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# HOMOGENEITY AND SYMMETRY ON ALMOST KENMOTSU 3-MANIFOLDS

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Abstract. In this paper, we give some classifications of almost Kenmotsu 3-manifolds under homogeneity and some symmetry conditions.

## 1. Introduction

In literature, Kenmotsu manifolds were firstly introduced and investigated by K. Kenmotsu [15] in 1972. Such manifolds were generalized to almost Kenmotsu manifolds by Janssens and Vanhecke [25] in 1981. It is known from [15] that any Kenmotsu manifold is locally isometric to a warped product of a line and a Kähler manifold. Conversely, a warped product of a line and a Kähler manifold admits a Kenmotsu structure. One of reasons that people are interested in Kenmotsu geometry lies in the fact that a Kenmotsu manifold of constant sectional curvature is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$, $n \geq 1$ (see [15]). This result was generalized to almost Kenmotsu manifolds (see [11]), namely an almost Kenmotsu manifold of constant sectional curvature is locally isometric to $\mathbb{H}^{2n+1}(-1)$, $n \geq 1$. This means that in some sense geometry of Kenmotsu manifolds corresponds to that of the hyperbolic spaces.

Among many interesting results in the framework of geometry of Kenmotsu manifolds (cf. Pitiş [20]), Kenmotsu [15] proved that a Kenmotsu manifold is locally symmetric if and only if it is semi-symmetric and this is equivalent to that the manifold is of constant sectional curvature $-1$. As a generalization of local symmetry, the notion of $\phi$-symmetry was introduced by Takahashi [24] in the study of Sasakian manifolds. In recent years, many authors studied Kenmotsu manifolds under certain geometric conditions weaker than local symmetry such as (local) $\phi$-symmetry and (local) $\phi$-recurrence. For examples, De, Yildiz and Yalniz [10] proved that a 3-dimensional normal almost contact metric manifold is locally $\phi$-symmetric if and only if the scalar curvature is a constant provided that both $\alpha$ and $\beta$ are constants. This result can be regarded

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as an extension of the above Kenmotsu's result since local symmetry implies $\phi$-recurrence. Also, De and Pathak [9] studied locally $\phi$-symmetric Kenmotsu 3-manifolds.

After almost Kenmotsu manifolds were introduced in [25], there were few results regarding such manifolds in a long time. Dileo and Pastore in [11, 12] started to study such manifolds and obtained some important results. For example, they proved that a locally symmetric $(2n+1)$-dimensional almost Kenmotsu manifold satisfying $R(X,Y)\xi = 0$ for any vector fields $X, Y$ orthogonal to the Reeb vector field $\xi$ is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Applying this result, the present author jointly with $X$. Liu in [30] proved that a locally symmetric $CR$-integrable almost Kenmotsu manifold of dimension $2n+1$, $n>1$, is locally isometric to either $\mathbb{H}^{2n+1}(-1)$ or $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Very recently, Cho [7] and the present author [26] independently obtained that a locally symmetric almost Kenmotsu 3-manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4)\times\mathbb{R}$. Semi-symmetric almost Kenmotsu manifolds satisfying some nullity conditions were studied in [29].

In this paper, we continue to study classification problem of almost Kenmotsu 3-manifolds under some additional conditions such as homogeneity, $\phi$-symmetry and semi-symmetry. Our main results can be viewed as some generalizations of the corresponding results obtained by Cho [7], De et al. [9–11] and Wang [26]. Some concrete examples illustrating our main results are given.

## 2. Almost Kenmotsu manifolds

An *almost contact structure* on a smooth differentiable manifold $M^{2n+1}$ of dimension $2n+1$ means a triple $(\phi,\xi,\eta)$ satisfying

$$\phi^2 = -\mathrm{id} + \eta\otimes\xi,\ \eta(\xi) = 1, \tag{2.1}$$

where $\phi$ is a $(1,1)$-type tensor field, $\xi$ is a vector field called the Reeb vector field and $\eta$ is a 1-form called the almost contact 1-form. If there exists a Riemannian metric $g$ on an almost contact manifold $M^{2n+1}$ such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y$, then $M^{2n+1}$ is said to be an *almost contact metric manifold* and $g$ is said to be a *compatible metric* with respect to the almost contact structure.

From Janssens and Vanhecke [25], in this paper by an *almost Kenmotsu manifold* we mean an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta\wedge\Phi$, where the *fundamental* 2*-form* $\Phi$ of the almost contact metric manifold $M^{2n+1}$ is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields $X$ and $Y$ on $M^{2n+1}$. We consider the product $M^{2n+1}\times\mathbb{R}$ of an almost contact metric manifold $M^{2n+1}$ and $\mathbb{R}$ and define on it an almost complex structure $J$ by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where $X$ denotes a vector field tangent to $M^{2n+1}$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of $\phi$. If

$$[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$$

holds, then the almost contact metric structure is said to be *normal*. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (cf. [15, 25]). It is well-known that an almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields $X, Y$.

Let $M^{2n+1}$ be an almost Kenmotsu manifold. We consider three tensor fields $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on $M^{2n+1}$, where $R$ is the Riemannian curvature tensor of $g$ and $\mathcal{L}$ is the Lie differentiation. From Dileo and Pastore [11,12], we know that the three $(1,1)$-type tensor fields $l$, $h'$ and $h$ are symmetric and satisfy $h\xi = 0$, $l\xi = 0$, $\mathrm{tr}h = 0$, $\mathrm{tr}(h') = 0$ and $h\phi + \phi h = 0$ and

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.5}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr}h^2, \tag{2.6}$$

for any vector fields $X, Y$, where $S$, $Q$ and $\nabla$ denote the Ricci curvature tensor, the Ricci operator with respect to $g$ and the Levi-Civita connection of $g$, respectively. Throughout the paper, we denote by $\mathcal{D}$ the contact distribution $\{\xi\}^\perp$ and all manifolds are assumed to be smooth and connected.

## 3. Homogeneity on almost Kenmotsu 3-manifolds

In 1958, Boothby and Wang [3] introduced the notion of homogeneous contact manifolds. Later, such notion was generalized on contact metric 3-manifolds (see [18]) and almost cosymplectic 3-manifolds (see [19]). In general, an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is said to be *homogeneous* if there exists a connected Lie group acting transitively as a group of diffeomorphisms on $M$ and leaving the almost contact form $\eta$ invariant.

Recall that a Lie group $G$ is said to be *unimodular* if its left invariant Haar measure is also right invariant. A Lie group $G$ is unimodular if and only if the endomorphism $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) = [X, Y]$ has trace equal to zero for any $X \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra associated to $G$.

