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NEUTROSOPHIC IDEALS IN SUBTRACTION ALGEBRAS

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Abstract. The notions of a neutrosophic subalgebra and a neutrosohic ideal of a subtraction algebra are introduced. Characterizations of a neutrosophic subalgebra and a neutrosophic ideal are investigated. We show that the homomorphic preimage of a neutrosophic subalgebra of a subtraction algebra is a neutrosophic subalgebra, and the onto homomorphic image of a neutrosophic subalgebra of a subtraction algebra is a neutrosophic subalgebra.

1. Introduction

B. M. Schein [8] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction " $\setminus$ " (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). B. Zelinka [11] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4, 5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. S. S. Ahn and Y. H. Kim [2] introduced the notions of an intersectional soft subalgebra and an intersectional soft ideal of a subtraction algebra and investigated some related properties of them.

Zadeh [10] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [3] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree

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of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Jun et. al [7] introduced the notions of a neutrosophic $\mathscr{N}$-subalgebras and a (closed) neutrosophic $\mathscr{N}$-ideal in a BCK/BCI-algebras and investigated some related properties.

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosohic ideal of a subtraction algebra. Characterizations of a neutrosophic subalgebra and a neutrosophic ideal are investigated. We show that the homomorphic preimage of a neutrosophic subalgebra of a subtraction algebra is a neutrosophic subalgebra, and the onto homomorphic image of a neutrosophic subalgebra of a subtraction algebra is a neutrosophic subalgebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results (see [5]).

By a *subtraction algebra* we mean an algebra $(X, -, 0)$ with a single binary operation " $-$ " that satisfies the following conditions: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on X: $a \leq b$ if and only if $a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebras in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Hence $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(a1) $(x - y) - y = x - y$,
(a2) $x - 0 = x$ and $0 - x = 0$,
(a3) $(x - y) - x = 0$,
(a4) $x - (x - y) \leq y$,

A non-empty subset A of a subtraction algebra X is called a *subalgebra* [4] of X if $x - y \in A$ for any $x, y \in A$. A non-empty subset I of a subtraction algebra X is called an *ideal* [4] of X if

(I1) $0 \in I$,

(I2) $(\forall x, y \in X)(x - y, y \in I$ imply $x \in I)$.

A mapping $f : X \to Y$ of subtraction algebras is called a *homomorphism* if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$.

Definition 2.1. *Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by*

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.2. ([7]) *Let A be a neutrosophic set in a subtraction algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as*

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

For any family $\{a_i | i \in \Lambda\}$, we define

$$\bigvee\{a_i | i \in \Lambda\} := \begin{cases} \max\{a_i | i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i | i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{a_i | i \in \Lambda\} := \begin{cases} \min\{a_i | i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i | i \in \Lambda\} & \text{otherwise.} \end{cases}$$

3. Neutrosophic ideals

In what follows, let X be a subtraction algebra unless otherwise specified.

Definition 3.1. *A neutrosophic set A in X is called a neutrosophic subalgebra of X if it satisfies:*

(3.1) $(\forall x, y \in X)(T_A(x-y) \geq \min\{T_A(x), T_A(y)\}, I_A(x-y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x-y) \leq \max\{F_A(x), F_A(y)\})$.

Proposition 3.2. *Every neutrosophic subalgebra of X satisfies the following conditions:*

(3.2) $(\forall x \in X)(T_A(0) \geq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x))$.

Proof. Straightforward. □

Example 3.3. *Let $X := \{0, 1, 2, 3\}$ be a subtraction algebra [6] with the following table:*

$-$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	1
3	3	2	1	0

Define a neutrosophic set A in X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.84 & \text{if } x \in \{0,3\} \\ 0.13 & \text{if } x \in \{1,2\}, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.84 & \text{if } x \in \{0,3\} \\ 0.13 & \text{if } x \in \{1,2\}, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.11 & \text{if } x \in \{0,3\} \\ 0.83 & \text{if } x \in \{1,2\}. \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $0 \leq \alpha+\beta+\gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \geq \alpha, T_A(y) \geq \alpha, I_A(x) \geq \beta, I_A(y) \geq \beta$ and $F_A(x) \leq \gamma, F_A(y) \leq \gamma$. Using (3.1), we have $T_A(x-y) \geq \min\{T_A(x), T_A(y)\} \geq \alpha$, $I_A(x-y) \geq \min\{I_A(x), I_A(y)\} \geq \beta$, and $F_A(x-y) \leq \max\{F_A(x), F_A(y)\} \leq \gamma$. Hence $x - y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $T_A(a_t - b_t) < \min\{T_A(a_t), T_A(b_t)\}, I_A(a_i - b_i)\} < \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f - b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t - b_t) <$

