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ON A CLASS OF QUASILINEAR ELLIPTIC EQUATION WITH INDEFINITE WEIGHTS ON GRAPHS

Shoudong Man and Guoqing Zhang

Abstract. Suppose that $G = (V, E)$ is a connected locally finite graph with the vertex set V and the edge set E. Let $\Omega \subset V$ be a bounded domain. Consider the following quasilinear elliptic equation on graph G

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u + f(x,u), \ x \in \Omega^\circ, \\ u = 0, \ x \in \partial\Omega, \end{cases}$$

where Ω° and $\partial\Omega$ denote the interior and the boundary of Ω, respectively, Δ_p is the discrete p-Laplacian, $K(x)$ is a given function which may change sign, λ is the eigenvalue parameter and $f(x,u)$ has exponential growth. We prove the existence and monotonicity of the principal eigenvalue of the corresponding eigenvalue problem. Furthermore, we also obtain the existence of a positive solution by using variational methods.

1. Introduction

In this paper, we consider the following quasilinear elliptic equation with indefinite weights on graph G

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u + f(x,u), \ x \in \Omega^\circ, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $G = (V, E)$ is a locally finite graph, $\Omega \subset V$ is a bounded domain, Ω° and $\partial\Omega$ denote the interior and the boundary of Ω, respectively, Δ_p denotes the discrete p-Laplacian, λ is the eigenvalue parameter and $K(x)$ is a given function which satisfies

$$K^+(x) \not\equiv 0, \ K \in L^1(\Omega), \ K^\pm(x) = \max\{\pm K(x), 0\}. \tag{1.2}$$

Quasilinear elliptic equations have been studied extensively on Euclidean domain and Riemannian manifold. Zhang and Liu [14] investigated the following

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critical elliptic equations with indefinite weights

$$\begin{cases} -\Delta u - \mu \frac{u}{(|x| \ln \frac{R}{|x|})^2} = \lambda K(x) u + f(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $K(x)$ satisfies (1.2), and proved the existence of a nontrivial solution by using the Mountain Pass Lemma. As for the p-Laplacian, in particular, Fan and Li [3] discussed

$$\begin{cases} -\triangle_p u = \lambda |u|^{p-2} u + f(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $0 < \lambda < \lambda_2$ and λ_2 is the second eigenvalue of the p-Laplacian with indefinite weights, and obtained the existence of a nontrivial solution for the problem (1.4). In [11], B. Xuan studied the following elliptic equation

$$\begin{cases} -\triangle_p u = \lambda V |u|^{p-2} u + f(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $V(x)$ is a given function satisfying

$$V^+ \not\equiv 0 \quad \text{and} \quad V \in L^s(\Omega).$$

B. Xuan obtained the existence of a nontrivial weak solution for the problem (1.5) in the case of $0 < \lambda < \lambda_1$ by the Mountain Pass Lemma and in the case of $\lambda_1 \le \lambda < \lambda_2$ by the Linking Argument Theorem, respectively. In [2], M. Degiovanni and S. Lancelotti studied the problem (1.5) when $V \in L^\infty(\Omega)$ and $f(x,u)$ is subcritical and superlinear at 0 and at infinity respectively, and they proved that there exists a nontrivial solution for the problem (1.5) for any $\lambda \in \mathbb{R}$.

Most recently, the investigation of discrete weighted Laplacians and various equations on graphs have attracted much attention [4–9, 13, 15, 16]. A. Grigor'yan, Y. Lin and Y. Y. Yang [7] studied the following Yamabe type equation

$$\begin{cases} -\Delta_p u + h(x)|u|^{p-2} u = f(x,u), \ x \in \Omega^\circ, \\ u \ge 0, \ x \in \Omega^\circ, \ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.6}$$

and obtained the existence of a positive solution if $h(x) > 0$. Ge [4] studied the following p-th Yamabe equation

$$-\Delta_p u + h(x) u^{p-1} = \lambda g u^{\alpha-1}, \quad x \in \Omega^\circ, \tag{1.7}$$

where $\alpha \ge p > 1$, and proved the existence of a positive solution. Zhang and Lin [16] proved that the problem (1.7) has at least a positive solution as $2 < \alpha \le p$.