**Theorem 3.1.** *A simply connected almost Kenmotsu* 3*-manifold $M$ is homogeneous if and only if it is isometric to a non-unimodular Lie group endowed with a left invariant almost Kenmotsu structure. More precisely, the Lie algebra of such Lie group is one of the following types*:

$A_1$ $[e_0, e_1] = -e_1$, $[e_1, e_2] = 0$, $[e_2, e_0] = e_2$. *In this case, $M$ is isometric to the hyperbolic* 3*-space* $\mathbb{H}^3(-1)$ *endowed with a Kenmotsu structure.*

$B_1$ $[e_0, e_1] = -e_1 + (\lambda + a)e_2$, $[e_1, e_2] = 0$, $[e_2, e_0] = (a - \lambda)e_1 + e_2$, *where* $\lambda, a \in \mathbb{R}$ *and* $\lambda \neq 0$. *In this case, $M$ is non-Kenmotsu. In particular, if* $\lambda = 1$ *and* $a = 0$, *$M$ is isometric to the product space* $\mathbb{H}^2(-4) \times \mathbb{R}$.

$B_2$ $[e_0, e_1] = -e_1 + (\lambda + a)e_2$, $[e_1, e_2] = be_1 - ce_2$, $[e_2, e_0] = (a - \lambda)e_1 + e_2$, *where* $\lambda, a, b, c \in \mathbb{R}$, $\lambda \neq 0$, $b = (\lambda - a)c \neq 0$ *and* $c = (\lambda + a)b \neq 0$. *In this case, $M$ is non-Kenmotsu.*

*In the last two types, $\lambda$ is the positive eigenvalue of the operator $h$.*

*Proof.* Let $(M, \phi, \xi, \eta, g)$ be a simply connected homogeneous almost Kenmotsu 3-manifold. Let $G$ be the Lie group acting transitively as a group of isometries which leaves the almost contact form $\eta$ invariant. Since $M$ is a simply connected homogeneous Riemannian 3-manifold, following Sekigawa [17] we see that $M$ is isometric to either a Lie group endowed with a left invariant Riemannian metric or a symmetric space.

For the first case, following a standard statement shown in [18, p. 247] or [19, p. 54] (see also [21]) we can consider $M$ as a Lie group $G$ and $(\phi, \xi, \eta, g)$ as a left invariant almost Kenmotsu structure. For simplicity, here we omit proof of the above statement. On the other hand, it was proved that a left invariant almost contact metric structure on a 3-dimensional Lie group $G$ is almost Kenmotsu if and only if $G$ is non-unimodular and its Lie algebra is one of the following two types (for more details see [4, p. 1379] and [4, Theorem 4.11]):

$$\text{(II)}\quad [\xi, e] = e + \beta\phi e,\ [e, \phi e] = 0,\ [\xi, \phi e] = \gamma e + \phi e.$$

$$\text{(IV)}\quad [\xi, e] = \frac{a_1 + a_3}{a_5}\Lambda,\ [e, \phi e] = -\Lambda,\ [\xi, \phi e] = \frac{a_4(a_1 + a_3)}{a_5^2}\Lambda,$$

where $\Lambda = a_4 e + a_5 \phi e$, $a_1, a_4, a_5 \neq 0$, $\{\xi, e, \phi e\}$ are eigenvector fields of $h$ corresponding eigenvalues $\{0, \lambda, -\lambda\}$. One can check that the above two Lie algebras are the same with the last two types in Theorem 3.1.

For the later case, note that a locally symmetric almost Kenmotsu 3-manifold is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$ (see [7,26]). Also, a simple calculation shows that $\mathbb{H}^3(-1)$ corresponds to the simply connected metric Lie group whose Lie algebra is given by $[\xi, e] = -e$, $[e, \phi e] = 0$, $[\phi e, \xi] = \phi e$ and $\mathbb{H}^2(-4) \times \mathbb{R}$ corresponds to a simply connected metric Lie group whose Lie algebra is given by $[\xi, e] = -e + \phi e$, $[e, \phi e] = 0$, $[\phi e, \xi] = -e + \phi e$. One observes that the later Lie algebra is just of type $B_1$ for $\lambda = 1$ and $a = 0$ (for more details see also [27, 28]).

We refer the reader to [6, p. 31] for the standard construction of Kenmotsu structure on a hyperbolic space. On the other hand, for the construction of the almost Kenmotsu structures defined on a non-unimodular Lie group we refer the reader to [12]. This completes the proof. □

An almost Kenmotsu 3-manifold is Kenmotsu if and only if $h$ vanishes (see Proposition 4.1). From (2.4) and (2.6) we see that on a Kenmotsu 3-manifold there holds $Q\xi = -2\xi$. For a non-Kenmotsu almost Kenmotsu structure of type $B_1$ in Theorem 3.1 we observe also that $Q\xi$ is a constant multiple of $\xi$. Thus, we have

*Remark* 3.1. The Reeb vector field of almost Kenmotsu 3-manifolds for types $A_1$ and $B_1$ is an eigenvector field of the Ricci operator. However, this does not hold for type $B_2$.

*Remark* 3.2. For simplicity, we state that a simply connected almost Kenmotsu 3-manifold is homogeneous if and only if it is isometric to a non-unimodular Lie group whose Lie algebra $G$ is given by

$$[e_0, e_1] = -e_1 + (\lambda + a)e_2,\ [e_1, e_2] = be_1 - ce_2,\ [e_2, e_0] = (a - \lambda)e_1 + e_2,$$

where $\lambda, a, b, c \in \mathbb{R}$. The type $A_1$ corresponds to $G$ satisfying $\lambda = a = b = c = 0$; the type $B_1$ corresponds to $G$ satisfying $b = c = 0$ and $\lambda \neq 0$ and the type $B_2$ corresponds to $G$ satisfying $\lambda, b, c \neq 0$.

## 4. Almost Kenmotsu 3-manifolds and local $\phi$-symmetry

In this section, we aim to give some local classifications of almost Kenmotsu 3-manifolds under condition of local $\phi$-symmetry. First, we need the following proposition.

**Proposition 4.1** ([11])**.** *An almost Kenmotsu* 3*-manifold is Kenmotsu if and only if $h$ vanishes.*

The notion of local $\phi$-symmetry is defined as the following.

**Definition 4.1** ([24])**.** An almost contact metric manifold is said to be locally $\phi$-symmetric if

$$\phi^2(\nabla_{\phi V} R)(\phi X, \phi Y)\phi Z = 0$$

for any vector fields $X, Y, Z, V$.

It is clear that local symmetry (i.e., $\nabla R = 0$) implies local $\phi$-symmetry, but in general the converse is not necessarily true. The following result shows that the converse mentioned above is not true even in Kenmotsu 3-manifolds.

