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USEFUL OPERATORS ON REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA OF TYPE $\mathfrak{sl}_n$

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Abstract. Let n denote an integer greater than 2 and let c denote a nonzero complex number. In this paper, we introduce a family of elements of the rational Cherednik algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ of type $\mathfrak{sl}_n$, which are analogous to the Dunkl-Cherednik elements of the rational Cherednik algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$ of type $\mathfrak{gl}_n$. We also introduce the raising and lowering element of $\mathbf{H}^{\mathfrak{sl}_n}(c)$ which are useful in the representation theory of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$, and provide simple results related to these elements.

1. Introduction

Throughout this paper, let n denote an integer greater than 2, and let c denote a nonzero complex number unless otherwise stated.

Roughly speaking, the rational Cherednik algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$ of type $\mathfrak{gl}_n$ associated with c is the algebra generated by two polynomial subalgebras $\mathbb{C}[\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n]$, $\mathbb{C}[\mathtt{y}_1, \mathtt{y}_2, \cdots, \mathtt{y}_n]$, and the symmetric group $\mathcal{S}_n$ subject to certain relations. The algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$ acts on the space $\mathbb{C}[\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n]$ of all polynomials in n variables $\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n$ in the following way. The element $\mathtt{x}_i$ acts as multiplication by $\mathtt{x}_i$, and the element $w \in \mathcal{S}_n$ permutes the variables $\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n$ naturally. On the other hand, the element $\mathtt{y}_i$ acts as the Dunkl operator:

$$D_i \;=\; \frac{\partial}{\partial \mathtt{x}_i} - c\sum_{j\neq i} \frac{1}{\mathtt{x}_i - \mathtt{x}_j}(1 - s_{ij}).$$

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In [3], Cherednik introduced a family of commuting elements of $\mathbf{H}^{\mathfrak{gl}_n}(c)$:

$$u_i \;=\; -\frac{1}{c}\mathtt{x}_i\mathtt{y}_i + \sum_{p=1}^{i-1} s_{pi} \quad (i \in \{1, 2, \cdots, n\}),$$

whose actions on $\mathbb{C}[\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n]$ are called the Dunkl-Cherednik operators. He proved that there exists a basis of the space $\mathbb{C}[\mathtt{x}_1, \mathtt{x}_2, \cdots, \mathtt{x}_n]$ consisting of simultaneous eigenvectors for the Dunkl-Cherednik operators and each nontrivial eigenspace is one-dimensional for a generic c. The simultaneous eigenvectors are also called non-symmetric Jack polynomials. Furthermore, Cherednik completely classified all isomorphism calsses of irreducible modules which admit a basis consisting of simultaneous eigenvectors for the actions of the elements u_i in [4]. Suzuki and Vazirani recovered his classification by combinatorial means in [6].

The rational Cherednik algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ of type $\mathfrak{sl}_n$, which is a subalgebra of $\mathbf{H}^{\mathfrak{gl}_n}(c)$ and its representations have been studied as well. For example, when $c = \frac{r}{n}$ with $\gcd(r, n) = 1$, the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ admits a finite dimensional representation. However, these results on the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ were obtained by totally different means from Cherednik's method for the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$. Indeed, it is not possible to apply his method to the representation theory of the subalgebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ naively. The main obstacle is that the elements u_i does not belong to the subalgebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.

The purpose of this paper is to introduce a family of commuting elements of $\mathbf{H}^{\mathfrak{sl}_n}(c)$ which is analogous to the Dunkl-Cherednik operators, and to provide several relations among the elements. We expect that the elements introduced in this paper allows us to show several results on the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ in a purely representation-theoretic way, which is analogous to Cherednik's method.

2. Preliminaries

In this section, we shortly review the definitions of the rational Cherednik algebras of type $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$. All details skipped in this section can be found in [1] or [2].