In this paper, we study the quasilinear elliptic equation (1.1) on graph G, which can be viewed as a discrete version of the equation (1.5) studied by B.

Xuan, etc. Firstly, we consider the following eigenvalue problem with indefinite weights on graph G

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u, \ x \in \Omega^\circ, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.8}$$

where $K(x)$ satisfies the assumption (1.2), and obtain the existence and monotonicity of the principal eigenvalue. Secondly, using the Mountain Pass Lemma and the Sobolev embedding theorem on graph G, we obtain the existence of a positive solution of the problem (1.1) as the nonlinear term $f(x,s)$ has exponential growth as $s \to +\infty$.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas on graph G, and state our main results. In Section 3, we prove our main theorems.

2. Preliminaries and main results

Let $G = (V, E)$ be a graph. The degree of vertex x, denoted by $\mu(x)$, is the number of edges connected to x. If $\mu(x)$ is finite for every vertex x of V, we say that G is a locally finite graph. We denote $x \sim y$ if vertex x is adjacent to vertex y, and $\omega_{xy} = \omega_{yx} > 0$ is the edge weight. The finite measure $\mu(x) = \sum_{y\sim x} \omega_{xy}$. The boundary of Ω is defined as $\partial\Omega = \{y \bar{\in} \Omega : \exists x \in \Omega$ such that $xy \in E\}$ and the interior of Ω is denoted by $\Omega^\circ = \Omega \setminus \partial\Omega$. A graph G is called connected if for any vertices $x, y \in V$, there exists $\{x_i\}_{i=0}^n$ that satisfies $x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n = y$. In this paper, we suppose that all graphs are connected.

From [7], for any function $u : \Omega \to \mathbb{R}$, the μ-Laplacian of u is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}[u(y) - u(x)]. \tag{2.1}$$

The associated gradient form reads

$$\Gamma(u,v)(x) = \frac{1}{2}\{\Delta(u(x)v(x)) - u(x)\Delta v(x) - v(x)\Delta u(x)\} \tag{2.2}$$

$$= \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{2.3}$$

The length of the gradient for u is

$$|\nabla u|(x) = \sqrt{\Gamma(u,u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{1/2}. \tag{2.4}$$

For any function $u : \Omega \to \mathbb{R}$, we denote

$$\int_\Omega u d\mu = \sum_{x\in\Omega} \mu(x)u(x), \tag{2.5}$$

and set

$$Vol(G) = \int_\Omega d\mu. \tag{2.6}$$

The p-Laplacian of $u : \Omega \to \mathbb{R}$, namely $\Delta_p u$, is defined in the distributional sense as

$$\int_\Omega (\Delta_p u)\phi d\mu = -\int_\Omega |\nabla u|^{p-2}\Gamma(u,\phi)d\mu, \ \forall \phi \in C_c(\Omega), \tag{2.7}$$

where $C_c(\Omega)$ is the set of all functions with compact support. So, $\Delta_p u$ can be written as

$$\Delta_p u(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))(u(y) - u(x)). \tag{2.8}$$

For any $p > 1$, $W^{1,p}(\Omega)$ is defined as a space of all functions $u : \Omega \to \mathbb{R}$ satisfying

$$||u||_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p d\mu + \int_\Omega |u|^p d\mu\right)^{1/p} < \infty. \tag{2.9}$$

Denote $C_0^1(\Omega)$ as a set of all functions $u : \Omega \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$, and $W_0^{1,p}(\Omega)$ as the completion of $C_0^1(\Omega)$ under the norm (2.9).