**Theorem 4.1.** *A Kenmotsu* 3*-manifold is locally $\phi$-symmetric if and only if it is locally isometric to the warped product $\mathbb{R} \times_{ce^t} N(k)$, where $N(k)$ denotes a Riemannian* 2*-manifold of constant Gaussian curvature $k$.*

*Proof.* We observe from De and Pathak [9, Eq. (22)] that a Kenmotsu 3-manifold is locally $\phi$-symmetric if and only if

$$\phi^2(\nabla_V R)(X,Y)Z = -\frac{V(r)}{2}(g(Y,Z)X - g(X,Z)Y) = 0$$

for any vector fields $X, Y, Z, V$ orthogonal to $\xi$, where $r$ is the scalar curvature. Here we omit the proof of the above relation for simplicity. The above relation implies that the scalar curvature $r$ is invariant along the contact distribution $\{\xi\}^\perp$. On the other hand, J. Inoguchi in [13] proved that a Kenmotsu 3-manifold is locally isometric to the warped product $\mathbb{R} \times_{ce^t} N(k)$ if and only if $\mathrm{grad}(r) \in \{\xi\}_{\mathbb{R}}$, where $c$ denotes a constant. This completes the proof. □

Next we study local $\phi$-symmetry on almost Kenmotsu 3-manifolds satisfying $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$. Almost contact metric 3-manifolds satisfying this condition were studied in [1, 27, 28]. We now construct an example of almost Kenmotsu 3-manifolds satisfying such condition.

From Milnor [16], if $G$ is a three-dimensional non-unimodular Lie group, then there exists a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_2, e_3] = 0,\ [e_1, e_3] = \gamma e_2 + \delta e_3 \tag{4.1}$$

and $\alpha + \delta = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We define a metric $g$ on $G$ by $g(e_i, e_j) = \delta_{ij}$ for $1 \le i, j \le 3$. We denote by $\xi = -e_1$ and by $\eta$ the dual 1-form of $\xi$. We define a $(1,1)$-type tensor field $\phi$ by $\phi(\xi) = 0$, $\phi(e_2) = e_3$ and $\phi(e_3) = -e_2$. It is easily check that $(G, \phi, \xi, \eta, g)$ admits a left invariant almost Kenmotsu structure. For more details regarding the above statements we refer to Dileo and Pastore [12].

**Example 4.1** ([27])**.** Let $G$ be a 3-dimensional non-unimodular Lie group with a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying (4.1) for $\alpha, \beta \in \mathbb{R}$. If either $\alpha \neq 1$ or $\beta \neq 0$, $G$ admits a left invariant non-Kenmotsu almost Kenmotsu structure satisfying $\nabla_\xi h = (\gamma - \beta)\phi h$.

Let us consider an almost Kenmotsu 3-manifold $M$. Let $\mathcal{U}_1$ be the open subset of $M$ such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of $M$ which is defined by $\mathcal{U}_2 = \{p \in M : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of $M$ and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of $h$ for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda\phi e$, where $\lambda$ is a positive function on $\mathcal{U}_1$. Note that the eigenvalue function $\lambda$ is smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

**Lemma 4.1** ([7, Lemma 6])**.** *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_\xi e = a\phi e,\ \nabla_\xi \phi e = -ae,\\
&\nabla_e \xi = e - \lambda\phi e,\ \nabla_e e = -\xi - b\phi e,\ \nabla_e \phi e = \lambda\xi + be,\\
&\nabla_{\phi e} \xi = -\lambda e + \phi e,\ \nabla_{\phi e} e = \lambda\xi + c\phi e,\ \nabla_{\phi e} \phi e = -\xi - ce,
\end{aligned}$$

*where $a, b, c$ are smooth functions.*

Applying Lemma 4.1 in the following well known Jacobi identity

$$[[\xi,e],\phi e]+[[e,\phi e],\xi]+[[\phi e,\xi],e]=0$$

yields that

$$\begin{cases} e(\lambda)-\xi(b)-e(a)+c(\lambda-a)-b=0,\\ \phi e(\lambda)-\xi(c)+\phi e(a)+b(\lambda+a)-c=0. \end{cases} \tag{4.2}$$

Moreover, applying Lemma 4.1 we have the following.

**Lemma 4.2.** *On $\mathcal{U}_1$, the Ricci operator can be written as*

$$\begin{aligned} Q\xi&=-2(\lambda^2+1)\xi-\sigma(e)e-\sigma(\phi e)\phi e,\\ Qe&=-\sigma(e)\xi-(A+2\lambda a)e+(\xi(\lambda)+2\lambda)\phi e,\\ Q\phi e&=-\sigma(\phi e)\xi+(\xi(\lambda)+2\lambda)e-(A-2\lambda a)\phi e, \end{aligned}$$

*with respect to the local basis $\{\xi,e,\phi e\}$, where for simplicity we set $A=e(c)+\phi e(b)+b^2+c^2+2$ and $\sigma(\cdot)=-g(Q\xi,\cdot)$.*

One can check that the following relation is true.

$$\sigma(e)=\phi e(\lambda)+2\lambda b,\ \sigma(\phi e)=e(\lambda)+2\lambda c. \tag{4.3}$$

From Lemma 4.2 we see that the scalar curvature is given by

$$r=-2(\lambda^2+1+A). \tag{4.4}$$

Applying Lemmas 4.1 and 4.2, (4.3), by a direct calculation we obtain the following nine equations.

$$(\nabla_\xi Q)\xi=-4\lambda\xi(\lambda)\xi-(\xi(\sigma(e))-a\sigma(\phi e))e-(\xi(\sigma(\phi e))+a\sigma(e))\phi e. \tag{4.5}$$

$$\begin{aligned} (\nabla_\xi Q)e=&-(\xi(\sigma(e))-a\sigma(\phi e))\xi\\ &-(\xi(A+2\lambda a)+2a(\xi(\lambda)+2\lambda))e+(\xi(\xi(\lambda)+2\lambda)-4\lambda a^2)\phi e. \end{aligned} \tag{4.6}$$

$$\begin{aligned} (\nabla_\xi Q)\phi e=&-(\xi(\sigma(\phi e))+a\sigma(e))\xi\\ &+(\xi(\xi(\lambda)+2\lambda)-4\lambda a^2)e-(\xi(A-2\lambda a)-2a(\xi(\lambda)+2\lambda))\phi e. \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\nabla_e Q)\xi=&\,2(\sigma(e)-\lambda\sigma(\phi e)-2\lambda e(\lambda))\xi\\ &+(A-2+2\lambda a+\lambda\xi(\lambda)-e(\sigma(e))-b\sigma(\phi e))e\\ &+(2\lambda^3+2\lambda^2a-A\lambda-\xi(\lambda)-e(\sigma(\phi e))+b\sigma(e))\phi e. \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\nabla_e Q)e=&\,(A-2+2\lambda a+\lambda\xi(\lambda)-e(\sigma(e))-b\sigma(\phi e))\xi\\ &+(2b(\xi(\lambda)+2\lambda)-2\sigma(e)-e(A+2\lambda a))e\\ &+(4\lambda ab-\sigma(\phi e)+\lambda\sigma(e)+e(\xi(\lambda)+2\lambda))\phi e. \end{aligned} \tag{4.9}$$

