Suppose that a set α of boxes and a box B are contained in a shifted diagram. When α contains a box which is strictly east and weakly north of B, we say that B lies *west* of α. Similarly, when α contains a box which is strictly south and weakly west of B, we say that B lies *north* of α. We can derive the same result with [1, Lemma 2] by replacing sliding paths by switching paths.

Lemma A.2 (cf. [1, Lemma 2]). *Let (A,B) be a shifted perforated $(\mathbf{a},\mathbf{b})$-pair such that B extends A. Suppose that the switching path β from (s,t) occurs directly after the switching path α from (i,j). For any $(s',t')\in\beta$,*

(a) *if (s',t') lies west of α, and (s',t') is not on the main diagonal of A, then at the next step (s',t') will lie west of α.*
(b) *if (s',t') lies north of α, then the same is true at the next step.*

Proof of Proposition 3.3. The only nontrivial part of the proof is to see that $\sigma^m(A\cup B)^{(a)}$ and $\sigma^m(A\cup B)^{(b)}$ are shifted perforated $\mathbf{a}$ and $\mathbf{b}$-tableau for all $m=1,\ldots,n$, respectively. For a fixed inner corner of B let us move this box until it becomes a removable corner by applying a sequence of shifted slides. In each step one can see that the corresponding filling satisfies the four conditions in Definition 3.1. Therefore the final filling obtained by discarding the new removable corner in the above is a b-border strip ShYT. By repeating this process, one can see that $\sigma^m(A\cup B)^{(b)}$ is a shifted perforated $\mathbf{b}$-tableau.

Next we will show that $\sigma^m(A\cup B)^{(a)}$ is a shifted perforated $\mathbf{a}$-tableau. For the box $\square(A\cup B)$, let α_1 be the switching path starting from this box. When α_1 is of type **T1**, the perforateness is obviously being preserved in the corresponding switching process. In case where α_1 is of type **T2**, if (S4) or (S7) does not occur in the switching process, then the perforateness is obviously being preserved. If (S4) or (S7) occurs in the switching process, then this switch will be used the final step. Note that $\alpha_1 = \{(i,j),(i+1,j),\dots,(j,j)\}$ for some $i<j$. Since **b**-boxes $(i+1,j+1),\dots,(j,j+1)$ cannot be contained in $A\cup B$ and the box $(j-1,j)$ will be filled a', so the perforateness is also being preserved. In case where α_1 is of type **T3** one can easily see the preservation of the perforateness similar to the case **T2**.

Assume that for some $r\geq 1$, $\sigma^r(A\cup B)$ is a shifted perforated pair such that the **a**-boxes in A corresponding from 1 to k as the order $\mathsf{stan}^{\mathbf{r}}(A)$ are fully switched. Let $(s,t) := \square((A\cup B)^r)$. Suppose that the switching path α occurs directly before the switching path β passing through (s,t). When β and α are intersected by (S4) or (S7), one can see that α is vertical whose final position is on the main diagonal and β is horizontal whose initial position is on the main diagonal. Lemma A.2(b) implies that all boxes in β lie north of α and they are filled with a' except the initial position of β. So the assertion holds. In the other cases Lemma A.2 says the perforateness is being preserved. This completes the proof. □

Appendix B. Proof of Lemma 4.4

We first give necessary lemmas. Let A be an ShYT consisting of $(k+1)$ **a**-boxes. Define $A^{[k]}$ to be the subtableau of A occupied by k in $\mathsf{stan}^{\mathbf{r}}(A)$ and

$$A^{[\leq k]} := A^{[k]}\cup A^{[k-1]}\cup\cdots\cup A^{[1]}.$$

Then, by definition, $A = A^{[k+1]}\cup A^{[\leq k]}$ and $A^{[\leq k]}$ extends $A^{[k+1]}$. If two shifted perforated $(\mathbf{a},\mathbf{b})$-pairs (A,B) and (A',B') are same when we ignore the (i,j) box, then we use the following expression

$$(A,B)\approx (A',B') \quad \text{up to } (i,j).$$

Lemma B.1. *Let (A,B) be a shifted perforated $(\mathbf{a},\mathbf{b})$-pair such that $|A| = k+1$ $(k\geq 1)$ and B extends A. Let $A^{[k+1]} = A_{i,j}$. If neither* (S4) *nor* (S7) *occurs in the shifted switching process corresponding to the switching path starting with (i,j), then*

$$\Sigma({}^{A}B, A_B)\approx \Sigma({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \textit{up to } (i,j).$$