2.1. Rational Cherednik algebra of type $\mathfrak{gl}_n$

Recall that n denotes an integer greater than 2 and $c \in \mathbb{C} \setminus \{0\}$. The rational Cherednik algebra of type $\mathfrak{gl}_n$ associated with c, denoted by $\mathbf{H}^{\mathfrak{gl}_n}(c)$, is the unital associative $\mathbb{C}$-algebra which has the following presentation.

Generators: $\mathtt{x}_1, \cdots, \mathtt{x}_{n-1}, \mathtt{x}_n, s_1, \cdots, s_{n-1}, \mathtt{y}_1, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n$

Relations:

$$\begin{array}{lll} s_i^2 = 1 & & (i \in \{1, \cdots, n-1\}), \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & & (i \in \{1, \cdots, n-2\}), \\ s_i s_j = s_j s_i & & (|i-j| > 1), \\ \mathtt{x}_i \mathtt{x}_j = \mathtt{x}_j \mathtt{x}_i, & \mathtt{y}_i \mathtt{y}_j = \mathtt{y}_j \mathtt{y}_i & (i, j \in \{1, \cdots, n-1, n\}), \\ s_i \mathtt{x}_i = \mathtt{x}_{i+1} s_i, & s_i \mathtt{y}_i = \mathtt{y}_{i+1} s_i & (i \in \{1, \cdots, n-1\}), \\ s_i \mathtt{x}_j = \mathtt{x}_j s_i, & s_i \mathtt{y}_j = \mathtt{y}_j s_i & (j \neq i, i+1), \end{array}$$

$$\mathtt{x}_i \mathtt{y}_j - \mathtt{y}_j \mathtt{x}_i = \begin{cases} -c s_{ij} & (i \neq j) \\ -1 + c \sum_{p \neq i} s_{ip} & (i = j) \end{cases},$$

where $s_{i,i+1} = s_{i+1,i} = s_i$ and $s_{ij} = \begin{cases} s_i s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i & (i < j) \\ s_i s_{i-1} \cdots s_{j+1} \cdots s_{i-1} s_i & (i > j) \end{cases}$ for $i, j \in \{1, \cdots, n-1\}$ with $i \neq j$.

Remark 2.1. *Many authors provide various presentations of the rational Cherednik algebra of type $\mathfrak{gl}_n$. We would like to inform that our presentation is compatible with the definition given in* [1].

2.2. Rational Cherednik algebra of type $\mathfrak{sl}_n$

We may define the rational Cherednik algebra of type $\mathfrak{sl}_n$ associated with c as a subalgebra of the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$.

Definition 2.2. *The **rational Cherednik algebra** (abbreviated RCA) **of type $\mathfrak{sl}_n$ associated with** $c \in \mathbb{C} - \{0\}$, denoted by $\mathbf{H}^{\mathfrak{sl}_n}(c)$, is the subalgebra of the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$ generated by $\mathtt{x}_1 - \mathtt{x}_n, \cdots, \mathtt{x}_{n-1} - \mathtt{x}_n$, $s_1, \cdots s_{n-1}$, and $\mathtt{y}_1 - \mathtt{y}_n, \cdots, \mathtt{y}_{n-1} - \mathtt{y}_n$.*

For each $i \in \{1, \cdots, n-1\}$, we denote $\mathtt{x}_i - \mathtt{x}_n$ by x_i for convenience. On the other hand, we denote by y_j the element

$$\mathtt{y}_j - \frac{1}{n}(\mathtt{y}_1 + \cdots + \mathtt{y}_{n-1} + \mathtt{y}_n) \quad (j \in \{1, \cdots, n-1, n\}).$$

It can be directly seen that the elements $y_1, \cdots, y_{n-1}, y_n$ belong to the subalgebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ and the following equation holds:

$$y_1 + \cdots + y_{n-1} + y_n = 0. \tag{1}$$

Let $\mathbb{C}[X_1, \cdots, X_{n-1}]$ (resp. $\mathbb{C}[Y_1, \cdots, Y_{n-1}]$) be the algebra of polynomials in $X_1, \cdots, X_{n-1}$ (resp. $Y_1, \cdots, Y_{n-1}$). For $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in (\mathbb{N} \cup \{0\})^{n-1}$, we denote $X_1^{a_1} \cdots X_{n-1}^{a_{n-1}}$ (resp. $Y_1^{a_1} \cdots Y_{n-1}^{a_{n-1}}$) by $\mathbf{X}^{\mathbf{a}}$ (resp. $\mathbf{Y}^{\mathbf{a}}$).