$$\begin{aligned} (\nabla_e Q)\phi e=&\,(2\lambda^3+2\lambda^2a-A\lambda-\xi(\lambda)-e(\sigma(\phi e))+b\sigma(e))\xi\\ &+(4\lambda ab-\sigma(\phi e)+\lambda\sigma(e)+e(\xi(\lambda)+2\lambda))e\\ &+(2\lambda\sigma(\phi e)-2b(\xi(\lambda)+2\lambda)-e(A-2\lambda a))\phi e. \end{aligned} \tag{4.10}$$

$$\begin{aligned}(\nabla_{\phi e}Q)\xi = {}& 2(\sigma(\phi e) - \lambda\sigma(e) - 2\lambda\phi e(\lambda))\xi \\ & + (2\lambda^3 - 2\lambda^2 a - A\lambda - \xi(\lambda) - \phi e(\sigma(e)) + c\sigma(\phi e))e \\ & + (A - 2 - 2\lambda a + \lambda\xi(\lambda) - \phi e(\sigma(\phi e)) - c\sigma(e))\phi e.\end{aligned} \tag{4.11}$$

$$\begin{aligned}(\nabla_{\phi e}Q)e = {}& (2\lambda^3 - 2\lambda^2 a - A\lambda - \xi(\lambda) - \phi e(\sigma(e)) + c\sigma(\phi e))\xi \\ & + (2\lambda\sigma(e) - 2c(\xi(\lambda) + 2\lambda) - \phi e(A + 2\lambda a))e \\ & + (\lambda\sigma(\phi e) - 4\lambda ac + \phi e(\xi(\lambda) + 2\lambda) - \sigma(e))\phi e.\end{aligned} \tag{4.12}$$

$$\begin{aligned}(\nabla_{\phi e}Q)\phi e = {}& (A - 2 - 2\lambda a + \lambda\xi(\lambda) - \phi e(\sigma(\phi e)) - c\sigma(e))\xi \\ & + (\lambda\sigma(\phi e) - 4\lambda ac + \phi e(\xi(\lambda) + 2\lambda) - \sigma(e))e \\ & + (2c(\xi(\lambda) + 2\lambda) - 2\sigma(\phi e) - \phi e(A - 2\lambda a))\phi e.\end{aligned} \tag{4.13}$$

Applying the above relations we prove the following.

**Theorem 4.2.** *Let $M$ be a non-Kenmotsu almost Kenmotsu 3-manifold such that $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$, and the scalar curvature is invariant along contact distribution. Then, $M$ is locally $\phi$-symmetric if and only if it is locally isometric to a 3-dimensional non-unimodular Lie group whose Lie algebra is of type $B_1$.*

*Proof.* Suppose that $M$ is a locally $\phi$-symmetric almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$, and $h \neq 0$. Then $\mathcal{U}_1 = M$ and hence Lemmas 4.1 and 4.2 are applicable.

It is known that the curvature tensor of any three-dimensional Riemannian manifold is given by

$$\begin{aligned}R(X,Y)Z = {}& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y)\end{aligned} \tag{4.14}$$

for any vector fields $X, Y, Z$, where $r$ is the scalar curvature. Taking the covariant derivative of the above relation along $V$ gives

$$\begin{aligned}&(\nabla_V R)(X,Y)Z \\ &= g(Y,Z)(\nabla_V Q)X - g(X,Z)(\nabla_V Q)Y + g((\nabla_V Q)Y,Z)X \\ &\quad - g((\nabla_V Q)X,Z)Y - \frac{1}{2}V(r)(g(Y,Z)X - g(X,Z)Y)\end{aligned} \tag{4.15}$$

for any vector fields $X, Y, Z, V$. Putting $X = Z = V = e$ and $Y = \phi e$ into (4.15) we get

$$\begin{aligned}&(\nabla_e R)(e,\phi e)e \\ &= -(\nabla_e Q)\phi e + g((\nabla_e Q)\phi e, e)e - g((\nabla_e Q)e, e)\phi e + \frac{1}{2}e(r)\phi e.\end{aligned}$$

Similarly, putting $X = V = e$ and $Y = Z = \phi e$ into (4.15) we have

$$\begin{aligned}&(\nabla_e R)(e,\phi e)\phi e \\ &= (\nabla_e Q)e + g((\nabla_e Q)\phi e, \phi e)e - g((\nabla_e Q)e, \phi e)\phi e - \frac{1}{2}e(r)e.\end{aligned}$$

Putting $X = Z = e$ and $Y = V = \phi e$ into (4.15) we obtain

$$\begin{aligned}&(\nabla_{\phi e}R)(e,\phi e)e\\ &= -(\nabla_{\phi e}Q)\phi e + g((\nabla_{\phi e}Q)\phi e, e)e - g((\nabla_{\phi e}Q)e, e)\phi e + \frac{1}{2}\phi e(r)\phi e.\end{aligned}$$

Putting $X = e$ and $Y = Z = V = \phi e$ into (4.15) gives

$$\begin{aligned}&(\nabla_{\phi e}R)(e,\phi e)\phi e\\ &= (\nabla_{\phi e}Q)e + g((\nabla_{\phi e}Q)\phi e, \phi e)e - g((\nabla_{\phi e}Q)e, \phi e)\phi e - \frac{1}{2}\phi e(r)e.\end{aligned}$$

From Definition 4.1 we see that an almost contact metric manifold is locally $\phi$-symmetric if and only if

$$(\nabla_V R)(X,Y)Z = \eta((\nabla_V R)(X,Y)Z)\xi \tag{4.16}$$

for any vector fields $X, Y, Z, V$ orthogonal to $\xi$. Applying the above four relations, it follows from (4.16) that an almost contact metric manifold is locally $\phi$-symmetric if and only if

$$\begin{aligned}g((\nabla_e Q)e, e) + g((\nabla_e Q)\phi e, \phi e) &= \frac{1}{2}e(r),\\ g((\nabla_{\phi e} Q)e, e) + g((\nabla_{\phi e} Q)\phi e, \phi e) &= \frac{1}{2}\phi e(r).\end{aligned} \tag{4.17}$$

On the other hand, applying Lemma 4.1 and by a direct calculation we obtain

$$(\nabla_\xi h)e = \xi(\lambda)e + 2a\lambda\phi e \text{ and } (\nabla_\xi h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e.$$

