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LIMIT THEOREMS FOR HAWKES PROCESSES WITH UNIFORM IMMIGRANTS

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Abstract. Hawkes process is a self-exciting simple point process with clustering effect whose jump rate depends on its entire past history. We consider Hawkes processes with uniform immigrants which is a special case of the Hawkes processes with renewal immigrants. We study the limit theorems for Hawkes processes with uniform immigrants. In particular, we obtain a law of large number, a central limit theorem, and a large deviation principle.

1. Introduction and main results

1.1. Introduction

We start with a general description of Hawkes process introduced by Brémaud and Massoulié [3].

Let N be a simple point process on $\mathbb{R}$ and let $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ be an increasing family of σ-algebras. Any nonnegative $\mathcal{F}_t^{-\infty}$-progressively measurable process λ_t with

$$E\left[N(a,b]|\mathcal{F}_a^{-\infty}\right] = E\left[\int_a^b \lambda_s ds|\mathcal{F}_a^{-\infty}\right]$$

a. s. for all interval $(a,b]$ is called an $\mathcal{F}_t^{-\infty}$-intensity of N. We use the notation $N_t := N(0,t]$ to denote the number of points in the interval $(0,t]$.

A general Hawkes process is a simple point process N admitting an $\mathcal{F}_t^{-\infty}$-intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)N(ds)\right),$$

where $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and left continuous, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and we always assume that $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. Here $\int_{-\infty}^t h(t-s)N(ds)$

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stands for $\int_{(-\infty,t)} h(t-s)N(ds)$. We always assume that $N(-\infty,0]=0$, i.e., the Hawkes process has empty history. In the literatures, $h(\cdot)$ and $\lambda(\cdot)$ are usually referred to as exciting function and rate function respectively. The Hawkes process is linear if $\lambda(\cdot)$ is linear and it is nonlinear otherwise. In general, the model described above is non-Markovian since the future evolution of a self-exciting simple point process is controlled by the timing of past events but it is Markovian for a special case. Hawkes process has wide applications in neuroscience, seismology, DNA modeling, finance and many other fields. It has both self-exciting and clustering properties, which is very appealing to some financial applications. In particular, self-exciting and clustering properties of Hawkes process make it a viable candidate in modeling the correlated defaults and evaluating the credit derivatives in finance, for example, see Errais et al. [7] and Dassios and Zhao [5].

Hawkes [10] introduced the linear case, and the linear Hawkes process can be studied via immigration-birth representation, see e.g. Hawkes and Oakes [11]. The stability, law of large numbers, central limit theorem, large deviations, Bartlett spectrum etc. have all been studied and understood very well. Almost all of the applications of Hawkes process in the literatures consider exclusively the linear case. Because of the lack of immigration-birth representation and computational tractability, nonlinear Hawkes process is much less studied. However, some efforts have already been made in this direction. Nonlinear case was first introduced by Brémaud and Massoulié [3]. Recently, Zhu [23–25,27,28] investigated several results for both linear and nonlinear model. The central limit theorem was investigated in Zhu [23] and the large deviation principles have been obtained in Zhu [27] and Zhu [27]. Limit theorems and rough fractional diffusions as scaling limits for nearly unstable Hawkes processes was obtained in Jaisson and Rosenbaum [13,14]. Zhu [25] have also studied for applications to financial mathematics. Some variations and extensions of Hawkes process have been studied in Dassios and Zhao [5], Zhu [26], Karabash and Zhu [15], Mehrdad and Zhu [16] and Ferro, Leiva and Møller [8]. In the recent paper of Seol [18], he considers the arrival time τ_n, i.e., the inverse process of Hawkes process, and studies the limit theorems (Law of Large numbers, Central limit theorem and Large deviations) for τ_n. Recently, Seol [17] studied for the law of large numbers, central limit theorem and invariance principles for discrete Hawkes processes starting from empty history. Moderate deviation principle for marked Hawkes processes was investigated in Seol [19] and limit theorems for the compensator of Hawkes processes was studied by Seol [20]. In the literature, there have been studies extending and modifying the classical Hawkes process. First, The baseline intensity can be chosen to be time-inhomogeneous (see Gao, Zhou, and Zhu [9]). Second, the immigrants can arrive according to a Cox process with shot noise intensity, in which case the model is known as the dynamic contagion model (see Dassios and Zhao [5]). Third, the immigrants can arrive according to a renewal process instead of a

Poisson process, which generalizes the classical Hawkes process. This is known as the renewal Hawkes process (see Wheatley, Filimonov, and Sorrette [22]).