We may regard the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ as a (infinite dimensional) vector space over $\mathbb{C}$. The following theorem, which is also known as the PBW theorem for the rational Cherednik algebra, gives a basis of the vector space $\mathbf{H}^{\mathfrak{sl}_n}(c)$ over $\mathbb{C}$. The proof can be found in [2].

Theorem 2.3 (PBW theorem for the rational Cherednik algebra)**.** *Let $\mathcal{X}$ (resp. $\mathcal{Y}$) denote the polynomial algebra $\mathbb{C}[X_1, \cdots, X_{n-1}]$ (resp. $\mathbb{C}[Y_1, \cdots, Y_{n-1}]$), and let $\mathbb{C}\mathcal{S}_n$ denote the group algebra of the symmetric group $\mathcal{S}_n$. As a vector space over $\mathbb{C}$, the rational Cherednik algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ is isomorphic to the tensor product $\mathcal{X} \otimes \mathbb{C}\mathcal{S}_n \otimes \mathcal{Y}$ of the vector spaces $\mathcal{X}$, $\mathbb{C}\mathcal{S}_n$ and $\mathcal{Y}$. More precisely, the linear map*

$$\iota \colon \mathcal{X} \otimes \mathbb{C}\mathcal{S}_n \otimes \mathcal{Y} \longrightarrow \mathbf{H}^{\mathfrak{sl}_n}(c)$$

given by

$$\iota(\mathbf{X}^{\mathbf{a}} \otimes w \otimes \mathbf{Y}^{\mathbf{b}}) = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} s_{i_1} s_{i_2} \cdots s_{i_k} y_1^{b_1} \cdots y_{n-1}^{b_{n-1}} \tag{2}$$

for $\mathbf{a} = (a_1, \cdots, a_{n-1})$, $\mathbf{b} = (b_1, \cdots, b_{n-1}) \in (\mathbb{N} \cup \{0\})^{n-1}$ and $w \in \mathcal{S}_n$, where w is represented as a product of the transpositions

$$w = (i_1\ i_1 + 1)(i_2\ i_2 + 1) \cdots (i_k\ i_k + 1), \tag{3}$$

is an isomorphism of vector spaces over $\mathbb{C}$.

Remark 2.4. *Due to the PBW theorem, we directly see that the group algebra $\mathbb{C}\mathcal{S}_n$ is embedded in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$. In this sense, if a permutation $w \in \mathcal{S}_n$ can be represented as in the equation (3), we will identify w with the element $s_{i_1} s_{i_2} \cdots s_{i_k}$ in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ in the paper.*

3. Useful Elements

In this section, we introduce a family of elements of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$, which are analogous to the Dunkl-Cherednik elements of the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$, introduce auxiliary elements (intertwining elements, raising element and lowering element), and prove that several relations among these elements hold. These elements are obtained from the corresponding elements of the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$ by modifying slightly.

3.1. Modified Dunkl-Cherednik elements

By modifying Dunkl-Cherednik operators slightly, we introduce a family of elements in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.

Definition 3.1. *For each $i \in \{1, 2, \cdots, n-1\}$, we define the* ***modified Dunkl-Cherednik element*** *g_i to be*

$$g_i \;=\; -\frac{1}{c}x_iy_i + \left(\sum_{j=1}^{i-1} s_{ji}\right) + s_{in} \;\in\; \mathbf{H}^{\mathfrak{sl}_n}(c). \tag{4}$$

The first property of the modified Dunkl-Cherednik elements we show here is that the elements $g_1, \cdots, g_{n-1}$ with $s_1, \cdots, s_{n-2}$ in $\mathbf{H}^{\mathfrak{sl}_n}(c)$ satisfy the defining relations for the degenerate affine Hecke algebra of type $\mathbf{A}_{n-2}$. We prove this property in Lemma 3.2 and Proposition 3.4.