$\alpha_1 \leq \min\{T_A(a_t), T_A(b_t)\}, I_A(a_i - b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f - b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \beta_1 \in (0,1]$ and $\gamma_1 \in [0,1)$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t - b_t, a_i - b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f - b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x-y) \geq \min\{T_A(x), T_A(y)\}, I_A(x-y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x-y) \leq \max\{T_A(x), T_A(y)\}$ for any $x, y, z \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 1, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 1, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \geq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \geq T_A(a_n)$, $I_A(0) \geq I_A(a_n)$, and $F_A(0) \leq F_A(a_n)$ for every positive integer n. Therefore $1 = \lim_{n\to\infty} T_A(a_n) \leq T_A(0) \leq 1, 1 = \lim_{n\to\infty} I_A(a_n) \leq I_A(0) \leq 1$, and $0 \leq F_A(0) \leq \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 1, I_A(0) = 1$, and $F_A(0) = 0$. □

Proposition 3.7. *If every neutrosophic subalgebra A of X satisfies the condition*

(3.3) $(\forall x, y \in X)(T_A(x-y) \geq T_A(y), I_A(x-y) \geq I_A(y)$, and $F_A(x-y) \leq F_A(y))$,

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.3) that $T_A(x) = T_A(x-0) \geq T_A(0), I_A(x) = I_A(x-0) \geq I_A(0)$, and $F_A(x) = F_A(x-0) \leq F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Theorem 3.8. *Every subalgebra of X can be represented as an (α, β, γ)-level set of a neutrosophic subalgebra A of X.*

Proof. Let S be a subalgebra of X and let A be a neutrosophic subalgebra of X. Define a neutrosophic set A in X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in S \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in S \\ \beta_2 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in S \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1]$ and $\gamma_1, \gamma_2 \in [0,1)$ with $\alpha_1 > \alpha_2, \beta_1 > \beta_2, \gamma_1 < \gamma_2$, and $0 \leq \alpha_1 + \beta_1 + \gamma_1 \leq 3, 0 \leq \alpha_2 + \beta_2 + \gamma_2 \leq 3$. Obviously, $S = A^{(\alpha_1,\beta_1,\gamma_1)}$. We now prove that A is a neutrosophic subalgebra of X. Let $x, y \in X$. If $x, y \in S$, then $x - y \in S$ because S is a subalgebra of X. Hence $T_A(x) = T_A(y) = T_A(x-y) = \alpha_1$, $I_A(x) = I_A(y) = I_A(x-y) = \beta_1$, $F_A(x) = F_A(y) = F_A(x-y) = \gamma_1$ and so $T_A(x-y) \geq \min\{T_A(x), T_A(y)\}$, $I_A(x-y) \geq \min\{I_A(x), I_A(y)\}$, $F_A(x-y) \leq \max\{F_A(x), F_A(y)\}$. If $x \in S$ and $y \notin S$, then $T_A(x) = \alpha_1, T_A(y) = \alpha_2$, $I_A(x) = \beta_1, I_A(y) = \beta_2$, $F_A(x) = \gamma_1, F_A(y) = \gamma_2$ and so $T_A(x-y) \geq \min\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x-y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x-y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Obviously, if $x \notin S$ and $y \notin S$, then $T_A(x-y) \geq \min\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x-y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x-y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Therefore A is a neutrosophic subalgebra of X. □

Theorem 3.9. *Let A be a neutrosophic set of X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Define a neutrosophic set A^* in X as follows:*

$$T_{A^*} : X \to [0,1],\ x \mapsto \begin{cases} T_A(x) & \text{if } x \in A^{(\alpha,\beta,\gamma)} \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{A^*} : X \to [0,1],\ x \mapsto \begin{cases} I_A(x) & \text{if } x \in A^{(\alpha,\beta,\gamma)} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_{A^*} : X \to [0,1],\ x \mapsto \begin{cases} F_A(x) & \text{if } x \in A^{(\alpha,\beta,\gamma)} \\ 1 & \text{otherwise.} \end{cases}$$

If A is a neutrosophic subalgebra of X, then so is A^*.