In this paper, we consider the Hawkes processes with uniform immigrants which is a special case of the renewal Hawkes process, by assuming that the immigrants arrive uniformly over time with intensity ν and an important statistics in many applications, and study the limit theorems for Hawkes processes with uniform immigrants.

The structure of this paper is organized as follows. Some auxiliary results to prove the main results are stated in Section 1.3 and the main results in Section 1.4. The proofs for the main theorems are contained in Section 2.

1.2. Preliminaries

In this section we are setting for main problems and introduce the classical results. We start with the assumptions which we will use throughout the paper.

Assumption 1.
1. $\lambda(z) = \nu + z$ for some $\nu > 0$,
2. $\|h\|_{L^1} < 1$ where $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$,
3. $t\nu \in \mathbb{N}$.

The first assumption of Assumption 1 says that λ is a linear and increasing function and so Hawkes process has a very nice immigration birth representation (see Hawkes and Oakes [11], 1974). The second assumption says that in the immigration birth representation, the total number of descendants of any given immigrant is finite with probability 1, and the third assumption is needed for the sake of simplicity due to the particular uniform immigrants assumption.

Here are some reviews for the results of Hawkes processes.

1.3. Limit theorems for Hawkes processes

The limit theorems for both linear and nonlinear Hawkes processes are well known and studied by many authors.

If $\lambda(\cdot)$ is linear, say $\lambda(z) = \nu + z$ for some $\nu > 0$, and $\|h\|_{L^1} < 1$, we can use a very nice immigration-birth representation and the limit theorems are well understood and more explicitly represented. Daley and Vere-Jones [4] proved the law of large numbers for linear Hawkes process. The functional central limit theorem for linear multivariate Hawkes process under certain assumptions have been obtained by Bacry et al. [1]. Bordenave and Torrisi [2] proved that if $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, then $\mathbb{P}(N_t/t \in \cdot)$ satisfies the large deviation principle. Moderate deviation principle for linear continuous time Hawkes processes is obtained by Zhu [24] and the limit theorems for linear marked Hawkes processes are obtained in Zhu [16]. In particular, Daley and Vere-Jones [4] proved the law of large numbers for linear Hawkes process as the following.

$$\frac{N_t}{t} \to \frac{\nu}{1 - \|h\|_{L^1}} \text{ as } t \to \infty. \tag{1}$$

The functional central limit theorem for linear multivariate Hawkes process under certain assumptions have been obtained by Bacry et al. [1] and they proved that

$$\frac{N_{\cdot t} - \cdot\mu t}{\sqrt{t}} \to \sigma B(\cdot) \text{ as } t \to \infty,$$

where $B(\cdot)$ is a standard Brownian motion and

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \text{ and } \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

The convergence used in above theorem is weak convergence on $D[0,1]$, the space of càdlàg function on $[0,1]$, equipped with Skorokhod topology. Bordenave and Torrisi [2] proved that if $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, then $\mathbb{P}(N_t/t \in \cdot)$ satisfies the large deviation principle with the good rate function $I(\cdot)$, which means that for any closed set $C \subset \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(N_t/t \in C) \leq - \inf_{x\in C} I(x),$$

and for any open set $G \subset \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(N_t/t \in G) \geq - \inf_{x\in G} I(x),$$

where

$$I(x) = \begin{cases} x\theta_x + \nu - \frac{\nu x}{\nu + \|h\|_{L^1} x} & \text{if } x \in (0,\infty), \\ \nu & \text{if } x = 0, \\ +\infty & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is the unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$, of

$$\mathbb{E}(e^{\theta S}) = \frac{x}{\nu + x\|h\|_{L^1}}, \ x > 0, \tag{2}$$

where S in the above equation denotes the total number of descendants of an immigrant, including the immigrant himself.