Lemma 2.1 ([7, Theorem 7])**.** *Let $G = (V, E)$ be a locally finite graph and Ω be a bounded domain of V such that $\Omega^0 \neq \emptyset$. For any $p > 1$, $W_0^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$. In particular, there exists a constant C depending only on p and Ω such that*

$$\left(\int_\Omega |u|^q d\mu\right)^{1/q} \leq C \left(\int_\Omega |\nabla u|^p d\mu\right)^{1/p} \tag{2.10}$$

for all $1 \leq q \leq +\infty$ and $u \in W_0^{1,p}(\Omega)$. Moreover, $W_0^{1,p}(\Omega)$ is pre-compact, namely, if $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$, then up to a subsequence, there exists some $u \in W_0^{1,p}(\Omega)$ such that $u_k \to u$ in $W_0^{1,p}(\Omega)$.

By Lemma 2.1, we obtain that $W_0^{1,p}(\Omega)$ is a Banach space.

Lemma 2.2 ([1, Mountain Pass Lemma])**.** *Let $(X, ||\cdot||)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $||e|| > r$ and $b = \inf_{||u||=r} J(u) > J(0) > J(e)$. If J satisfies the $(PS)_c$ condition with $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}$, then c is a critical value of J.*

For the well-know Trudinger-Moser inequality on Euclidean domain and complete Riemannian manifold [10, 12], by Lemma 2.1, we have:

Lemma 2.3 (Trudinger-Moser inequality on locally finite graphs)**.** *Suppose that $G = (V, E)$ is a locally finite graph. Let $\Omega \subset V$ be a bounded domain.*

Then there exists a constant C which depends only on p and Ω such that

$$\sup_{||u||_{W_0^{1,p}(\Omega)}\leq 1}\int_\Omega \exp(\alpha|u|^{\frac{p}{p-1}})d\mu \leq C|\Omega| \quad for\ any\quad \alpha>1 \ and\ p>2, \tag{2.11}$$

where $|\Omega| = \int_\Omega d\mu(x) = Vol\,\Omega$, and $Vol\,\Omega$ denotes the volume of the subgraph Ω.

Proof. For any function u which satisfies $||u||_{W_0^{1,p}(\Omega)} \leq 1$, by Lemma 2.1 and $\frac{p}{p-1} > 1$, we obtain that there exists a constant C_0 such that

$$\left(\int_\Omega |u|^{\frac{p}{p-1}}d\mu\right)^{\frac{p-1}{p}} \leq C_0\left(\int_\Omega |\nabla u|^p d\mu\right)^{\frac{1}{p}} = C_0||u||_{W_0^{1,p}(\Omega)} \leq C_0. \tag{2.12}$$

Denote $\mu_{\min} = \min_{x\in\Omega}\mu(x)$. Then (2.12) leads to

$$||u||_{L^\infty(\Omega)} \leq \frac{C_0}{\mu_{\min}}. \tag{2.13}$$

Thus for any $\alpha > 1$ and $p > 2$, we have

$$\left(\int_\Omega \exp(\alpha|u|^{\frac{p}{p-1}})d\mu\right)^{\frac{p-1}{p}} \leq \exp\left(\frac{\alpha C_0}{\mu_{\min}}\right)|\Omega|^{\frac{p-1}{p}}. \tag{2.14}$$

So, we have

$$\sup_{||u||_{W_0^{1,p}(\Omega)}\leq 1}\int_\Omega \exp(\alpha|u|^{\frac{p}{p-1}})d\mu \leq C|\Omega|, \tag{2.15}$$

where $C = \left(\exp(\frac{\alpha C_0}{\mu_{\min}})\right)^{\frac{p}{p-1}}$. □

In order to find the principal eigenvalue of the problem (1.8), we solve the following minimization problem

$$(P)\quad \text{minimize}\quad \int_\Omega |\nabla u|^p d\mu,\quad u\in W_0^{1,p}(\Omega)\quad \text{and}\quad \int_\Omega K(x)|u|^p d\mu = 1. \tag{2.16}$$

Now, we state our main theorems.

Theorem 2.4. *Under the assumption* (1.2)*, Problem* (P) *has a solution* $e_1 \geq 0$*. Moreover,* e_1 *is an eigenvalue of the problem* (1.8) *corresponding to the principal eigenvalue* $\lambda_1 = \int_\Omega |\nabla e_1|^p d\mu$.