Proof. Let A be a neutrosophic subalgebra of X. By Theorem 3.4, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X. If $x, y \in A^{(\alpha,\beta,\gamma)}$, then $x - y \in A^{(\alpha,\beta,\gamma)}$. Hence we have $T_{A^*}(x-y) = T_A(x-y) \geq \min\{T_A(x), T_A(y)\} = \min\{T_{A^*}(x), T_{A^*}(y)\}, I_{A^*}(x-y) = I_A(x-y) \geq \min\{I_A(x), I_A(y)\} = \min\{I_{A^*}(x), I_{A^*}(y)\}$, and $F_{A^*}(x-y) = F_A(x-y) \leq \max\{F_A(x), F_A(y)\} = \max\{F_{A^*}(x), F_{A^*}(y)\}$ for any $x, y \in X$. If $x \notin A^{(\alpha,\beta,\gamma)}$ or $y \notin A^{(\alpha,\beta,\gamma)}$, then $T_{A^*}(x) = 0, I_{A^*}(x) = 0, F_{A^*}(x) = 1$

or $T_{A^*}(y) = 0, I_{A^*}(y) = 0, F_{A^*}(y) = 1$. Therefore we get $T_{A^*}(x-y) \geq \min\{T_{A^*}(x), T_{A^*}(y)\} = 0, I_{A^*}(x-y) \geq \min\{I_{A^*}(x), I_{A^*}(y)\} = 0$, and $F_{A^*}(x-y) \leq \max\{T_{A^*}(x), T_{A^*}(y)\} = 1$ for any $x, y \in X$. Thus A^* is a neutrosophic subalgebra of X. □

Definition 3.10. *A neutrosophic set A in X is called a neutrosophic ideal of X if it satisfies (3.2) and*

(3.4) $(\forall x, y \in X)(T_A(x) \geq \min\{T_A(x-y), T_A(y)\}, I_A(x) \geq \min\{I_A(x-y), I_A(y)\},$ *and* $F_A(x) \leq \max\{F_A(x-y), F_A(y)\})$.

Proposition 3.11. *Every neutrosophic ideal of X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic ideal of X. Put $x := x-y$ and $y := x$ in (3.4). Then we have $T_A(x-y) \geq \min\{T_A((x-y)-x), T_A(x)\}, I_A(x-y) \geq \min\{I_A((x-y)-x), I_A(x)\}$, and $F_A(x-y) \leq \max\{F_A((x-y)-x), F_A(x)\}$. It follows from (a3) and (3.2) that $T_A(x-y) \geq \min\{T_A((x-x)-y), T_A(x)\} = \min\{T_A(0), T_A(x)\} \geq \min\{T_A(x), T_A(y)\}, I_A(x-y) \geq \min\{I_A((x-y)-x), I_A(x)\} = \min\{I_A(0), I_A(x)\} \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x-y) \leq \max\{F_A((x-y)-x), F_A(x)\} = \max\{F_A(0), F_A(x)\} \leq \max\{F_A(x), F_A(y)\}$, for any $x, y \in X$. Thus A is a neutrosophic subalgebra of X. □

The converse of Proposition 3.11 may not be true in general (see Example 3.12.)

Example 3.12. *(a) Let $X := \{0, a, b, c\}$ be a subtraction algebra [2] with the following table:*

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Define a neutrosophic set A in X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.72 & \text{if } x \in \{0, a\} \\ 0.11 & \text{if } x \in \{b, c\}, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.72 & \text{if } x \in \{0, a\} \\ 0.11 & \text{if } x \in \{b, c\}, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0, a\} \\ 0.71 & \text{if } x \in \{b, c\}. \end{cases}$$

It is easy to check that A is a neutrosophic ideal of X.
(b) Let $X = \{0,1,2,3\}$ be a subtraction algebra as in Example 3.3. Define a neutrosophic set B in X as follows:

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.53 & \text{if } x = 0 \\ 0.22 & \text{if } x \in \{1,2\} \\ 0.13 & \text{if } x = 3, \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.53 & \text{if } x = 0 \\ 0.22 & \text{if } x \in \{1,2\} \\ 0.13 & \text{if } x = 3, \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.11 & \text{if } x = 0 \\ 0.25 & \text{if } x \in \{1,2\} \\ 0.46 & \text{if } x = 3. \end{cases}$$

It is easy to check that B is a neutrosophic subalgebra of X. But it is not a neutrosophic ideal of X, since $T_B(3) = 0.13 \ngeq \min\{T_B(3-1), T_B(1)\} = \max\{T_B(2), T_B(1)\} = 0.22$.