Remark 2. The rate function described above $I(x)$ can be represented as more explicit form. Note that (see [12] for details), for all $\theta \in (-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$, $\mathbb{E}(e^{\theta S})$ satisfies

$$\mathbb{E}(e^{\theta S}) = e^{\theta} e^{\|h\|_{L^1}(\mathbb{E}(e^{\theta S})-1)}, \tag{3}$$

which implies that $\theta_x = \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - \|h\|_{L^1}\left(\frac{x}{\nu + x\|h\|_{L^1}} - 1\right)$. Substituting into the formula, we have

$$I(x) = \begin{cases} x \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - x + \|h\|_{L^1} x + \nu & \text{if } x \in (0,\infty), \\ \nu & \text{if } x = 0, \\ +\infty & \text{if } x \in (-\infty, 0). \end{cases}$$

If $\lambda(\cdot)$ is nonlinear, the usual immigration-birth representation no longer works and so nonlinear model is much harder to study. Brémaud and Massoulié [3] proved that there exists a unique stationary version of nonlinear hawkes processes under certain conditions and the convergence to equilibrium of a non-stationary version. Central limit theorem is obtained in Zhu [23] and Zhu [28] proved large deviation for a special case for nonlinear case when $h(\cdot)$ is exponential or sums of exponentials. Zhu [27] proved a process-level, i.e., level-3 large deviation principle for nonlinear Hawkes processes for general $h(\cdot)$ and hence by contradiction principle, the level-1 large deviation principle for $\mathbb{P}(N_t/t \in \cdot)$.

1.4. Statement of the main results

This section states the main results of this paper. We obtain the law of large numbers, central limit theorem and large deviation principle for the N_t process. In particular, we discover that the limit for the law of large numbers is the same as in the classical Hawkes process case, but the variance in the central limit theorem and the rate function in the large deviation principle differ from those for the classical Hawkes process.

We recall that the linear Hawkes process with empty history has the intensity:

$$\lambda_t = \nu + \int_0^{t-} h(t-s)dN_s, \tag{4}$$

where $\nu > 0$ is the baseline intensity, $h(\cdot)$ is the exciting function.

In the intensity, there are two parts: ν and $\int_0^{t-} h(t-s)dN_s$. The baseline intensity ν can be interpreted as the external or exogenous factor, and the term $\int_0^{t-} h(t-s)dN_s$ can be interpreted as the internal or endogenous factor. The latter explains the self-exciting phenomenon of the Hawkes process.

It is well known that a linear Hawkes process can be represented by using a nice immigration-birth representation and the immigration-birth representation. The followings are general description of the tool. The immigrant arrives according to a standard homogeneous Poisson process with constant intensity $\nu > 0$, and then each immigrant generates children according to a Galton-Waston tree (see [11] for details). Let η be the number of children of an immigrant and η has Poisson distribution with parameter $\|h\|_{L^1}$. Then we know that conditional on the number of the children of an immigrant, the time that a child was born has probability density function

$$\frac{h(\cdot)}{\|h\|_{L^1}}. \tag{5}$$

The immigration-birth representation says that N_t equals to the total number of immigrants and their descendants up to time t. In our case, immigrants arrive uniformly over time with intensity ν instead of Poisson with intensity ν.

In the literature, there have been studies extending and modifying the classical Hawkes process. First, The baseline intensity can be chosen to be time-inhomogeneous (see Gao, Zhou, and Zhu [9]). Second, the immigrants can arrive according to a Cox process with shot noise intensity, in which case the model is known as the dynamic contagion model (see Dassios and Zhao [5]). Third, the immigrants can arrive according to a renewal process instead of a Poisson process, which generalizes the classical Hawkes process. This is known as the renewal Hawkes process (see Wheatley, Filimonov, and Sorrette [22]).

In this paper, we consider a special case of the renewal Hawkes process, by assuming that the immigrants arrive uniformly over time with intensity ν, and we compare our limit theorems with the ones for the classical Hawkes process. To be more precise, the immigrants arrive at times:

$$\frac{i}{\nu}, \qquad i = 1, 2, 3, \ldots. \tag{6}$$

Each immigrant generates descendants according to the immigration-birth representation of the Hawkes process.

We first recall that $t\nu \in \mathbb{N}$.

Then, we have the following decomposition:

$$N_t = \sum_{i=1}^{\nu t} X_i, \tag{7}$$

where X_i are independent, and X_i is the number of descendants of the ith immigrant that arrives at time $\frac{i}{\nu}$ on the time interval $[\frac{i}{\nu}, \frac{\nu t}{\nu}]$ plus the ith immigrant. Note that X_i has the same distribution as the number of descendants of an immigrant, including the immigrant, that arrives at time 0 on the time interval $[0, \frac{\nu t - i}{\nu}]$. The followings are our results.