Lemma 3.2. *The following relations hold in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ for all $i, j \in \{1, \cdots, n-2\}$:*

$$s_ig_is_i \;=\; g_{i+1} - s_i, \tag{5}$$

$$s_ig_js_i \;=\; g_j \qquad (j \neq i, i+1). \tag{6}$$

Proof. Let $i \in \{1, \cdots, n-2\}$. Then we have

$$\begin{aligned} s_ig_is_i &= s_i\left(-\frac{1}{c}x_iy_i + \left(\sum_{j=1}^{i-1} s_{ji}\right) + s_{in}\right)s_i \\ &= -\frac{1}{c}x_{i+1}y_{i+1} + \left(\sum_{j=1}^{i-1} s_{j,i+1}\right) + s_{i+1,n} \\ &= g_{i+1} - s_{i,i+1} \;=\; g_{i+1} - s_i. \end{aligned}$$

In the case of $j < i$, we directly see that the equation $s_ig_j = g_js_i$ holds because s_i commutes with x_j, y_j, $s_{1j}, \cdots, s_{j-1,j}$, and s_{jn}.

Now assume $i + 1 < j \leq n-2$. Then we have

$$\begin{aligned} s_ig_js_i &= s_i\left(-\tfrac{1}{c}x_jy_j + s_{1j} + \cdots + s_{ij} + s_{i+1,j} + \cdots + s_{j-1,j} + s_{jn}\right)s_i \\ &= -\tfrac{1}{c}x_jy_j + s_{1j} + \cdots + s_{i+1,j} + s_{ij} + \cdots + s_{j-1,j} + s_{jn} \;=\; g_j. \end{aligned}$$

□

Remark 3.3. *From (5), one can easily show that the following relation holds:*

$$s_ig_{i+1}s_i \;=\; g_i + s_i$$

for all $i, j \in \{1, \cdots, n-2\}$.

Proposition 3.4. *The modified Dunkl-Cherednik elements $g_1, g_2, \cdots, g_{n-1}$ commute with each other in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$; that is, the equation $g_ig_j = g_jg_i$ holds for every $i, j \in \{1, 2, \cdots, n-1\}$.*

Proof. Without loss of generality, we may assume $i < j$. We first show a special case directly: $g_1 g_2 = g_2 g_1$.

$$\begin{aligned}
g_1 g_2 &= \tfrac{1}{c^2} x_1 y_1 x_2 y_2 - \tfrac{1}{c} x_1 y_1 s_{12} - \tfrac{1}{c} x_1 y_1 s_{2n} \\
&\qquad - \tfrac{1}{c} s_{1n} x_2 y_2 + s_{1n} s_{12} + s_{1n} s_{2n} \\
&= \tfrac{1}{c^2} x_1 (x_2 y_1 + c s_{12} - c s_{1n}) y_2 - \tfrac{1}{c} x_1 s_{12} y_2 - \tfrac{1}{c} x_1 s_{2n} y_1 \\
&\qquad - \tfrac{1}{c} (x_2 - x_1) s_{1n} y_2 + s_{1n} s_{12} + s_{1n} s_{2n} \\
&= \tfrac{1}{c^2} x_1 x_2 y_1 y_2 + \tfrac{1}{c} x_1 s_{12} y_2 - \tfrac{1}{c} x_1 s_{1n} y_2 \\
&\qquad - \tfrac{1}{c} x_1 s_{12} y_2 - \tfrac{1}{c} x_1 s_{2n} y_1 - \tfrac{1}{c} x_2 s_{12} y_2 \\
&\qquad + \tfrac{1}{c} x_1 s_{1n} y_2 + s_{1n} s_{12} + s_{1n} s_{2n} \\
&= \tfrac{1}{c^2} x_1 x_2 y_1 y_2 - \tfrac{1}{c} x_1 s_{2n} y_1 - \tfrac{1}{c} x_2 s_{1n} y_2 + s_{2n} s_{1n} + s_{1n} s_{2n}
\end{aligned}$$