Proof. Let u be the smallest integer such that $\sigma^u(A^{[\leq k]},B)$ gets fully switched. Since $A^{[k+1]}$ remains unchanged in applying σ^u to (A,B), it follows that $\square(\sigma^u(A,B)) = A^{[k+1]} = A_{i,j}$. When $A^{[k+1]}$ is fully switched in $\sigma^u(A,B)$, there is nothing to prove. Hence we assume that $A^{[k+1]}$ is not fully switched. Then, by Lemma A.1, the switching path at (i,j) has one of the following types:

T1 $\{(i,j),\dots,(i,j+y)\}$ for some $y\geq 1$
T2 $\{(i,j),\dots,(i+x,j)\}$ for some $x\geq 1$
T3 $\{(i,j),(i,j+1),(i+1,j+1)\}$

In case of **T1**, $\sigma^{u+y}(A,B)$ is fully switched and $(\sigma^{u+y}(A,B))_{i-1,j+y}$ is a b'-box or an empty box since $(\sigma^u(A,B))_{i,j+y}$ is a **b**-box. For $1\leq s\leq y$, let v_s be the integer uniquely determined by the condition

$$\square(\sigma^{v_s}({}^A B, A_B)) = (i, j+y-s).$$

Then we have

$$\sigma^{v_1+1}({}^A B, A_B) \approx \sigma^{v_1}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,j), \ldots, (i, j+y-1).$$

More generally, for $1 \leq s \leq y$ and $v_s \leq r \leq v_{s+1} - 1$, it is not difficult to see that

$$\sigma^{r+1}({}^A B, A_B) \approx \sigma^{r+1-s}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,j), \ldots, (i, j+y-s).$$

It should be noticed that we are regarding v_{y+1} as a sufficiently large integer. This implies that $\big(\Sigma({}^A B, A_B)\big)_{i,j} = A^{[k+1]}$ and

$$\Sigma\left({}^A B, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,j)$$

because the (i,j) box is not in $\Sigma({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B)$.

In case of **T2**, $\sigma^{u+x}(A,B)$ is fully switched. For $1 \leq s \leq x$, let v_s be the integer uniquely determined by the condition

$$\square(\sigma^{v_s}({}^A B, A_B)) = (i+x-s, j).$$

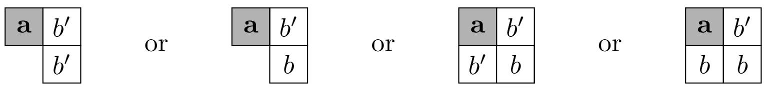
For $1 \leq s \leq x$ and $v_s \leq r \leq v_{s+1} - 1$,

$$\sigma^{r+1}({}^A B, A_B) \approx \sigma^{r+1-s}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,j), \ldots, (i+x-s, j).$$

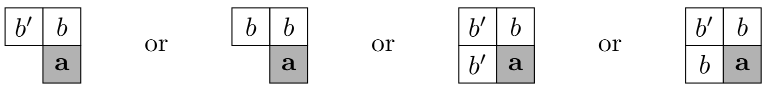
Here we are regarding v_{x+1} as a sufficiently large integer. This equivalence holds since $\square(\sigma^{v_1}({}^A B, A_B)) = (i+x-1, j)$ is switched with an **a**-box placed at $(i+x, j)$ regardless of the $(i+x, j-1)$ box. As in case of **T1**, it holds that $\big(\Sigma({}^A B, A_B)\big)_{i,j} = A^{[k+1]}$ and

$$\Sigma\left({}^A B, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,j).$$

Finally, in case of **T3**, $\sigma^u(A,B)$ must contain one of the following subfillings:



It implies that $({}^A B, A_B)$ must contain one of the following subfillings:



Note that in the first two subfillings, the upper leftmost box denotes the (i,i) box in $\sigma^u(A,B)$ and $\left({}^A B, A_B\right)$, respectively. In particular, $A^{[k+1]} = A_{i,i}$. Let v be the integer uniquely determined by the condition

$$\square\left(\sigma^v\left({}^A B, A_B\right)\right) = (i, i+1).$$

Applying (S4) to $\square(\sigma^v({}^AB, A_B))$, we can deduce that

$$\sigma^{v+1}\left({}^AB, A_B\right) \approx \sigma^v\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,i).$$

Furthermore, since the (i,i) **a**-box in $\sigma^{v+1}\left({}^AB, A_B\right)$ cannot be switched with any **b**-box in $\sigma^{v+1}\left({}^AB, A_B\right)$, it follows that for $r \geq 1$,

$$\sigma^{v+1+r}\left({}^AB, A_B\right) \approx \sigma^{v+r}\left({}^{A^{[k]}}B, A^{[k]}{}_B\right) \quad \text{up to } (i,i).$$