Theorem 3 (Law of Large Numbers)**.** *Assume that Assumption 1 is satisfied. Then we have*

$$\frac{N_t}{t} \to \frac{\nu}{1 - \|h\|_{L^1}}, \tag{8}$$

in probability as $t \to \infty$.

Next, let us state the central limit theorem.

Theorem 4 (Central Limit Theorem)**.** *Assume that Assumption 1 is satisfied and*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_0^t \int_u^\infty h(s)dsdu = 0. \tag{9}$$

Then, we have

$$\frac{N_t - \frac{\nu t}{1-\|h\|_{L^1}}}{\sqrt{t}} \to N\left(0, \frac{\nu\|h\|_{L^1}}{(1-\|h\|_{L^1})^3}\right), \tag{10}$$

in distribution as $t \to \infty$.

Finally, let us show the large deviation principle.

Theorem 5 (Large Deviation Principle)**.** *Assume that Assumption 1 is satisfied. Then we say that $\mathbb{P}(N_t/t \in \cdot)$ satisfies a large deviation principle with the rate function*

$$I(x) = \begin{cases} (x-\nu)\log\left(\frac{x-\nu}{x\|h\|_{L^1}}\right) - x + \nu + x\|h\|_{L^1} & \text{if } x \geq \nu, \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

2. Proofs of the main results

In this section, we give the proofs of the main theorems.

2.1. Law of large numbers

Proof of Theorem 3. Let Y_i be the number of descendants of ith immigrant, including the immigrant, that arrives at time $\frac{i}{\nu}$ on the time interval $[\frac{i}{\nu}, \infty)$. Then Y_i are i.i.d. with moment generating function:

$$x(\theta) := \mathbb{E}[e^{\theta Y_i}] \tag{12}$$

satisfies the equation (See [12] for details)

$$x(\theta) = e^{\theta + (x(\theta)-1)\|h\|_{L^1}} \tag{13}$$

for $\theta \leq \theta_c = \|h\|_{L^1} - 1 - \log\|h\|_{L^1}$ and it is infinity otherwise.

Thus, from the moment generating function, Y_i has the mean

$$\mathbb{E}[Y_i] = \frac{1}{1-\|h\|_{L^1}}. \tag{14}$$

The strong law of large numbers shows that

$$\frac{1}{t}\sum_{i=1}^{\nu t} Y_i \to \frac{\nu}{1-\|h\|_{L^1}}, \tag{15}$$

a.s. as $t \to \infty$.

On the other hand,

$$N_t = \sum_{i=1}^{\nu t} X_i \leq \sum_{i=1}^{\nu t} Y_i, \tag{16}$$

and we can compute that

$$\mathbb{E}\left[\sum_{i=1}^{\nu t} Y_i - \sum_{i=1}^{\nu t} X_i\right] = \sum_{i=1}^{\nu t} \mathbb{E}[Y_i - X_i], \tag{17}$$

and

$$\mathbb{E}[X_i] = f\left(\frac{\nu t - i}{\nu}\right), \tag{18}$$

where $f(t)$ satisfies the renewal equation:

$$f(t) = 1 + \int_0^t h(t-s)f(s)ds. \tag{19}$$

This can be derived from the fact that for any $\theta \leq \|h\|_{L^1} - 1 - \log\|h\|_{L^1}$,

$$x(t;\theta) = \mathbb{E}[e^{\theta S_t}], \tag{20}$$

where S_t is distributed as the number of descendants of an immigrant, including the immigrant, that arrives at time 0 on the time interval $[0,t]$, satisfies the equation

$$x(t;\theta) = e^{\theta + \int_0^t (x(s;\theta)-1)h(t-s)ds}. \tag{21}$$

Moreover, we notice that

$$\mathbb{E}[Y_i] = f(\infty). \tag{22}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{\nu t} Y_i - \sum_{i=1}^{\nu t} X_i\right] &= \sum_{i=1}^{\nu t}\left[f(\infty) - f\left(\frac{\nu t - i}{\nu}\right)\right] \\ &= \sum_{i=0}^{\nu t-1}\left[f(\infty) - f\left(\frac{i}{\nu}\right)\right]. \end{aligned} \tag{23}$$