$$\begin{aligned}
g_2 g_1 &= \tfrac{1}{c^2} x_2 y_2 x_1 y_1 - \tfrac{1}{c} x_2 y_2 s_{1n} - \tfrac{1}{c} s_{12} x_1 y_1 \\
&\qquad + s_{12} s_{1n} - \tfrac{1}{c} s_{2n} x_1 y_1 + s_{2n} s_{1n} \\
&= \tfrac{1}{c^2} x_2 (x_1 y_2 + c s_{12} - c s_{2n}) y_1 - \tfrac{1}{c} x_2 s_{1n} y_2 - \tfrac{1}{c} x_2 s_{12} y_1 \\
&\qquad + s_{12} s_{1n} - \tfrac{1}{c} (x_1 - x_2) s_{2n} y_1 + s_{2n} s_{1n} \\
&= \tfrac{1}{c^2} x_1 x_2 y_1 y_2 + \tfrac{1}{c} s_2 s_{12} y_1 - \tfrac{1}{c} x_2 s_{2n} y_1 \\
&\qquad - \tfrac{1}{c} x_2 s_{1n} y_2 - \tfrac{1}{c} x_2 s_{12} y_1 + s_{12} s_{1n} \\
&\qquad - \tfrac{1}{c} x_1 s_{2n} y_1 + \tfrac{1}{c} x_2 s_{2n} y_1 + s_{2n} s_{1n} \\
&= \tfrac{1}{c^2} x_1 x_2 y_1 y_2 - \tfrac{1}{c} x_2 s_{1n} y_2 - \tfrac{1}{c} x_1 s_{2n} y_1 + s_{1n} s_{2n} + s_{2n} s_{1n}
\end{aligned}$$

Now assume $j \geq 3$. Then by applying Lemma 3.2 repeatedly, we have

$$g_j = s_{j-1} \cdots s_2 g_2 s_2 \cdots s_{j-1} + \sum_{k=2}^{j-1} s_{kj}.$$

Since the element g_1 commutes with the element g_2 and $s_2, \cdots s_{n-1}$, we obtain $g_1 g_j = g_j g_1$.

Let us assume $1 < i < j$ additionally. Then we also express g_i as

$$g_i = s_{i-1} \cdots s_1 g_1 s_1 \cdots s_{i-1} + \sum_{k=1}^{i-1} s_{ki},$$

from which it follows that the equation $g_i g_j = g_j g_i$ holds because the element g_j commutes with g_1 and $s_1, \cdots, s_i$. $\square$

Let $\mathcal{G}$ denote the subalgebra of $\mathbf{H}^{\mathfrak{sl}_n}(c)$ generated by the elements $g_1, g_2, \cdots, g_{n-1}$. Note that it follows from the previous proposition that the subalgebra $\mathcal{G}$ is a commutative algebra.

3.2. Intertwing elements

Here we introduce auxiliary elements of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$, and give relations between these elements and the elements g_i. In Section 4, we will see that these auxiliary elements are useful in the representation theory of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.

Definition 3.5. *For each $i \in \{1, \cdots, n-2\}$, we define the **intertwining element** φ_i to be the element*

$$\varphi_i \;=\; 1 + s_i(g_i - g_{i+1}) \;\in\; \mathbf{H}^{\mathfrak{sl}_n}(c). \tag{7}$$

The next proposition shows how well the intertwining elements behave with the elements $g_1, g_2, \cdots, g_{n-1}$.

Proposition 3.6. *The following relation holds in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$:*

$$g_j\varphi_i = \varphi_i g_{s_i(j)}$$

for $i \in \{1, \cdots, n-2\}$ and $j \in \{1, \cdots, n-1\}$.