For the remaining two subfillings, we let v_1 and v_2 be the integer uniquely determined by

$$\square\left(\sigma^{v_1}\left({}^AB, A_B\right)\right) = (i, j+1)$$

and

$$\square\left(\sigma^{v_2}\left({}^AB, A_B\right)\right) = (i, j),$$

respectively. Here the (i,j) box is the upper leftmost box in the each subfilling. Then

$$\sigma^{v_1+1}\left({}^AB, A_B\right) \approx \sigma^{v_1}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,j) \text{ and } (i, j+1).$$

Now consider any **b**-box on the southwest of (i,j). The switching path at (i,j) does not contain $(i, j+1)$. Moreover, the (i,j) **a**-box in $\sigma^{v_2+1}({}^AB, A_B)$ cannot be switched with any **b**-box in $\sigma^{v_2+1}({}^AB, A_B)$. Therefore, for $s = 1, 2$ and $v_s \leq r \leq v_{s+1} - 1$,

$$\sigma^{r+1}\left({}^AB, A_B\right) \approx \sigma^{r+1-s}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,j) \text{ and } (i, j+2-s).$$

Here we are regarding v_3 as a sufficiently large integer. Hence we showed for all subfillings above that

$$\Sigma\left({}^AB, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,j),$$

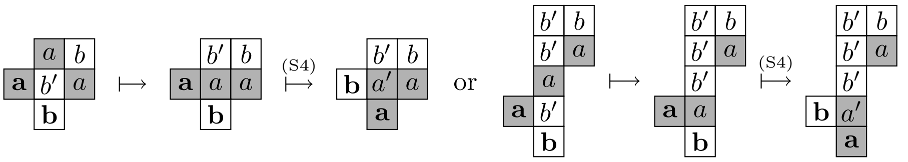
where $A^{[k+1]} = A_{i,j}$. $\square$

Lemma B.2. *Let (A,B) be a shifted perforated $(\mathbf{a},\mathbf{b})$-pair such that $|A| = k+1$ $(k \geq 1)$ and B extends A. Let $A^{[k+1]} = A_{i,i}$. If either* (S4) *or* (S7) *occurs in the shifted switching process corresponding to the switching path starting with (i,i), then*

$$\Sigma\left({}^AB, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \textit{up to } (i,i).$$

Proof. Let u be the smallest integer such that $\sigma^u(A^{[\leq k]}, B)$ is fully switched. Then the uth switch should be (S4) or (S7) and by which $A^{[k+1]}$ is moved from (i,i) to $(i, i+1)$.

When (S4) is applied to $\sigma^{u-1}(A,B)$, the $(i, i+1)$ box in $\sigma^{u-1}(A,B)$ is filled with a and this box is in the switching path at $A^{[k]}$. In particular, the $(i, i+2)$ box in $\sigma^{u-1}(A,B)$ is either an a-box or an empty box and the ith column in $A \cup B$ has only a single box at (i,i). Refer to the following figure:



Hence $\sigma^u(A,B)$ is fully switched and

$$\left({}^{A}B, A_B\right) \approx \left({}^{A^{[\le k]}}B, {A^{[\le k]}}_B\right) \quad \text{up to } (i,i),\ (i,i+1) \text{ and } (i+1,i+1).$$

Let v be the integer such that $\square(\sigma^v({}^{A}B, A_B)) = (i,i)$. We see that $\square(\sigma^v({}^{A^{[\le k]}}B, {A^{[\le k]}}_B)) = (i,i+1)$ and for $0 \le r \le v$,

$$\sigma^r({}^{A}B, A_B) \approx \sigma^r({}^{A^{[\le k]}}B, {A^{[\le k]}}_B) \quad \text{up to } (i,i),\ (i,i+1) \text{ and } (i+1,i+1).$$

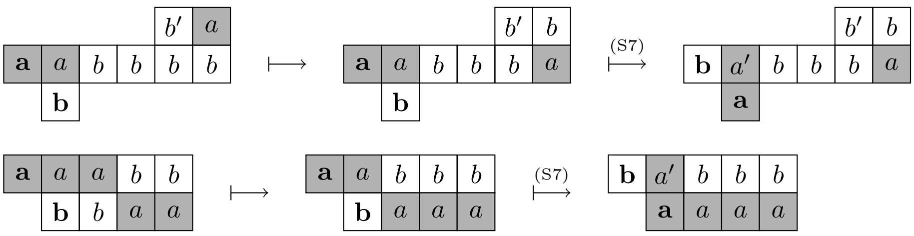
Since (S2) and (S3) are applied to $\square(\sigma^v({}^{A^{[\le k]}}B, {A^{[\le k]}}_B))$ and $\square(\sigma^v({}^{A}B, A_B))$, respectively, it follows that

$$\sigma^{v+1}({}^{A}B, A_B) \approx \sigma^{v+1}({}^{A^{[\le k]}}B, {A^{[\le k]}}_B) \quad \text{up to } (i,i).$$