Theorem 2.5. *Let $K_1(x)$ and $K_2(x)$ be two weights which satisfy the assumption* (1.2)*. Assume $K_1(x) < K_2(x)$ for all $x \in \Omega$ and $\{x\in\Omega : K_1 < K_2\} \neq \emptyset$. Then $\lambda_1(K_2) < \lambda_1(K_1)$.*

Theorem 2.6. *Let Ω_1 be a proper bounded open subset of a bounded domain $\Omega_2 \subset K$. Then $\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1)$.*

Theorem 2.7. *Let $G = (V, E)$ be a locally finite graph and $\Omega \subset V$ be a bounded domain with $\Omega^0 \neq \emptyset$. Let λ_1 be defined as in Theorem* 2.4*. Set $0 < \lambda < \lambda_1$ and $p > 2$. Suppose that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses*:

(H1) *For any* $x \in \Omega$, $f(x,t)$ *is continuous in* $t \in \mathbb{R}$;
(H2) *For all* $(x,t) \in \Omega \times [0,+\infty)$, $f(x,t) \geq 0$, *and* $f(x,0) = 0$ *for all* $x \in \Omega$;
(H3) $f(x,t)$ *has exponential growth at* $+\infty$, *that is, for all* $\alpha > 1$,

$$\lim_{t\to+\infty} \frac{f(x,t)}{\exp(\alpha|t|^{\frac{p}{p-1}})} = 0; \tag{2.17}$$

(H4) *For any* $x \in \Omega$, *there holds* $\lim_{t\to 0+} \frac{f(x,t)}{t^{p-1}} = 0$;
(H5) *There exist* $q > p > 2$ *and* $s_0 > 0$ *such that if* $s \geq s_0$, *then there holds* $0 < qF(x,s) < f(x,s)s$ *for any* $x \in \Omega$, *where* $F(x,s) = \int_0^s f(x,t)dt$.

Then there exists a positive solution for the problem (1.1).

3. The proof of main Theorems

Proof of Theorem 2.4. Now, we introduce the functional ϕ, ψ: $W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\phi(u) = \int_\Omega |\nabla u|^p d\mu, \quad \text{and} \quad \psi(u) = \int_\Omega K(x)|u|^p d\mu. \tag{3.1}$$

Let us also introduce the set

$$M = \{u \in W_0^{1,p}(\Omega) : \psi(u) = 1\}. \tag{3.2}$$

By the assumption (1.2), we have $M \neq \emptyset$ and M is a manifold of C^1 in $W_0^{1,p}(\Omega)$. Obviously the functional $\phi(u)$ is bounded from below. Hence, let $\{u_n\}$ be a minimizing sequence for the problem (P) such that

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p d\mu \leq \lambda_1 + o_k(1)||u_k||_{W_0^{1,p}(\Omega)}, \tag{3.3}$$

$$\int_\Omega |\nabla u_k|^p d\mu = o_k(1)||u_k||_{W_0^{1,p}(\Omega)}. \tag{3.4}$$

Calculate (3.3)$-\frac{1}{\theta}\times$(3.4), we have

$$\left(\frac{1}{p} - \frac{1}{\theta}\right)||u_m||^p_{W_0^{1,p}(\Omega)} \leq M, \tag{3.5}$$

where $\theta > p$ is a constant. This implies that $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus by Lemma 2.1, there exists some $e_1 \in W_0^{1,p}(\Omega)$ such that $u_n \to e_1$ in $W_0^{1,p}(\Omega)$ and

$$\phi(e_1) = \int_\Omega |\nabla e_1|^p d\mu \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p d\mu = \liminf_{n\to\infty} \phi(u_n) = \inf(P) \tag{3.6}$$

and $\int_\Omega K(x)|e_1|^p d\mu = 1$. It is clear that e_1 is a solution of the problem (P). Moreover, since $|e_1|$ is also a solution of the problem (P), we may assume $e_1 \geq 0$.