Proof. Let $i \in \{1, \cdots, n-2\}$. From Lemma 3.2, we directly see that the following equations hold:

$$g_i s_i \;=\; s_i s_i g_i s_i \;=\; s_i(g_{i+1} - s_i) \;=\; s_i g_{i+1} - 1, \tag{8}$$

$$g_{i+1} s_i \;=\; (s_i g_i s_i + s_i)s_i \;=\; s_i g_i + 1. \tag{9}$$

Thus, we obtain

$$\begin{aligned} g_i\varphi_i &= g_i(1 + s_i(g_i - g_{i+1})) \\ &= g_i + g_i s_i(g_i - g_{i+1}) \\ &= g_i + (s_i g_{i+1} - 1)(g_i - g_{i+1}) \\ &= g_i + s_i g_{i+1}(g_i - g_{i+1}) - (g_i - g_{i+1}) \\ &= s_i(g_i - g_{i+1})g_{i+1} + g_{i+1} \\ &= (s_i(g_i - g_{i+1}) + 1)g_{i+1} \;=\; \varphi_i g_{i+1}. \end{aligned}$$

Similarly, one can easily check the relation $g_{i+1}\varphi_i = \varphi_i g_i$ holds.

For $j \in \{1, \cdots, n-1\}$ with $j \neq i, i+1$, note that the element g_j commutes with s_i as well as $g_i - g_{i+1}$, from which it follows that the relation $g_j\varphi_i = \varphi_i g_j$ holds. □

From the following proposition, we see $\varphi_i \in \mathcal{G}$ for each $i \in \{1, \cdots, n-2\}$. In particular, φ_i^2 commutes with the elements $g_1, g_2, \cdots, g_{n-1}$.

Proposition 3.7. *For each $i \in \{1, \cdots, n-2\}$, the following equation holds in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$:*

$$\varphi_i^2 \;=\; (1 + g_i - g_{i+1})(1 - g_i + g_{i+1}). \tag{10}$$

Proof. It is straightforward. Indeed, by proposition 3.6 and equation (7), we obtain

$$\begin{aligned}\varphi_i^2 &= (1+s_i(g_i-g_{i+1}))\varphi_i \\ &= \varphi_i + s_i\varphi_i(g_{i+1}-g_i) \\ &= \varphi_i - s_i(1+s_i(g_i-g_{i+1}))(g_i-g_{i+1}) \\ &= \varphi_i - s_i(g_i-g_{i+1}) - (g_i-g_{i+1})^2 \\ &= 1-(g_i-g_{i+1})^2 \\ &= (1+g_i-g_{i+1})(1-g_i+g_{i+1}).\end{aligned}$$

□

3.3. Raising element

Together with the intertwing elements, we introduce two more auxiliary elements in this paper. One is the raising element and the other is the lowering element.

Definition 3.8. *The element*

$$r \;=\; x_1 s_1 \cdots s_{n-2} \;\in\; \mathbf{H}^{\mathfrak{sl}_n}(c). \tag{11}$$

is said to be the ***raising element****.*

Remark 3.9. *The element x_1 is not a zero-divisor in $\mathbf{H}^{\mathfrak{sl}_n}(c)$ by the PBW theorem, and $s_1, \cdots, s_{n-2}$ are all invertible elements in $\mathbf{H}^{\mathfrak{sl}_n}(c)$. Hence, the raising element $r = x_1 s_1 \cdots s_{n-2}$ is not a zero-divisor.*

The following lemma gives relations between the raising element r and the elements g_i in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.

Proposition 3.10. *The raising element r with the $n-1$ elements $g_1, g_2, \cdots, g_{n-1}$ satisfies the following relations in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.*

$$\begin{aligned} g_1 r &= r\left(g_{n-1} - \tfrac{1}{c}\right), \\ g_i r &= r g_{i-1} \qquad\qquad (i \in \{2, \cdots, n-1\}) \end{aligned}$$

Proof. Let $w = s_1 \cdots s_{n-2}$. Note $x_1 w = w x_{n-1}$ and $y_1 w = w y_{n-1}$. Hence, we see that the first relation holds by a direct computation as follows.