Theorem 3.13. *Let A be a neutrosophic set in X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic ideal of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in X$ be such that $x - y, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x-y) \geq \alpha, T_A(y) \geq \alpha, I_A(x-y) \geq \beta, I_A(y) \geq \beta$, and $F_A(x-y) \leq \gamma, F_A(y) \leq \gamma$. By Definition 3.10, we have $T_A(0) \geq T_A(x) \geq \min\{T_A(x-y), T_A(y)\} \geq \alpha, I_A(0) \geq I_A(x) \geq \min\{I_A(x-y)), I_A(y)\} \geq \beta$, and $F_A(0) \leq F_A(x) \leq \max\{F_A(x-y), T_A(y)\} \leq \gamma$. Hence $0, x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is an ideal of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(0) < T_A(a), I_A(0) < I_A(b)$, and $F_A(0) > F_A(c)$. Then there exist $a_t, b_t \in (0,1]$ and $c_t \in [0,1)$ such that $T_A(0) < a_t \leq T_A(a), I_A(0) < b_t \leq I_A(b)$ and $F_A(0) > c_t \geq F_A(c)$. Hence $0 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(0) \geq T_A(x), I_A(0) \geq I_A(x)$ and $F_A(0) \leq F_A(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_A(a_t) < \min\{T_A(a_t-b_t), T_A(b_t)\}, I_A(a_i) < \min\{I_A(a_i-b_i), I_A(b_i)\}$, and $F_A(a_f) > \max\{T_A(a_f - b_f), T_A(b_f)\}$. Then there exist $s_t, s_i \in (0,1]$ and $s_f \in [0,1)$ such that $T_A(a_t) < s_t \leq \min\{T_A(a_t - b_t), T_A(b_t)\}, I_A(a_i) < s_i \leq \min\{I_A(a_i - b_i), I_A(b_i)\}$, and $F_A(a_f) > s_f \geq \max\{T_A(a_f - b_f), T_A(b_f)\}$.

Hence $a_t - b_t, b_t, a_i - b_i, a_f - b_f \in A^{(s_t,s_i,s_f)}$, and $b_t, b_i, b_f \in A^{(s_t,s_i,s_f)}$. But $a_t, a_i \notin A^{(s_t,s_i,s_f)}$ and $a_f \notin A^{(s_t,s_i,s_f)}$. This is a contradiction. Therefore $T_A(x) \geq \min\{T_A(x-y), T_A(y)\}, I_A(x) \geq \min\{I_A(x-y)), I_A(y)\}$ and $F_A(x) \leq \max\{F_A(x-y), F_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic ideal of X. □

Proposition 3.14. *Every neutrosophic ideal A of X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow T_A(x) \geq T_A(y), I_A(x) \geq I_A(y), F_A(x) \leq F_A(y))$,

(ii) $(\forall x, y, z \in X)(x - y \leq z \Rightarrow T_A(x) \geq \min\{T_A(y), T_A(z)\}, I_A(x) \geq \min\{I_A(y), I_A(z)\}, F_A(x) \leq \max\{F_A(y), F_A(z)\})$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$. Using (3.4) and (3.2), we have $T_A(x) \geq \min\{T_A(x-y), T_A(y)\} = \min\{T_A(0), T_A(y)\} = T_A(y), I_A(y) \geq \min\{I_A(x-y), I_A(y)\} = \min\{I_A(0), I_A(y)\} = I_A(y)$, and $F_A(x) \leq \max\{F_A(x-y), F_A(y)\} = \max\{F_A(0), F_A(y)\} = F_A(y)$.
(ii) Let $x, y, z \in X$ be such that $x - y \leq z$. By (3.4) and (3.2), we get $T_A(x-y) \geq \min\{T_A((x-y)-z), T_A(z)\} = \min\{T_A(0), T_A(z)\} = T_A(z), I_A(x-y) \geq \min\{I_A((x-y)-z), I_A(z)\} = \min\{I_A(0), I_A(z)\} = I_A(z)$, and $F_A(x-y) \leq \max\{F_A((x-y)-z), F_A(z)\} = \max\{F_A(0), F_A(z)\} = F_A(z)$. Hence $T_A(x) \geq \min\{T_A(x-y), T_A(y)\} \geq \min\{T_A(y), T_A(z)\}$, $I_A(x) \geq \min\{I_A(x-y), I_A(y)\} \geq \min\{I_A(y), I_A(z)\}$, and $F_A(x) \leq \max\{F_A(x-y), F_A(y)\} \leq \max\{F_A(y), F_A(z)\}$, for any $x, y, z \in X$. □

The following corollary is easily proved by induction.