Since $\lim_{t\to\infty} f(t) = f(\infty)$, we conclude that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\sum_{i=1}^{\nu t} Y_i - \sum_{i=1}^{\nu t} X_i\right] = 0. \tag{24}$$

By Markov's inequality, we have proved that

$$\frac{N_t}{t} = \frac{1}{t}\sum_{i=1}^{\nu t} X_i \to \frac{\nu}{1-\|h\|_{L^1}}, \tag{25}$$

in probability as $t \to \infty$. The proof of Theorem 3 is completed. □

2.2. Central limit theorems

Proof of Theorem 4. We recall that for any $\theta \leq \theta_c$,

$$\mathbb{E}[e^{\theta Y_i}] = x(\theta) = e^{\theta + (x(\theta)-1)\|h\|_{L^1}}. \tag{26}$$

We can compute that

$$\mathbb{E}[Y_i^2] = x''(0), \tag{27}$$

and

$$x'(\theta) = (1 + x'(\theta)\|h\|_{L^1})x(\theta), \tag{28}$$

and

$$x''(\theta) = x''(\theta)\|h\|_{L^1}x(\theta) + (1 + x'(\theta)\|h\|_{L^1})x'(\theta). \tag{29}$$

Note that $x(0) = 1$ and thus $x'(0) = \frac{1}{1-\|h\|_{L^1}}$ and

$$x''(0) = \frac{1}{(1-\|h\|_{L^1})^3}. \tag{30}$$

Hence,

$$\mathrm{Var}(Y_i) = x''(0) - (x'(0))^2 = \frac{\|h\|_{L^1}}{(1-\|h\|_{L^1})^3}. \tag{31}$$

By the classical central limit theorem for i.i.d. random variables,

$$\frac{1}{\sqrt{t}}\left[\sum_{i=1}^{\nu t} Y_i - \frac{\nu t}{1-\|h\|_{L^1}}\right] \to N\left(0, \frac{\nu\|h\|_{L^1}}{(1-\|h\|_{L^1})^3}\right), \tag{32}$$

in distribution as $t \to \infty$. The conclusion follows if

$$\frac{1}{\sqrt{t}}\mathbb{E}\left[\sum_{i=1}^{\nu t} Y_i - \sum_{i=1}^{\nu t} X_i\right] \to 0 \tag{33}$$

in probability as $t \to \infty$. That is equivalent to

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \sum_{i=0}^{\nu t-1} \left[f(\infty) - f\left(\frac{i}{\nu}\right)\right] = 0. \tag{34}$$

By the definition of $f(t)$:

$$f(t) = 1 + \int_0^t h(t-s)f(s)ds, \tag{35}$$

and by $f(\infty) = \frac{1}{1-\|h\|_{L^1}}$, we can compute that

$$f(\infty) - f(t) = \frac{\int_t^\infty h(s)ds}{1-\|h\|_{L^1}} + \int_0^t h(t-s)(f(\infty)-f(s))ds, \tag{36}$$

and by using the equation (36), we get

$$\begin{aligned}
&\int_0^{\nu t}[f(\infty)-f(s)]ds \\
&= \frac{\int_0^{\nu t}\int_u^\infty h(s)dsdu}{1-\|h\|_{L^1}} + \int_0^{\nu t}\int_0^s h(s-u)(f(\infty)-f(u))duds \\
&= \frac{\int_0^{\nu t}\int_u^\infty h(s)dsdu}{1-\|h\|_{L^1}} + \int_0^{\nu t}\Big[\int_u^{\nu t} h(s-u)ds\Big](f(\infty)-f(u))du \\
&\leq \frac{\int_0^{\nu t}\int_u^\infty h(s)dsdu}{1-\|h\|_{L^1}} + \|h\|_{L^1}\int_0^{\nu t}(f(\infty)-f(u))du,
\end{aligned} \tag{37}$$

which implies that

$$\int_0^{\nu t}[f(\infty)-f(s)]ds \sim \frac{\int_0^{\nu t}\int_u^\infty h(s)dsdu}{(1-\|h\|_{L^1})^2} \tag{38}$$

as $t \to \infty$. The equation (38) implies that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_0^{\nu t} f(\infty) - f(s) ds = \lim_{t\to\infty} \frac{1}{\sqrt{t}} \frac{\int_0^{\nu t} \int_u^\infty h(s) ds du}{(1 - \|h\|_{L^1})^2} = 0. \tag{39}$$