This implies that whenever $r \ge v+1$, the switch to be applied to $\sigma^r({}^{A}B, A_B)$ should be same to that to be applied to $\sigma^r({}^{A^{[\le k]}}B, {A^{[\le k]}}_B)$. Thus we see that

$$\Sigma\left({}^{A}B, A_B\right) \approx \Sigma\left({}^{A^{[\le k]}}B, {A^{[\le k]}}_B\right) \quad \text{up to } (i,i).$$

When (S7) is applied to $\sigma^{u-1}(A,B)$, then $\square(\sigma^{u-1}(A,B)) = (i,i+1)$. Moreover, $(i+1,i+1)$ and $(i,i+2)$ in $\sigma^{u-1}(A,B)$ respectively are filled with $\mathbf{b}$ and b. We claim that $A^{[k]} = A_{i,i+1}$. Suppose that $A^{[k]} \ne A_{i,i+1}$. Then $(i-1,i+1)$ is in the switching path at $A^{[k]}$ and filled with a in $\sigma^{u-2}(A,B)$. Now let us investigate the $(i-1,i+2)$ box in $\sigma^{u-2}(A,B)$. If it is filled with a, then $\square(\sigma^{u-2}(A,B)) = (i-1,i+2)$ and thus $\square(\sigma^{u-1}(A,B)) = (i-1,i+1)$ or $(i-1,i+3)$. If it is filled with b', then the $(u-1)$st switch is (S1) and $\square(\sigma^{u-1}(A,B)) = (i-1,i+2)$. This shows that the $(i-1,i+2)$ box should be filled with b in $\sigma^{u-2}(A,B)$, thus in $\sigma^{u-1}(A,B)$. It is obviously a contradiction to the fact that the $(i,i+2)$ box in $\sigma^{u-1}(A,B)$ is filled with b. Hence we verified the claim. In addition, by using this property, we see that the ith column in $A \cup B$ has only a single box at (i,i). Refer to the following figure:



Suppose that the switching path at (i,i) is $\{(i,i),\ldots,(i,i+y)\}$ for some $y \geq 2$. Then $\sigma^{u+y-1}(A,B)$ is fully switched and

$$({}^{A}B, A_B) \approx ({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \text{ up to } (i,i),\ldots,(i,i+y) \text{ and } (i+1,i+1).$$

For $1 \leq s \leq y$, let v_s be the integer determined by the condition

$$\Box(\sigma^{v_s}({}^{A}B, A_B)) = (i, i+y-s).$$

Then, for $1 \leq s \leq y-1$ and $v_s \leq r \leq v_{s+1}-1$,

$$\sigma^{r+1}({}^{A}B, A_B) \approx \sigma^{r+1-s}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \text{ up to } (i,i),\ldots,(i,i+y-s) \text{ and } (i+1,i+1).$$

Applying σ to the case where $s = y-1$ and $r = v_y - 1$, one can see that

$$\sigma^{v_y+1}({}^{A}B, A_B) \approx \sigma^{v_y-y+2}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B) \quad \text{up to } (i,i).$$

This equivalence implies that for each $r \geq 1$, the switch to be applied to $\sigma^{v_y+r}({}^{A}B, A_B)$ is equal to that to be applied to $\sigma^{v_y-y+r+1}({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B)$. Thus we see that

$$\Sigma\left({}^{A}B, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,i).$$

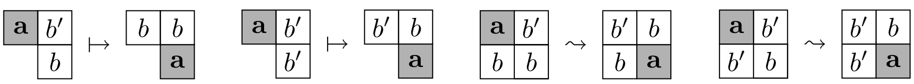
$\Box$

Proof of Lemma 4.4. We prove the assertion by induction on the number of boxes in A. To begin with, consider the case where A consists of a single box. Let (i,j) be its position and α be the switching path starting with (i,j). As in the proof of Lemma B.1, the type of α is one of **T1**, **T2** and **T3**. Let u be the smallest integer such that $\sigma^u(A,B)$ is fully switched.