Corollary 3.15. *Every neutrosophic ideal A of X satisfies the following property:*

(3.5) $(\forall x, a_1, \cdots, a_n \in X)((\cdots(x - a_1) - \cdots) - a_n = 0 \Rightarrow T_A(x) \geq \bigwedge_{k=1}^{n} T_A(a_k), I_A(x) \geq \bigwedge_{k=1}^{n} I_A(a_k)$, *and* $F_A(x) \leq \bigvee_{k=1}^{n} F_A(a_k))$.

Definition 3.16. *Let A and B be neutrosophic sets of a set X. The union of A and B is defined to be a neutrosophic set*

$$A \tilde{\cup} B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}$, $F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, *for all* $x \in X$. *The intersection of A and B is defined to be a neutrosophic set*

$$A \tilde{\cap} B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}$, $F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, *for all* $x \in X$.

Theorem 3.17. *The intersection of two neutrosophic ideals of X is also a neutrosophic ideal of X.*

Proof. Let A and B be neutrosophic ideals of X. For any $x \in X$, we have $T_{A\cap B}(0) = \min\{T_A(0), T_B(0)\} \geq \min\{T_A(x), T_B(x)\} = T_{A\cap B}(x), I_{A\cap B}(0) = \min\{T_A(0), T_B(0)\} \geq \min\{I_A(x), I_B(x)\} = I_{A\cap B}(x)$, and $F_{A\cap B}(0) = \max\{F_A(0), F_B(0)\} \leq \max\{F_A(x), F_B(x)\} = F_{A\cap B}(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned} T_{A\cap B}(x) &= \min\{T_A(x), T_B(x)\} \\ &\geq \min\{\min\{T_A(x-y), T_A(y)\}, \min\{T_B(x-y), T_B(y)\}\} \\ &= \min\{\min\{T_A(x-y), T_B(x-y)\}, \min\{T_A(y), T_B(y)\}\} \\ &= \min\{T_{A\cap B}(x-y), T_{A\cap B}(y)\}, \end{aligned}$$

$$\begin{aligned} I_{A\cap B}(x) &= \min\{I_A(x), I_B(x)\} \\ &\geq \min\{\min\{I_A(x-y), I_A(y)\}, \min\{I_B(x-y), I_B(y)\}\} \\ &= \min\{\min\{I_A(x-y), I_B(x-y)\}, \min\{I_A(y), I_B(y)\}\} \\ &= \min\{I_{A\cap B}(x-y), I_{A\cap B}(y)\}, \end{aligned}$$

and

$$\begin{aligned} F_{A\cap B}(x) &= \max\{F_A(x), F_B(x)\} \\ &\leq \max\{\max\{F_A(x-y), F_A(y)\}, \max\{F_B(x-y), F_B(y)\}\} \\ &= \max\{\max\{F_A(x-y), F_B(x-y)\}, \max\{F_A(y), F_B(y)\}\} \\ &= \max\{F_{A\cap B}(x-y), F_{A\cap B}(y)\}. \end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic ideal of X. □

Corollary 3.18. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic ideals of X, then so is $\tilde{\cap}_{i\in\mathbb{N}} A_i$.*

Proposition 3.19. *Let A be a neutrosophic ideal of X. Then $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_T := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$ are ideals of X.*

Proof. Clearly, $0 \in X_T$. Let $x - y, y \in X_T$. Then $T_A(x-y) = T_A(0)$ and $T_A(y) = T_A(0)$. It follows from (3.4) that $T_A(x) \geq \min\{T_A(x-y), T_A(y)\} = T_A(0)$. By (3.2), we get $T_A(x) = T_A(0)$. Hence $x \in X_T$. Therefore X_T is an ideal of X. By a similar way, X_I and X_F are ideals of X. □

Let $f : X \to Y$ be a function of sets. If

$$M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$$

is a neutrosophic set of a set Y, then the preimage of M under f is defined to be a neutrosophic set

$$f^{-1}(M) := \{\langle x, f^{-1}(T_M)(x), f^{-1}(I_M)(x), f^{-1}(F_M)(x)\rangle | x \in X\}$$

of X, where $f^{-1}(T_M)(x) = T_M(f(x)), f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 3.20. *Let $f : X \to Y$ be a homomorphism of subtraction algebras. If $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$ is a neutrosophic subalgebra of Y, then the preimage of M under f is a neutrosophic subalgebra of X.*