By monotonicity of $f(\infty) - f(t)$ as a function of t, we get

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \sum_{i=0}^{\nu t-1} \left[f(\infty) - f\left(\frac{i}{\nu}\right) \right] = 0. \tag{40}$$

This completes the proof of Theorem 4. □

2.3. Large deviations principle

We start with the basic definitions in large deviations theory (e.g. See Dembo and Zeitouni [6] or Varadhan [21] for detail). Recall that a sequence $(\phi_n)_{n\in\mathbb{N}}$ of probability measures on a topological space X satisfies the large deviation principle with rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semi-continuous and for any measurable set B, we have

$$-\inf_{x\in B^\circ} I(x) \le \liminf_{n\to\infty} \frac{1}{n} \log \phi_n(B) \le \limsup_{n\to\infty} \frac{1}{n} \log \phi_n(B) \le -\inf_{x\in \bar{B}} I(x),$$

where B° is the interior of B and $\bar{B}$ is its closure.

Proof of Theorem 5. For any $\theta \le \theta_c = \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, we can compute that

$$\mathbb{E}[e^{\theta N_t}] = \prod_{i=1}^{\nu t} \mathbb{E}[e^{\theta X_i}] = \prod_{i=0}^{\nu t-1} x\left(\frac{i}{\nu}; \theta\right), \tag{41}$$

where $x(t;\theta)$ is defined in Equation (20). Therefore, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] = \nu \log x(\infty;\theta) = \nu \log x(\theta), \tag{42}$$

where $x(\theta)$ satisfies the equation

$$x(\theta) = e^{\theta + (x(\theta)-1)\|h\|_{L^1}}. \tag{43}$$

Note that $\nu \log x(\theta)$ is differentiable in θ for any $\theta < \theta_c$, and its derivative is given by

$$\nu \frac{x'(\theta)}{x(\theta)} \to \infty \tag{44}$$

as $\theta \to \theta_c$. This can be seen from the fact that

$$x'(\theta) = (1 + x'(\theta)\|h\|_{L^1}) x(\theta), \tag{45}$$

and at $\theta_c = \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$,

$$x(\theta_c) = \frac{1}{\|h\|_{L^1}}. \tag{46}$$

We also know that (See [12] for details) for $\theta > \theta_c$,

$$x(\theta) = \mathbb{E}[e^{\theta Y_i}] = \infty. \tag{47}$$

Hence we proved the essential smoothness for Gärtner-Ellis theorem (see [2] for details), and by the Gärtner-Ellis theorem, $\mathbb{P}(N_t/t \in \cdot)$ satisfies a large deviation principle with the rate function

$$I(x) = \sup_{\theta\in\mathbb{R}}\{\theta x - \nu \log x(\theta)\}. \tag{48}$$

Note that $\frac{N_t}{t} \geq \nu$ since the number of immigrants is νt. Therefore $I(x) = \infty$ for any $x < \nu$. Now, we assume that $x \geq \nu$. At optimality in the Legendre transform expression above

$$x = \nu\frac{x'(\theta)}{x(\theta)} = \nu(1 + x'(\theta)\|h\|_{L^1}), \tag{49}$$

which implies that

$$x'(\theta) = \frac{\frac{x}{\nu} - 1}{\|h\|_{L^1}}, \tag{50}$$

which implies that

$$x(\theta) = \frac{x'(\theta)}{1 + x'(\theta)\|h\|_{L^1}} = \frac{x - \nu}{x\|h\|_{L^1}}. \tag{51}$$

Therefore this optimal θ is given by

$$\theta = \log x(\theta) - (x(\theta) - 1)\|h\|_{L^1} = \log\left(\frac{x-\nu}{x\|h\|_{L^1}}\right) - \frac{x-\nu}{x} + \|h\|_{L^1} \tag{52}$$

and hence we conclude that for any $x \geq \nu$,

$$I(x) = x\log\left(\frac{x-\nu}{x\|h\|_{L^1}}\right) - x + \nu + x\|h\|_{L^1} - \nu\log\left(\frac{x-\nu}{x\|h\|_{L^1}}\right). \tag{53}$$

Hence, we have proved the desired result. □

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