In case of **T1**, when the $(i+1,j)$ box is filled with $\mathbf{b}$ in B, the $(i,j+1)$ box is filled with b' and all boxes $(i,j+2)$ through $(i,j+y)$ should be filled with b. On the other hand, when the $(i+1,j)$ box is not in B, the $(i,j+1)$ box is filled with $\mathbf{b}$ and all boxes $(i,j+2)$ through $(i,j+y)$ should be filled with b. In either case, since the $(i-1,j+y)$ box is a b'-box or an empty box, it follows that $\sigma^i({}^{A}B, A_B) = \sigma^{u-i}(A,B)$ for all $1 \leq i \leq u$. Thus $\sigma^u({}^{A}B, A_B) = (A,B)$.

In case of **T2**, all boxes $(i+1,j)$ through $(i+x-1,j)$ are filled with b', whereas the $(i+x,j)$ box is filled with $\mathbf{b}$. When the $(i+x,j-1)$ box is filled with $\mathbf{b}$ in B, the $(i+x,j)$ box is filled with b. On the other hand, when the $(i+x,j-1)$ box is not in B, it is filled with $\mathbf{b}$. In either case, since $\Box({}^{A}B, A_B) = (i+x-1,j)$, it follows that $\sigma^i({}^{A}B, A_B) = \sigma^{u-i}(A,B)$ for all $1 \leq i \leq u$. Thus $\sigma^u({}^{A}B, A_B) = (A,B)$.

Finally, one can easily see that the **T3** case can occur only in the following subfillings:



For all subfillings above, it is straightforward to see that $\Sigma({}^{A}B, A_B) = (A,B)$.

We now assume that the desired result holds for all ShYTs of border strip shape with k boxes. Suppose that A has $k+1$ boxes. Letting $A^{[k+1]} = A_{i,j}$, by Lemma B.1 and Lemma B.2 we have

$$\Sigma\left({}^{A}B, A_B\right) \approx \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) \quad \text{up to } (i,j).$$

Moreover, from induction hypothesis it follows that

$$\Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right) = \left(A^{[\leq k]}, B\right).$$

It follows that

$$\begin{aligned}\Sigma\left({}^{A}B, A_B\right) &= A^{[k+1]} \cup \Sigma\left({}^{A^{[\leq k]}}B, A^{[\leq k]}{}_B\right)\\ &= A^{[k+1]} \cup \left(A^{[\leq k]}, B\right)\\ &= (A, B),\end{aligned}$$

as required. □

Appendix C. Proof of Proposition 6.1

Proof of Proposition 6.1. (a) Suppose that $\widetilde{\sigma}$ is applied to $\sigma^k(A,B)$ for some $k \geq 0$. Here $\sigma^k(A,B)$ is a shifted perforated $(\mathbf{a},\mathbf{b})$-pair by Proposition 3.3.

When $\widetilde{\sigma}$ =(S1′) is applied to the (i,i) a-box in $\sigma^k(A,B)$, then $(\widetilde{\sigma}\circ\sigma^k(A,B))_{i,i} = b$ and $(\widetilde{\sigma}\circ\sigma^k(A,B))_{i,i+1} = a'$. Moreover, the $(i+1,i+1)$ box is not contained in $\sigma^k(A,B)$ since if it is filled with b, then the switch to be applied to $\sigma^k(A,B)$ is not (S1′) but (S3). Now suppose that another modified switch is applied to $\sigma^j \circ \widetilde{\sigma} \circ \sigma^k(A,B)$ for some $j \geq 0$. Then $\Box(\sigma^j \circ \widetilde{\sigma} \circ \sigma^k(A,B))$ is $(i-1,i)$ and has the neighbor to the south. Since $(\sigma^k(A \cup B))_{i,i+1} = b'$, it follows that $(\sigma^j \circ \widetilde{\sigma} \circ \sigma^k(A,B))_{i-1,i+1} = b'$. Therefore the switch to be applied to $\sigma^j \circ \widetilde{\sigma} \circ \sigma^k(A,B)$ is not a modified switch but (S5), which is absurd.

When $\widetilde{\sigma}$ =(S2′) or (S6′) is applied to the $(i-1,i)$ a'-box in $\sigma^k(A,B)$, then $(\widetilde{\sigma}\circ\sigma^k(A,B))_{i,i} = a$, $(\widetilde{\sigma}\circ\sigma^k(A,B))_{i-1,i} = b'$, and the $(i-1,i-1)$ box is not contained in $\sigma^k(A,B)$ by Definition 3.1. By the occurrence of (S2′) or (S6′), the $(i,i+1)$ box in $\sigma^k(A,B)$ cannot be filled with $\mathbf{b}$, so the $(i,i+1)$ box in $\sigma^k(A,B)$ is an a-box or an empty box. This implies that the (i,i) a-box in $(\widetilde{\sigma}\circ\sigma^k(A,B))$ is fully switched and is a unique box on the main diagonal in $(\widetilde{\sigma}\circ\sigma^k(A,B))$. As a result, no modified switches can be applied to $(\sigma^j \circ \widetilde{\sigma} \circ \sigma^k)(A,B)$ for all $j \geq 0$.