Since for every $v \in C_c^1(\Omega)$, we have

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \frac{\phi(e_1 + \varepsilon v)}{\psi(e_1 + \varepsilon v)} = 0. \tag{3.7}$$

So e_1 is an eigenfunction of the problem (P) corresponding to the principal eigenvalue $\lambda_1 = \int_\Omega |\nabla e_1|^p d\mu$. □

Proof of Theorem 2.5. By Theorem 2.4, we have

$$\lambda_1 = \inf_{u\not\equiv 0, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^p d\mu}{\int_\Omega K(x)|u|^p d\mu} = \inf_{u\in M} \phi(u). \tag{3.8}$$

Let u_1 be an eigenfunction associated to $\lambda_1(K_1)$. So we have

$$\lambda_1(K_1) = \frac{\int_\Omega |\nabla u_1|^p d\mu}{\int_\Omega K(x)|u_1|^p d\mu}. \tag{3.9}$$

Since $K_1 < K_2$ and $\mu(x) > 0$ for all $x \in \Omega$, we have

$$\begin{aligned} \int_\Omega K_1(x)|u_1|^p d\mu &= \sum_{x\in\Omega} K_1(x)|u_1(x)|^p \mu(x) \\ &< \sum_{x\in\Omega} K_2(x)|u_1(x)|^p \mu(x) \\ &= \int_\Omega K_2(x)|u_1|^p d\mu. \end{aligned} \tag{3.10}$$

Using $u_2 = \frac{u_1}{(\int_\Omega K_2(x)|u_1|^p d\mu)^{1/p}}$ as an admissible function in (3.8) for $\lambda_1(K_2)$, we have

$$\begin{aligned} \lambda_1(K_2) &\le \frac{\int_\Omega |\nabla u_2|^p d\mu}{\int_\Omega K_2(x)|u_2|^p d\mu} = \frac{\int_\Omega |\nabla u_1|^p d\mu}{\int_\Omega K_2(x)|u_1|^p d\mu} \\ &< \frac{\int_\Omega |\nabla u_1|^p d\mu}{\int_\Omega K_1(x)|u_1|^p d\mu} = \lambda_1(K_1). \end{aligned} \tag{3.11}$$

Thus we have $\lambda_1(K_2) < \lambda_1(K_1)$. □

Proof of Theorem 2.6. Let $u \in W_0^{1,p}(\Omega_1)$ be an eigenfunction associated to $\lambda_1(\Omega_1)$, that is,

$$\lambda_1(\Omega_1) = \frac{\int_{\Omega_1} |\nabla u|^p d\mu}{\int_{\Omega_1} K(x)|u|^p d\mu}. \tag{3.12}$$

Let $\tilde{u} \in W_0^{1,p}(\Omega_2)$ satisfy

$$\tilde{u} = \begin{cases} u, & x \in \Omega_1, \\ 0, & x \in \Omega_2 \setminus \Omega_1. \end{cases} \tag{3.13}$$

So we have

$$\int_{\Omega_2} K|\tilde{u}|^p d\mu = \sum_{x\in\Omega_1} K(x)|u(x)|^p \mu(x) = \int_{\Omega_1} K|u|^p d\mu, \tag{3.14}$$

$$\int_{\Omega_2} |\nabla\tilde{u}|^p d\mu = \int_{\Omega_1} |\nabla\tilde{u}|^p d\mu. \tag{3.15}$$

Combining (3.14) and (3.15), and taking $\tilde{u}/(\int_{\Omega_2} K\tilde{u}^p d\mu)^{1/p}$ as an admissible function for $\lambda_1(\Omega_2)$, we have

$$\lambda_1(\Omega_2) \leq \frac{\int_{\Omega_2} |\nabla \tilde{u}|^p d\mu}{\int_{\Omega_2} K(x)|\tilde{u}|^p d\mu} = \frac{\int_{\Omega_1} |\nabla \tilde{u}|^p d\mu}{\int_{\Omega_1} K(x)|\tilde{u}|^p d\mu} = \lambda_1(\Omega_1). \tag{3.16}$$