Since we have assumed that $M$ satisfies $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$, and $h \neq 0$, then it follows from the above relation that

$$\xi(\lambda) = 0. \tag{4.18}$$

Suppose that the scalar curvature of $M$ is invariant along the contact distribution, then from (4.4) and (4.18) we obtain

$$e(A) = -2\lambda e(\lambda),\ \phi e(A) = -2\lambda\phi e(\lambda). \tag{4.19}$$

Using relations (4.3), (4.9), (4.10), (4.12), (4.13), (4.18) and (4.19) in (4.17) we see that $M$ is locally $\phi$-symmetric if and only if

$$\begin{aligned}3\lambda e(\lambda) - \phi e(\lambda) + 2\lambda^2 c - 2\lambda b &= 0,\\ 3\lambda\phi e(\lambda) - e(\lambda) + 2\lambda^2 b - 2\lambda c &= 0.\end{aligned} \tag{4.20}$$

In terms of $e(\lambda)$ and $\phi e(\lambda)$, from the above relation, $b$ and $c$ can be expressed as the following

$$\begin{aligned}2\lambda(\lambda^2 - 1)c &= (1 - 3\lambda^2)e(\lambda) - 2\lambda\phi e(\lambda),\\ 2\lambda(\lambda^2 - 1)b &= (1 - 3\lambda^2)\phi e(\lambda) - 2\lambda e(\lambda).\end{aligned} \tag{4.21}$$

From Lemma 4.1 we have

$$[\xi, e] = -e + (\lambda + a)\phi e,\ [e, \phi e] = be - c\phi e,\ [\xi, \phi e] = (\lambda - a)e - \phi e. \tag{4.22}$$

In view of (4.18), it follows from the above relation that

$$\xi(e(\lambda)) = -e(\lambda) + (\lambda + a)\phi e(\lambda),$$
$$\xi(\phi e(\lambda)) = (\lambda - a)e(\lambda) - \phi e(\lambda). \tag{4.23}$$

Since $a$ is a constant, (4.2) becomes

$$\xi(b) = e(\lambda) + (\lambda - a)c - b,$$
$$\xi(c) = \phi e(\lambda) + (\lambda + a)b - c. \tag{4.24}$$

In view of (4.18) and the assumption $\lambda > 0$, differentiating the first term of (4.21) along the Reeb vector field $\xi$ gives an equation; in the resulting equation using (4.21), (4.23) and the second term of (4.24) we have

$$(\lambda^2 - 1)\phi e(\lambda) - 2ae(\lambda) = 0. \tag{4.25}$$

Similarly, applying (4.18) and the assumption $\lambda > 0$, and differentiating the second term of (4.21) along the Reeb vector field $\xi$ gives an equation; in the resulting equation using (4.21), (4.23) and the first term of (4.24) we have

$$(\lambda^2 - 1)e(\lambda) + 2a\phi e(\lambda) = 0. \tag{4.26}$$

Comparing this relation with (4.25) gives

$$(4a^2 + (\lambda^2 - 1)^2)e(\lambda) = 0. \tag{4.27}$$

We separate the proof into the following main two cases.

**Case i.** In view of the above relation and the assumption $\lambda > 0$, firstly, we consider the case $\lambda = 1$ and $a = 0$. Using this in (4.20) we have $b = c$ and using this in (4.2) we have $\xi(b) = 0$. Then, it follows from Lemma 4.2 that

$$Q\xi = -4\xi - 2be - 2b\phi e,$$
$$Qe = -2b\xi - Ae + 2\phi e,$$
$$Q\phi e = -2b\xi + 2e - A\phi e,$$

where $A = 2 + e(b) + \phi e(b) + 2b^2$. By using the above relations we obtain

$$Qh\phi - h\phi Q + g(Q\xi, \cdot)\xi - \eta \otimes Q\xi = 0. \tag{4.28}$$

In view of $\xi(b) = 0$, applying (4.22) we have $\xi(e(b) + \phi e(b)) = 0$ and hence we have $\xi(A) = 0$. Thus, from (4.19) we see that $A$ is a constant. Applying this, $\lambda = 1$, $a = 0$ and $\xi(b) = 0$ in (4.5)-(4.7) we have

$$\nabla_\xi Q = 0. \tag{4.29}$$

Combining (4.28) with (4.29) and applying (2.3) we see that the Ricci tensor is invariant along the Reeb flow, i.e., $\mathcal{L}_\xi Q = 0$. J. T. Cho in [8] proved that a three-dimensional non-Kenmotsu almost Kenmotsu manifold $M^3$ satisfies $\mathcal{L}_\xi Q = 0$ if and only if the manifold is locally isometric to a non-unimodular Lie group and in this case whose Lie algebra is given by

$$[\xi, e] = -e + \phi e,\ [e, \phi e] = 0,\ [\xi, \phi e] = e - \phi e.$$

Moreover, it was proved that $b = c = 0$ on $M^3$ under the condition $\mathcal{L}_\xi Q = 0$ (cf. [8, p. 272]). Finally, we may use $\lambda = 1$ and $a = b = c = 0$ in relations (4.5)-(4.13) and observe that the Ricci tensor is parallel and therefore the manifold is locally symmetric. The present author in [26, Theorem 3.4] and Cho [7] proved that a three-dimensional locally symmetric strictly almost Kenmotsu manifold is locally isometric to the product space $\mathbb{H}^2(-4) \times \mathbb{R}$.

**Case ii.** In view of (4.27), next we consider the other case, i.e., $4a^2 + (\lambda^2 - 1)^2 \neq 0$ holds on some open subset. It follows directly that $e(\lambda) = 0$ and hence from (4.26) we have either $a = 0$ or $\phi e(\lambda) = 0$.

**Case ii-1.** Firstly, we consider $a = 0$. In view of $4a^2 + (\lambda^2 - 1)^2 \neq 0$, from (4.25) we have that $\phi e(\lambda) = 0$ and hence from (4.18) we see that $\lambda$ is a positive constant not equal to 1. Now, (4.21) becomes $b = c = 0$, were we used $\lambda \neq 1$. In this case, from Lemma 4.1 we have

$$[\xi, e] = -e + \lambda\phi e,\ [e, \phi e] = 0,\ [\xi, \phi e] = \lambda e - \phi e.$$

We say that $M^3$ is locally isometric to a non-unimodular Lie group.