(b) If a modified switch is not used in the modified shifted switching process, there is nothing to prove. Otherwise, by (a), each $(\mathbf{a},\mathbf{b})$-pair appearing in the modified shifted switching process can be written as $\sigma^j \circ \widetilde{\sigma} \circ \sigma^k(A,B)$ for some nonnegative integers j,k. Due to Proposition 3.3, we have only to see that there exists a pair of ShYTs $(\widehat{A},\widehat{B})$ such that $\widehat{B}$ extends $\widehat{A}$ and

$$\widetilde{\sigma}\circ\sigma^k(A,B) = \sigma^{k+1}(\widehat{A},\widehat{B}).$$

Set $(A', B') := \sigma^k(A, B)$.

First, suppose that $\widetilde{\sigma} =$ (S1$'$) is applied to the (i,i) a-box in A'. Then

$$\widetilde{\sigma}(A', B') \approx \sigma(A', B') \quad \text{up to } (i,i) \text{ and } (i, i+1),$$

where $\sigma =$ (S1). Let B'' be the filling obtained from B' by changing b' into b at $(i, i+1)$. It is a shifted perforated **b**-tableau since the $(i+1, i+1)$ box is not contained in B'. Similarly, let A'' be the filling obtained from A' by changing a into a' at (i,i). It is also a shifted perforated **a**-tableau with a primed letter on the main diagonal since the (i,i) a-box is the westernmost box in A'. Moreover, since the (i,i) a'-box is the southwesternmost box in A'', it follows that $\square(A'' \cup B'') = (i,i)$ and thus

$$\widetilde{\sigma}(A', B') = \sigma(A'', B'').$$

Now define $(\widehat{A}, \widehat{B})$ to be the pair of ShYTs obtained from (A, B) by changing a into a' at the westernmost a-box in A and by changing b' into b at the southernmost b'-box in B. We claim that $A'' \cup B'' = \sigma^k(\widehat{A}, \widehat{B})$. Since the westernmost (i,i) a-box in A is not affected by σ^k and the southernmost b'-box of B has been moved to the $(i, i+1)$ b'-box in B' through σ^k,

$$\sigma^k(\widehat{A}, \widehat{B}) \approx A' \cup B' (\approx A'' \cup B'') \quad \text{up to } (i,i) \text{ and } (i, i+1).$$

The claim follows from the fact that the (i,i) box and $(i, i+1)$ in $\sigma^k(\widehat{A}, \widehat{B})$ are a' and b, respectively.

Second, suppose that $\widetilde{\sigma} =$(S2$'$) is applied to the $(i-1, i)$ a'-box in A'. Then

$$\widetilde{\sigma}(A', B') \approx \sigma(A', B') \quad \text{up to } (i-1, i) \text{ and } (i,i),$$

where $\sigma =$ (S2). Let A'' be the filling obtained from A' by changing a' into a at $(i-1, i)$. It is a shifted perforated **a**-tableau since no a'-boxes are in the ith row in A'. Similarly, let B'' be the filling obtained from B' by changing b into b' at (i,i). It is also a shifted perforated **b**-tableau with a primed letter on the main diagonal since the (i,i) b-box is the westernmost box in B'. In addition, since the $(i-1, i)$ box in A' is the southernmost a'-box and the $(i-1, i)$ box in A'' is the westernmost a-box, it follows that $\square(A' \cup B') = \square(A'' \cup B'') = (i-1, i)$, and thus

$$\widetilde{\sigma}(A', B') = \sigma(A'', B'').$$

Define $(\widehat{A}, \widehat{B})$ to be the pair of ShYTs obtained from (A, B) by changing a' into a at the southernmost a'-box in A and b into b' at the westernmost b-box in B. We claim that $A'' \cup B'' = \sigma^k(\widehat{A}, \widehat{B})$. Notice that the $(i, i+1)$ box is clearly not in $A \cup B$, so the (i,i) box is filled with b in $A \cup B$. Since (S2$'$) is applied to the $(i-1, i)$ a'-box, the (i,i) box in $A' \cup B'$ is filled with b. It means that the westernmost (i,i) b-box in B is not affected by σ^k. On the other hand, no boxes are in the $(i-1)$st column of $A \cup B$, so the southernmost a'-box in the ith column of A has been moved to the $(i-1, i)$ a'-box in A' through σ^k. Combining these properties yields that

$$\sigma^k(\widehat{A}, \widehat{B}) \approx A' \cup B' (\approx A'' \cup B'') \quad \text{up to } (i-1, i) \text{ and } (i,i).$$

The claim now follows from the fact that the $(i-1,i)$ box and (i,i) in $\sigma^k(\widehat{A},\widehat{B})$ are a and b', respectively.