Thus, the proof is complete. □

Proof of Theorem 2.7. Now, we define the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p d\mu - \frac{\lambda}{p}\int_\Omega K(x)|u|^p d\mu - \int_\Omega F(x, u^+) d\mu, \tag{3.17}$$

where $u^+(x) = \max\{u(x), 0\}$. By the definition of λ_1, for $u \in W_0^{1,p}(\Omega)$ we have

$$\lambda_1 \int_\Omega K(x)|u|^p d\mu \leq \int_\Omega |\nabla u|^p d\mu. \tag{3.18}$$

Since $0 < \lambda < \lambda_1$, we have

$$J(u) \geq \frac{1}{p}(1 - \frac{\lambda}{\lambda_1}) \int_\Omega |\nabla u|^p d\mu - \int_\Omega F(x, u^+) d\mu. \tag{3.19}$$

Indeed, from (H4), there exist $\tau, \delta > 0$ such that if $|u| \leq \delta$ we have

$$f(x, u^+) \leq \tau (u^+)^{p-1}. \tag{3.20}$$

On the other hand, by (H3), there exist c, β such that

$$f(x, u^+) \leq c \exp(\beta |u|^{\frac{p}{p-1}}), \ \forall |u| \geq \delta. \tag{3.21}$$

Then we obtain that, for $q > p$,

$$F(x, u^+) \leq c \exp(\beta |u|^{\frac{p}{p-1}})|u|^q, \ \forall |u| \geq \delta. \tag{3.22}$$

Combining (3.20) and (3.22), we obtain that

$$F(x, u^+) \leq \tau \frac{|u|^p}{p} + c \exp(\beta |u|^{\frac{p}{p-1}})|u|^q. \tag{3.23}$$

By the Hölder inequality, we have

$$\begin{aligned} J(u) \geq & \frac{1}{p}(1 - \frac{\lambda}{\lambda_1})||u||^p_{W_0^{1,p}(\Omega)} - \frac{\tau}{p}\int_\Omega |u|^p d\mu \\ & - c\left(\int_\Omega \exp(\beta p |u|^{\frac{p}{p-1}}) d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{qp'} d\mu\right)^{\frac{1}{p'}}, \end{aligned} \tag{3.24}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By Lemma 2.3, when $||u||_{W_0^{1,p}(\Omega)} \leq 1$ for any $u \in W_0^{1,p}(\Omega)$ we obtain that

$$\int_\Omega \exp(\beta p |u|^{\frac{p}{p-1}}) d\mu < C|\Omega|. \tag{3.25}$$

By Lemma 2.1, there exists some constant C that depends only on p and Ω such that

$$\left(\int_\Omega |u|^p d\mu\right)^{1/p} \le C\left(\int_\Omega |\nabla u|^p d\mu\right)^{1/p} = C||u||_{W_0^{1,p}(\Omega)}, \tag{3.26}$$

$$\left(\int_\Omega |u|^{qp'} d\mu\right)^{1/p'} \le C\left(\int_\Omega |\nabla u|^p d\mu\right)^{q/p} = C||u||^q_{W_0^{1,p}(\Omega)}. \tag{3.27}$$

By (3.24), (3.25), (3.26) and (3.27), we can find some sufficiently small $r > 0$ such that if $||u||_{W_0^{1,p}(\Omega)} = r$ we have

$$J(u) \ge \frac{1}{p}(1 - \frac{\lambda}{\lambda_1} - C\tau)||u||^p_{W_0^{1,p}(\Omega)} - C||u||^q_{W_0^{1,p}(\Omega)}. \tag{3.28}$$