**Case ii-2.** Now we consider the other subcase, $a \neq 0$, or equivalently, $\phi e(\lambda) = 0$. From (4.18) we know that $\lambda$ is a positive constant. Thus, it follows from (4.21) that either $\lambda = 1$ or $b = c = 0$. We observe that the former subcase implies the later one. In fact, using $\lambda = 1$ in equation (4.20) we have $b = c$. Using this in (4.2) we have $\xi(b) = -ab$ and $\xi(c) = ab$. Because of $b = c$ and $a \neq 0$, we obtain $b = 0$. In both cases, (4.22) can be written as the following form

$$[\xi, e] = -e + (\lambda + a)\phi e,\ [e, \phi e] = 0,\ [\xi, \phi e] = (\lambda - a)e - \phi e,$$

where $a \neq 0$, $\lambda \in \mathbb{R}$. We still say that $M$ is locally isometric to a non-unimodular Lie group.

Conversely, we need to check that any left invariant non-Kenmotsu almost Kenmotsu structure defined on a three-dimensional non-unimodular Lie group satisfying $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$, is locally $\phi$-symmetric. In fact, from Example 4.1 we obtain the Levi-Civita connection given by (for more details see [12])

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_{e_2}\xi = \alpha e_2 + \frac{1}{2}(\beta + \gamma)e_3,\ \nabla_{e_3}\xi = \frac{1}{2}(\beta + \gamma)e_2 + (2 - \alpha)e_3,\\
&\nabla_\xi e_2 = \frac{1}{2}(\gamma - \beta)e_3,\ \nabla_{e_2}e_2 = -\alpha\xi,\ \nabla_{e_3}e_2 = -\frac{1}{2}(\beta + \gamma)\xi,\\
&\nabla_\xi e_3 = \frac{1}{2}(\beta - \gamma)e_2,\ \nabla_{e_2}e_3 = -\frac{1}{2}(\beta + \gamma)\xi,\ \nabla_{e_3}e_3 = (\alpha - 2)\xi,
\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ and either $\alpha \neq 1$ or $\beta \neq 0$. Hence the Ricci tensor can be written as the following

$$\begin{aligned}
Q\xi &= -2\left(\alpha^2 - 2\alpha + \frac{1}{4}(\beta + \gamma)^2 + 2\right)\xi,\\
Qe_2 &= \left(\frac{\gamma^2 - \beta^2}{2} - 2\alpha\right)e_2 - (\alpha\gamma + \beta(2 - \alpha))e_3,
\end{aligned}$$

$$Qe_3 = -(\alpha\gamma + \beta(2-\alpha))e_2 + \left(2(\alpha-2) + \frac{\beta^2-\gamma^2}{2}\right)e_3.$$

A simple calculation gives that $(\nabla_{e_i}Q)e_j$ is collinear with the Reeb vector field $\xi$ for any $i, j \in \{2, 3\}$. Then, the proof for necessary case follows from (4.17). $\square$

*Remark* 4.1. Since local symmetry condition on almost Kenmotsu 3-manifolds implies that the scalar curvature is a constant, $\nabla_\xi h = 0$ (cf. [11, Proposition 6]) and local $\phi$-symmetry, therefore, Theorems 4.1 and 4.2 are generalizations of Cho [7, Theorem 5], [15, Corollary 6] for three-dimensional case and Wang [26, Theorem 3.4].

*Remark* 4.2. According to Theorems 4.1 and 4.2 we observe that there exist many locally $\phi$-symmetric almost Kenmotsu 3-manifolds which are not locally symmetric.

Before closing this section, we give an example of locally $\phi$-symmetric almost Kenmotsu 3-manifold which is not homogeneous. Although the scalar curvature of such manifold is invariant along the contact distribution but the manifold does not satisfy $\nabla_\xi h = 2a\phi h$ for certain $a \in \mathbb{R}$.

**Example 4.2.** We denote by $(x, y, z)$ the usual canonical coordinates of $\mathbb{R}^3$. We set

$$M^3 := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}.$$

On $M^3$ we define an almost contact metric structure $(\phi, \xi, \eta, g)$ as the following:

$$\xi := \frac{\partial}{\partial z},\ \eta := dz,\ g = ze^{2z}dx^2 + \frac{e^{2z}}{z}dy^2 + dz^2,$$
$$\phi(\xi) = 0,\ \phi\left(\frac{\partial}{\partial x}\right) = z\frac{\partial}{\partial y},\ \phi\left(\frac{\partial}{\partial y}\right) = -\frac{1}{z}\frac{\partial}{\partial x}.$$

It was shown in [17] that $M^3$ is a generalized $(k, \nu)'$-almost Kenmotsu 3-manifold with $k = -1 - \frac{1}{4z^2}$ and $\nu = -2 + \frac{1}{z}$. Moreover, using the well-known Koszul formula the Levi-Civita connection of $M^3$ is given as the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi \frac{\partial}{\partial x} = \left(1 + \frac{1}{2z}\right)\frac{\partial}{\partial x},$$
$$\nabla_\xi \frac{\partial}{\partial y} = \left(1 - \frac{1}{2z}\right)\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial x}}\xi = \left(1 + \frac{1}{2z}\right)\frac{\partial}{\partial x},$$
$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -\frac{1+2z}{2}e^{2z}\xi,\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 0,$$
$$\nabla_{\frac{\partial}{\partial y}}\xi = \left(1 - \frac{1}{2z}\right)\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{1-2z}{2z^2}e^{2z}\xi.$$

By a direct calculation, the Ricci operator of $M^3$ is given by

$$Q\xi = -\left(2 + \frac{1}{2z^2}\right)\xi,$$

$$Q\frac{\partial}{\partial x} = \left(\frac{1}{2z^2} - \frac{1}{z} - 2\right)\frac{\partial}{\partial x},$$
$$Q\frac{\partial}{\partial y} = \left(-\frac{1}{2z^2} + \frac{1}{z} - 2\right)\frac{\partial}{\partial y}.$$

Using the above relations we have

$$r = -6 - \frac{1}{2z^2}.$$

Obviously, the scalar curvature is invariant along the contact distribution. We set

$$e = \frac{1}{\sqrt{z}e^z}\frac{\partial}{\partial x},\ \phi e = \frac{\sqrt{z}}{e^z}\frac{\partial}{\partial y}.$$

Then $\{\xi, e, \phi e\}$ is a local orthonormal basis of $M^3$ at each point. One can check that $(\nabla_e Q)e$, $(\nabla_e Q)\phi e$, $(\nabla_{\phi e} Q)e$, $(\nabla_{\phi e} Q)\phi e$ are all collinear with the Reeb vector field. Therefore, from (4.17) we see that $M^3$ is locally $\phi$-symmetric. By a direct calculation, we have $he = \frac{1}{2z}e$, $h\phi e = -\frac{1}{2z}\phi e$ and $\nabla_\xi h = -\frac{1}{z}h$.

## 5. Almost Kenmotsu 3-manifolds and semi-symmetry

In this section, we study the other kind of symmetry condition named semi-symmetry on almost Kenmotsu 3-manifolds.