Finally, in case where $\widetilde{\sigma} = (\mathrm{S6}')$ is applied to the $(i-1,i)$ a'-box in A', $(\widehat{A},\widehat{B})$ can be obtained in a similar way as in the second case. In this case, the $(i,i+1)$ box may be or may not be in $A \cup B$. In any case, it can be easily seen that the westernmost (i,i) b-box in B is not affected by σ^k and the southernmost a'-box in the ith column of A has been moved to the $(i-1,i)$ a'-box in A' through σ^k. □

Appendix D. Proof of Theorem 6.2

We give two auxiliary lemmas to prove this theorem. By using the following lemmas, it can be proved essentially in the same way as in the proof of Theorem 3.6.

Let A and B be ShYTs consisting of $\mathbf{a}$-boxes and $\mathbf{b}$-boxes, respectively. Define $\omega(A)$ by the ShYT obtained by priming the southwesternmost box in A. Recall that $(a')'$ is set to be a. In particular, for (A,B) such that B extends A, define $\Omega(A,B)$ by $(\omega(A),\omega(B))$.

Lemma D.1. *Suppose that (A,B) is a shifted perforated $(\mathbf{a},\mathbf{b})$-pair such that B extends A. If $\Sigma(A,B)$ is not equal to $\widetilde{\Sigma}(A,B)$, then we have*

(a) $\widetilde{\Sigma}(A,B) = \Sigma \circ \Omega(A,B)$.

(b) $\Omega \circ \widetilde{\Sigma}(A,B) = \Sigma(A,B)$, *or equivalently,* $\Omega\circ\Sigma = \Sigma\circ\Omega$.

Proof. (a) The proof of Proposition 6.1(b) shows that if a modified switch is applied to $\sigma^k(A,B)$ for some $k \geq 0$, then

$$\widetilde{\sigma}\circ\sigma^k(A,B) = \sigma^{k+1}(\omega(A),\omega(B)). \tag{5}$$

By Proposition 6.1(a), only nonmodified switches can be applied to either side of (5). Therefore we can conclude that

$$\sigma^l\circ\widetilde{\sigma}\circ\sigma^k(A,B) = \sigma^l\circ\sigma^{k+1}(\omega(A),\omega(B)) \quad \text{for all } l \geq 0.$$

(b) Note that

$$\begin{aligned}\Omega\circ\Sigma\circ\Omega(A,B) &= \Omega\circ\Sigma(\omega(A),\omega(B))\\ &= \Omega({}^{\omega(A)}\omega(B), \omega(A)_{\omega(B)})\\ &= (\omega({}^{\omega(A)}\omega(B)), \omega(\omega(A)_{\omega(B)})).\end{aligned}$$

Since $\Sigma(A,B) \neq \widetilde{\Sigma}(A,B)$ by assumption, the switches (S3), (S4) and (S7) cannot occur in the modified shifted switching process. This again implies that they cannot occur in the shifted switching process, too. In particular, since neither (S4) nor (S7) is not used in obtaining $({}^{A}\omega(B), A_{\omega(B)})$ from $(A,\omega(B))$, the southwesternmost box in A is moved to that in $A_{\omega(B)}$ by Lemma A.1 and these boxes are both primed or both unprimed. It follows that $\omega(A_{\omega(B)}) = \omega(A)_{\omega(B)}$. In addition, since (S3) is not used in the shifted jeu de taquin of B

depending on $\omega(A)$, the southwesternmost box in B is also moved to that in ${}^{\omega(A)}B$ and these two boxes are both primed or both unprimed. This implies that $\omega({}^{\omega(A)}B) = {}^{\omega(A)}\omega(B)$. Now combining Remark 3.8(iv) with the fact that $\mathsf{stan}(A) = \mathsf{stan}(\omega(A))$ and $\mathsf{stan}(B) = \mathsf{stan}(\omega(B))$, we can derive that

$$
\begin{aligned}
{}^{\omega(A)}B = {}^{\mathsf{stan}(\omega(A))}B = {}^{\mathsf{stan}(A)}B = {}^{A}B \quad \text{and} \\
A_{\omega(B)} = A_{\mathsf{stan}(\omega(B))} = A_{\mathsf{stan}(B)} = A_B.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\Omega \circ \widetilde{\Sigma}(A, B) &= \Big(\omega({}^{\omega(A)}\omega(B)), \omega(\omega(A)_{\omega(B)})\Big) \\
&= \Big({}^{\omega(A)}B, A_{\omega(B)}\Big) = ({}^{A}B, A_B).
\end{aligned}
$$