By (H4), we set $\tau < \frac{1}{C}(1 - \frac{\lambda}{\lambda_1})$. Since $q > p > 2$, we have

$$\inf_{||u||_{W_0^{1,p}(\Omega)}=r} J(u) > 0. \tag{3.29}$$

By (H5), there exist two positive constants c_1 and c_2 such that

$$F(x, u^+) \ge c_1(u^+)^q - c_2. \tag{3.30}$$

Take $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0 \ge 0$ and $u_0 \not\equiv 0$. For any $t > 0$, we have

$$J(tu_0) \le \frac{t^p}{p}||u_0||^p_{W_0^{1,p}(\Omega)} - \frac{t^p}{p}\lambda\int_\Omega K(x)|u_0|^p d\mu - c_1 t^q \int_\Omega u_0^q d\mu + c_2|\Omega|. \tag{3.31}$$

Since $q > p > 2$, we have $J(tu_0) \to -\infty$ as $t \to +\infty$. Hence there exists some $u_1 \in W_0^{1,p}(\Omega)$ satisfying

$$J(u_1) < 0,\ ||u_1||_{W_0^{1,p}(\Omega)} > r. \tag{3.32}$$

Now we prove that $J(u)$ satisfies the $(PS)_c$ condition for any $c \in R$. To see this, we assume $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to \infty$, that is

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p d\mu - \frac{1}{p}\lambda\int_\Omega K(x)|u_k|^p d\mu - \int_\Omega F(x, u_k^+)d\mu = c + o_k(1), \tag{3.33}$$

$$\int_\Omega |\nabla u_k|^p d\mu - \lambda\int_\Omega K(x)|u_k|^p d\mu - \int_\Omega u_k f(x, u_k^+)d\mu = o_k(1)||u_k||_{W_0^{1,p}(\Omega)}. \tag{3.34}$$

By the definition of λ_1, (3.33) and (H5), we have

$$\begin{aligned}\frac{1}{p}(1 - \frac{\lambda}{\lambda_1})\int_\Omega |\nabla u_k|^p d\mu &\le \int_\Omega F(x, u_k^+)d\mu + c + o_k(1)\\ &\le \frac{1}{q}\int_\Omega u_k f(x, u_k^+)d\mu + c + o_k(1).\end{aligned} \tag{3.35}$$

By (3.34) and (3.35), we get

$$(\frac{1}{p} - \frac{1}{q})(1 - \frac{\lambda}{\lambda_1})\int_\Omega |\nabla u_k|^p d\mu \le C - \frac{1}{q}o_k(1)||u_k||_{W_0^{1,p}(\Omega)}. \tag{3.36}$$

Since $q > p$, we have $||u_k||_{W_0^{1,p}(\Omega)} \leq M$. Thus $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Then the $(PS)_c$ condition follows by Lemma 2.1.

Combining (3.29), (3.32) and the obvious fact that $J(0) = 0$, we conclude by Lemma 2.2 that there exists a function $u \in W_0^{1,p}(\Omega)$ such that

$$J(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) > 0 \tag{3.37}$$

and $J'(u) = 0$, where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1\}$. Hence there exists a nontrivial solution $u \in W_0^{1,p}(\Omega)$ to the equation

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u + f(x,u), \ x \in \Omega^\circ, \\ u = 0, \ x \in \partial\Omega. \end{cases}$$

Testing the above equation by $u^- = \min\{u, 0\}$ and noting that

$$\Gamma(u^-, u) = \Gamma(u^-, u^-) + \Gamma(u^-, u^+) \geq |\nabla u^-|^2, \tag{3.38}$$

since $0 < \lambda < \lambda_1$, we have

$$\begin{aligned} \int_\Omega |u^-|^p d\mu - \lambda \int_\Omega K(x)|u^-|^p d\mu &\leq - \int_\Omega u^- \Delta_p u d\mu - \lambda \int_\Omega K(x) u^- |u^-|^{p-2} u d\mu \\ &= \int_\Omega u^- f(x, u^+) d\mu = 0. \end{aligned}$$

This implies that $u^- \equiv 0$ and thus $u \geq 0$. □

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Shoudong Man
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China
Email address: manshoudong@163.com; shoudongmantj@tjufe.edu.cn

Guoqing Zhang
College of Sciences
University of Shanghai for Science and Technology
Shanghai, 200093, P. R. China
Email address: shzhangguoqing@126.com