A Riemannian manifold $(M, g)$ is said to be semi-symmetric if its Riemannian curvature tensor $R$ satisfies

$$R(X, Y)\cdot R = 0 \tag{5.1}$$

for any vector fields $X, Y$ on $M$, where the endomorphism $R(X, Y)$ acts on $R$ as a derivation. Riemannian manifolds satisfying (5.1) were first introduced by E. Cartan in [5] and first named "semi-symmetric" by N. S. Sinjukov in [22]. It is clear that a locally symmetric space is semi-symmetric, but the converse is not necessarily true. Looking for a non-locally symmetric semi-symmetric space and determining on what condition a semi-symmetric space is locally symmetric are interesting topics in history (see [23]).

Before giving our main results, we give a necessary and sufficient condition for an almost Kenmotsu 3-manifold to be semi-symmetric.

**Lemma 5.1.** *A non-Kenmotsu almost Kenmotsu* 3*-manifold is semi-symmetric if and only if*

$$\begin{aligned}
&(\xi(\lambda) + 2\lambda)\sigma(e) - (\lambda^2 - 2\lambda a + 1)\sigma(\phi e) = 0,\\
&(\xi(\lambda) + 2\lambda)\sigma(\phi e) - (\lambda^2 + 2\lambda a + 1)\sigma(e) = 0,\\
&(\xi(\lambda) + 2\lambda)(r + 4A) - 2\sigma(e)\sigma(\phi e) = 0,\\
&(\xi(\lambda) + 2\lambda)^2 + (\lambda^2 + 2\lambda a + 1)(A + 2\lambda a - 2\lambda^2 - 2) = (\sigma(\phi e))^2,\\
&(\xi(\lambda) + 2\lambda)^2 + (\lambda^2 - 2\lambda a + 1)(A - 2\lambda a - 2\lambda^2 - 2) = (\sigma(e))^2.
\end{aligned} \tag{5.2}$$

*Proof.* Let $M$ be an almost Kenmotsu 3-manifold with $h \neq 0$. Then $\mathcal{U}_1 = M$ and Lemmas 4.1, 4.2 are applicable. Applying Lemma 4.2 in (4.14), by (4.4), the curvature tensor of $M$ can be given as the following.

$$R(e,\xi)\xi = -(\lambda^2 + 2\lambda a + 1)e + (\xi(\lambda) + 2\lambda)\phi e.$$
$$R(\phi e,\xi)\xi = (\xi(\lambda) + 2\lambda)e - (\lambda^2 - 2\lambda a + 1)\phi e.$$
$$R(e,\phi e)\xi = -\sigma(\phi e)e + \sigma(e)\phi e.$$
$$R(e,\xi)e = (\lambda^2 + 2\lambda a + 1)\xi + \sigma(\phi e)\phi e.$$
$$R(e,\xi)\phi e = -(\xi(\lambda) + 2\lambda)\xi - \sigma(\phi e)e.$$
$$R(\phi e,\xi)e = -(\xi(\lambda) + 2\lambda)\xi - \sigma(e)\phi e.$$
$$R(e,\phi e)e = \sigma(\phi e)\xi + \frac{1}{2}(r + 4A)\phi e.$$
$$R(e,\phi e)\phi e = -\sigma(e)\xi - \frac{1}{2}(r + 4A)e.$$
$$R(\phi e,\xi)\phi e = (\lambda^2 - 2\lambda a + 1)\xi + \sigma(e)e.$$

By a direct calculation, using the above nine relations we have

$$\begin{aligned}
(R(e,\xi)\cdot R)(\xi,\phi e,\xi) ={}& \{(\xi(\lambda) + 2\lambda)\sigma(e) - (\lambda^2 - 2\lambda a + 1)\sigma(\phi e)\}e \\
&+ 2\{(\lambda^2 + 2\lambda a + 1)\sigma(e) - (\xi(\lambda) + 2\lambda)\sigma(\phi e)\}\phi e. \\
(R(e,\xi)\cdot R)(e,\phi e,\xi) ={}& \left\{\frac{1}{2}(\xi(\lambda) + 2\lambda)(r + 4A) - \sigma(e)\sigma(\phi e)\right\}e - \{(\sigma(\phi e))^2 \\
&- (\xi(\lambda)+2\lambda)^2 - (\lambda^2 + 2\lambda a+1)(A+2\lambda a-2\lambda^2-2)\}\phi e. \\
(R(\phi e,\xi)\cdot R)(\phi e,e,\xi) ={}& \left\{\frac{1}{2}(\xi(\lambda) + 2\lambda)(r + 4A) - \sigma(e)\sigma(\phi e)\right\}\phi e - \{(\sigma(e))^2 \\
&- (\xi(\lambda)+2\lambda)^2 - (\lambda^2-2\lambda a+1)(A-2\lambda a-2\lambda^2-2)\}e.
\end{aligned}$$

If $M^3$ is semi-symmetric, then (5.2) follows directly from (5.1) and the previous three relations. Conversely, one can check that if (5.2) is true then (5.1) holds for any vector fields $X, Y$. $\square$

**Theorem 5.1.** *Let $M$ be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$. Then $M$ is semi-symmetric if and only if one of the following cases occurs*:

- *$M$ is locally symmetric and in this case $M$ is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$.*
- *$M$ is semi-symmetric and in this case in a neighbourhood $U$ for every point $p \in M$, there exist coordinates $x, y, z$ and an orthonormal frame $\{\xi, e, \phi e\}$ of eigenvectors of $h$ corresponding eigenvalues $\{0, 1, -1\}$ such that*

$$\xi = \frac{\partial}{\partial x}, e = \frac{f_1}{\sqrt{2}}\frac{\partial}{\partial x} + \frac{1 + f_2}{\sqrt{2}}\frac{\partial}{\partial y} + \frac{f_3}{\sqrt{2}}\frac{\partial}{\partial z}, \phi e = -\frac{f_1}{\sqrt{2}}\frac{\partial}{\partial x} + \frac{1 - f_2}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{f_3}{\sqrt{2}}\frac{\partial}{\partial z},$$

*where* $f_i = e^{-2x-\sqrt{2}b(z)y}k_i(z)$, $k_i(z) : i = 1, 2, 3$ *are all smooth functions of* $z$ *and* $b(z)$ *is a non-constant function of* $z$.