Proof. Let $f^{-1}(M)$ be the preimage of M under f. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(T_M(x-y)) =& T_M(f(x-y)) = T_M(f(x) - f(y)) \\ &\geq \min\{T_M(f(x)), T_M(f(y))\} \\ &= \min\{f^{-1}(T_M)(x), f^{-1}(T_M)(y)\}, \\ f^{-1}(I_M(x-y)) =& I_M(f(x-y)) = I_M(f(x) - f(y)) \\ &\geq \min\{I_M(f(x)), I_M(f(y))\} \\ &= \min\{f^{-1}(I_M)(x), f^{-1}(I_M)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M(x-y)) =& F_M(f(x-y)) = F_M(f(x) - f(y)) \\ &\leq \max\{F_M(f(x)), F_M(f(y))\} \\ &= \max\{f^{-1}(F_M)(x), f^{-1}(F_M)(y)\}. \end{aligned}$$

Hence $f^{-1}(M)$ is a neutrosophic subalgebra of X. □

Let $f : X \to Y$ be an onto function of sets. If A is a neutrosophic set of X, then the image of A under f is defined to be a neutrosophic set

$$f(A) := \{\langle y, f(T_A)(y), f(I_A)(y), f(F_A)(y)\rangle | y \in Y\}$$

of Y, where $f(T_A)(y) = \bigvee_{x \in f^{-1}(y)} T_A(x), f(I_A)(y) = \bigvee_{x \in f^{-1}(y)} I_A(x)$, and $f(F_A)(y) = \bigwedge_{x \in f^{-1}(y)} F_A(x)$.

Theorem 3.21. *For an onto homomorphism $f : X \to Y$ of subtraction algebras, let A be a neutrosophic set of X such that*

(3.6) $(\forall C \subseteq X)(\exists x_0 \in C)(T_A(x_0) = \bigvee_{z \in C} T_A(z), I_A(x_0) = \bigvee_{z \in C} I_A(z), F_A(x_0) = \bigwedge_{z \in C} F_A(z))$.

If A is a neutrosophic subalgebra of X, then the image of A under f is a neutrosophic subalgebra of Y.

Proof. Let $f(A)$ be the image of A under f. Let $a, b \in Y$. Then $f^{-1}(a) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$ in X. By (3.6), there exist $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that

$$\begin{aligned} T_A(x_a) &= \bigvee_{z\in f^{-1}(a)} T_A(z), I_A(x_a) = \bigvee_{z\in f^{-1}(a)} I_A(z), F_A(x_a) \\ &= \bigwedge_{z\in f^{-1}(a)} F_A(z), \end{aligned}$$

$$\begin{aligned} T_A(x_b) &= \bigvee_{w\in f^{-1}(b)} T_A(w), I_A(x_b) = \bigvee_{w\in f^{-1}(b)} I_A(w), F_A(x_b) \\ &= \bigwedge_{w\in f^{-1}(b)} F_A(w). \end{aligned}$$

Thus

$$\begin{aligned} f(T_A)(a-b) &= \bigvee_{x\in f^{-1}(a-b)} T_A(x) \geq T_A(x_a - x_b) \geq \min\{T_A(x_a), T_A(x_b)\} \\ &= \min\{\bigvee_{z\in f^{-1}(a)} T_A(z), \bigvee_{w\in f^{-1}(b)} T_A(w)\} \\ &= \min\{f(T_A)(a), f(T_A)(b)\}, \end{aligned}$$

$$\begin{aligned} f(I_A)(a-b) &= \bigvee_{x\in f^{-1}(a-b)} I_A(x) \geq I_A(x_a - x_b) \geq \min\{I_A(x_a), I_A(x_b)\} \\ &= \min\{\bigvee_{z\in f^{-1}(a)} I_A(z), \bigvee_{w\in f^{-1}(b)} I_A(w)\} \\ &= \min\{f(I_A)(a), f(I_A)(b)\}, \end{aligned}$$

and

$$\begin{aligned} f(F_A)(a-b) &= \bigwedge_{x\in f^{-1}(a-b)} F_A(x) \leq F_A(x_a - x_b) \leq \max\{F_A(x_a), F_A(x_b)\} \\ &= \max\{\bigwedge_{z\in f^{-1}(a)} F_A(z), \bigwedge_{w\in f^{-1}(b)} F_A(w)\} \\ &= \max\{f(F_A)(a), f(F_A)(b)\}. \end{aligned}$$

Hence $f(A)$ is a neutrosophic subalgebra of Y. □

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