□

Lemma D.2. *Let (A, B) be a shifted perforated $(\mathbf{a}, \mathbf{b})$-pair such that B extends A. Then we have*

(a) $\widetilde{A_B} \cup \widetilde{{}^{B}A}$ *has the same shape as $A \cup B$.*

(b) $\widetilde{A_B}$ *extends* $\widetilde{{}^{A}B}$. *In addition,* $\widetilde{A_B}$ *and* $\widetilde{{}^{A}B}$ *are SShYTs of border strip shape.*

Proof. (a) The assertion follows from the fact that neither a switch nor a modified switch does not affect on the shape.

(b) When $\Sigma(A, B) = \widetilde{\Sigma}(A, B)$, there is nothing to prove. When $\Sigma(A, B) \neq \widetilde{\Sigma}(A, B)$, $\widetilde{\Sigma}(A, B)$ is equal to $\Omega \circ \Sigma(A, B)$ by Lemma D.1(b). It follows that $\widetilde{\Sigma}(A, B)^{(a)}$ extends $\widetilde{\Sigma}(A, B)^{(b)}$. Since no primed letters are on the main diagonal in $\widetilde{\Sigma}(A, B)$, we obtain the desired result. □

References

[1] G. Benkart, F. Sottile, and J. Stroomer, *Tableau switching: algorithms and applications*, J. Combin. Theory Ser. A **76** (1996), no. 1, 11–43.

[2] A. S. Buch, A. Kresch, and H. Tamvakis, *Littlewood-Richardson rules for Grassmannians*, Adv. Math. **185** (2004), no. 1, 80–90.

[3] S. Cho, *A new Littlewood-Richardson rule for Schur P-functions*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 939–972.

[4] S.-I. Choi, S.-Y. Nam, and Y.-T. Oh, *Bijections among combinatorial models for shifted Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **128** (2014), 56–83.

[5] E. A. DeWitt, *Identities Relating Schur s-Functions and Q-Functions*, ProQuest LLC, Ann Arbor, MI, 2012.

[6] D. Grantcharov, J. H. Jung, S.-J. Kang, M. Kashiwara, and M. Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 457–489.

[7] M. D. Haiman, *On mixed insertion, symmetry, and shifted Young tableaux*, J. Combin. Theory Ser. A **50** (1989), no. 2, 196–225.

[8] P. N. Hoffman and J. F. Humphreys, *Projective Representations of the Symmetric Groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1992.

[9] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, **16**, Addison-Wesley Publishing Co., Reading, MA, 1981.

[10] O. Pechenik and A. Yong, *Genomic tableaux*, J. Algebraic Combin. **45** (2017), no. 3, 649–685.
[11] B. E. Sagan, *Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), no. 1, 62–103.
[12] M. P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Canad. J. Math. **13** (1961), 117–128.
[13] ———, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94.
[14] L. Serrano, *The shifted plactic monoid*, Math. Z. **266** (2010), no. 2, 363–392.
[15] K. Shigechi, *Shifted tableaux and product of Schur's symmetric functions*, arXiv:1705.06437v1.
[16] J. R. Stembridge, *On symmetric functions and the spin characters of S_n*, in Topics in algebra, Part 2 (Warsaw, 1988), 433–453, Banach Center Publ., 26, Part 2, PWN, Warsaw, 1990.
[17] ———, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134.
[18] H. Thomas and A. Yong, *A combinatorial rule for (co)minuscule Schubert calculus*, Adv. Math. **222** (2009), no. 2, 596–620.
[19] D. R. Worley, *A theory of shifted Young tableaux*, ProQuest LLC, Ann Arbor, MI, 1984.

Seung-Il Choi
Research Institute of Mathematics
Seoul National University
Seoul 08826, Korea
Email address: `ignatioschoi@snu.ac.kr`

Sun-Young Nam
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: `synam.math@gmail.com`

Young-Tak Oh
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: `ytoh@sogang.ac.kr`