*Proof.* It was proved in [15] that any semi-symmetric Kenmotsu manifold is of constant sectional curvature $-1$. Thus, next we consider only the non-Kenmotsu case. By Proposition 4.1, let $M$ be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 2a\phi h$, $a \in \mathbb{R}$, $h \neq 0$. As seen in proof of Theorem 4.1, $\nabla_\xi h = 2a\phi h$ is equivalent to $\xi(\lambda) = 0$. Now (5.2) becomes

$$\begin{aligned} &2\lambda\sigma(e) - (\lambda^2 - 2\lambda a + 1)\sigma(\phi e) = 0,\\ &2\lambda\sigma(\phi e) - (\lambda^2 + 2\lambda a + 1)\sigma(e) = 0,\\ &\lambda(r + 4A) - \sigma(e)\sigma(\phi e) = 0,\\ &4\lambda^2 + (\lambda^2 + 2\lambda a + 1)(A + 2\lambda a - 2\lambda^2 - 2) = (\sigma(\phi e))^2,\\ &4\lambda^2 + (\lambda^2 - 2\lambda a + 1)(A - 2\lambda a - 2\lambda^2 - 2) = (\sigma(e))^2. \end{aligned} \tag{5.3}$$

Multiplying the first term of (5.3) by $\sigma(e)$ gives that $2\lambda(\sigma(e))^2 = (\lambda^2-2\lambda a+1)\sigma(e)\sigma(\phi e)$. Putting the third and fifth terms of (5.3) into this equation gives

$$(\lambda^2 - 1 - 2\lambda a)(\lambda^2 - 1 + 2\lambda a) = 0, \tag{5.4}$$

where we used (4.4) and $\lambda > 0$. In view of (5.4), for simplicity, in what follows we consider only the case $\lambda^2 - 1 - 2\lambda a = 0$ since the proof for the other case is similar with this one. As $a \in \mathbb{R}$, it follows that $\lambda$ is a positive constant. From (4.3) we have $\sigma(e) = 2\lambda b$ and $\sigma(\phi e) = 2\lambda c$. Using this and $\lambda^2 - 1 - 2\lambda a = 0$ in the first or the second term of (5.3) gives $c = \lambda b$. This implies $\xi(c) = \lambda\xi(b)$ because of $\lambda$ a constant. On the other hand, using $\lambda, a \in \mathbb{R}$, from (4.2) we get

$$\xi(b) = (\lambda - a)c - b,\ \xi(c) = ab, \tag{5.5}$$

where in the second term of (5.5) we used $c = \lambda b$. Comparing (5.5) with $\xi(c) = \lambda\xi(b)$ and using $c = \lambda b$ gives

$$ab = \lambda^2 c - \lambda ac - c. \tag{5.6}$$

Multiplying (5.6) by $\lambda$, using $\lambda a = \frac{\lambda^2-1}{2}$ and $c = \lambda b$ we obtain $2ac = \lambda^3 c - \lambda c$. Multiplying this equation by $\lambda$ and using $\lambda a = \frac{\lambda^2-1}{2}$ we have

$$(\lambda^2 - 1)^2 c = 0. \tag{5.7}$$

It follows that either $\lambda = 1$ or $c = 0$. For the later case, we also have $b = 0$ and this implies that $\sigma(e) = \sigma(\phi e) = 0$ and $A = 2$. Using this in the third term of (5.3) gives $r = -8$. On the other hand, from (4.4) we have $r = -2(\lambda^2 + 3)$. Comparing this with $r = -8$ we have $\lambda = 1$ and this implies $a = 0$. Now, by relations (4.5)-(4.13) one can check that the Ricci operator is parallel, i.e., the manifold is locally symmetric.

For the first case $\lambda = 1$, from $\lambda^2 - 1 - 2\lambda a = 0$ and $c = \lambda b$ we obtain still $a = 0$, $b = c$ and hence $\sigma(e) = \sigma(\phi e) = 2b$. Next we only consider the case $b \neq 0$ on some open subset set since the case $b = 0$ has been discussed. Moreover, from (4.2) we have $\xi(b) = 0$. From Lemma 4.2 we have $A = 2+2b^2+e(b)+\phi e(b)$. However, using $\lambda = 1$, $a = 0$ and $\sigma(e) = \sigma(\phi e) = 2b$ in the forth or fifth term of (5.3) gives $A = 2(b^2+1)$ and hence we have $e(b) + \phi e(b) = 0$. Moreover, we state that $b$ is not a constant. In fact, if $b$ is a constant, from Lemma 4.1 we see that $M$ is locally isometric to a non-unimodular Lie group. It was proved in [14] that a non-unimodular Lie group is semi-symmetric if and only if it is locally isometric. This implies $b = 0$ and contradicts the assumption.

Now, from Lemma 4.2 the Ricci operator can be expressed with respect to local orthonormal basis $\{\xi, e, \phi e\}$ by the following form

$$Q = \begin{pmatrix} -4 & -2b & -2b \\ -2b & -2(b^2+1) & 2 \\ -2b & 2 & -2(b^2+1) \end{pmatrix}.$$

It follows that the eigenvalues of the Ricci operator are $\{-2(b^2+2), -2(b^2+2), 0\}$ and hence $M$ is semi-symmetric (see [2]).

Next, we set $e_1 := \frac{e-\phi e}{\sqrt{2}}$ and $e_2 := \phi e_1 = \frac{e+\phi e}{\sqrt{2}}$. Now from Lemma 4.1 we get

$$[\xi, e_1] = -2e_1,\ [e_1, e_2] = \sqrt{2}be_1,\ [\xi, e_2] = 0. \tag{5.8}$$

Since the distribution spanned by $\{\xi, e_2\}$ is integrable, then for any point $p$ there exists a chart $\{U, (x, y, z)\}$ such that

$$\xi = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_1 = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}, \tag{5.9}$$

where $f_1$, $f_2$ and $f_3$ are smooth functions. Using (5.8), from (5.9) we have

$$\begin{aligned} &\frac{\partial f_1}{\partial x} = -2f_1,\ \frac{\partial f_2}{\partial x} = -2f_2,\ \frac{\partial f_3}{\partial x} = -2f_3, \\ &\frac{\partial f_1}{\partial y} = -\sqrt{2}bf_1,\ \frac{\partial f_2}{\partial y} = -\sqrt{2}bf_2,\ \frac{\partial f_3}{\partial y} = -\sqrt{2}bf_3. \end{aligned} \tag{5.10}$$

From the above discussion we also have $\xi(b) = 0$, $e(b) + \phi e(b) = 0$, i.e.,

$$\frac{\partial b}{\partial x} = 0,\ \frac{\partial b}{\partial y} = 0. \tag{5.11}$$

By (5.11) and the above discussion, we know that $b$ is a non-constant function of $z$, denoted by $b(z)$. Therefore, a solution for (5.10) is given by

$$f_i = e^{-2x-\sqrt{2}b(z)y}k_i(z),$$

where $k_i : i = 1, 2, 3$ are smooth functions of $z$. This completes the proof. $\square$

*Remark* 5.1. According to Theorem 5.1 one can construct many semi-symmetric almost Kenmotsu 3-manifolds which are neither Kenmotsu nor locally symmetric.

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