$$\begin{aligned}
g_1 r &= \left(-\tfrac{1}{c}x_1y_1 + s_{1n}\right) x_1 w \\
&= \left(-\frac{1}{c}x_1\left(x_1y_1 + 1 - c\sum_{j=2}^{n} s_{1j} - cs_{1n}\right) + s_{1n}x_1\right) w \\
&= \left(-\frac{1}{c}x_1^2y_1 - \frac{1}{c}x_1 + x_1\sum_{j=2}^{n} s_{1j} + x_1s_{1n} - x_1s_{1n}\right) w \\
&= x_1\left(-\frac{1}{c}x_1y_1 - \frac{1}{c} + \sum_{j=2}^{n} s_{1j}\right) w \\
&= x_1w\left(-\frac{1}{c}x_{n-1}y_{n-1} - \frac{1}{c} + \sum_{j=1}^{n-2} s_{j,n-1} + s_{n-1,n}\right) \\
&= r\left(g_{n-1} - \tfrac{1}{c}\right)
\end{aligned}$$

We now suppose $i \in \{2, \cdots, n-1\}$. Since $x_iw = wx_{i-1}$ and $y_iw = wy_{i-1}$, we obtain

$$\begin{aligned}
g_i r &= \left(-\frac{1}{c}x_iy_i + s_{1i} + \left(\sum_{j=2}^{i-1} s_{ji}\right) + s_{in}\right) x_1 w \\
&= \left(-\frac{1}{c}x_i(x_1y_i + cs_{1i}) + x_is_{1i} + x_1\left(\sum_{j=2}^{i-1} s_{ji}\right) + x_1s_{in}\right) w \\
&= x_1\left(-\frac{1}{c}x_iy_i + \left(\sum_{j=2}^{i-1} s_{ji}\right) + s_{in}\right) w \\
&= x_1w\left(-\frac{1}{c}x_{i-1}y_{i-1} + \left(\sum_{j=1}^{i-2} s_{j,i-1}\right) + s_{i-1,n}\right) = rg_{i-1}.
\end{aligned}$$

□

3.4. Lowering operator

Definition 3.11. *The element*

(12) $$\ell = -\tfrac{1}{c}(s_{n-1}\cdots s_2s_1y_1 - s_{n-2}\cdots s_2s_1y_1g_1) \in \mathbf{H}^{\mathfrak{sl}_n}(c).$$

is said to be the ***lowering element****.*

We will see that both $r\ell$ and ℓr belong to $\mathcal{G}$. In particular, the products $r\ell, \ell r$ commute with the elements $g_1, g_2, \cdots, g_{n-1}$.

Lemma 3.12. *The raising element r and the lowering element ℓ satisfy the relations:*

$$\begin{aligned} r\ell &= (1+g_1)(1-g_1), \\ \ell r &= (1+g_{n-1}-\tfrac{1}{c})(1-g_{n-1}+\tfrac{1}{c}). \end{aligned}$$

Proof. Recall that we denote by s_{1n} the element

$$s_1 \cdots s_{n-2}s_{n-1}s_{n-2}\cdots s_1 \in \mathbf{H}^{\mathfrak{sl}_n}(c).$$

Thus, we obtain

$$\begin{aligned} r\ell &= -\tfrac{1}{c}(x_1 s_{1n} y_1 - x_1 y_1 g_1) \\ &= -\tfrac{1}{c}(-s_{1n} x_1 y_1 - x_1 y_1 g_1) \\ &= -\tfrac{1}{c}(c s_{1n}(g_1 - s_{1n}) + c(g_1 - s_{1n})g_1) \\ &= 1 - g_1^2 \;=\; (1+g_1)(1-g_1). \end{aligned}$$

On the other hand, together with Proposition 3.10, the relation

$$r\ell = (1+g_1)(1-g_1)$$

implies

$$\begin{aligned} r\ell r &= (1+g_1)(1-g_1)r \\ &= r\left(1+g_{n-1}-\tfrac{1}{c}\right)\left(1-g_{n-1}+\tfrac{1}{c}\right). \end{aligned}$$

Hence, it follows that the relation

$$\ell r \;=\; \left(1+g_{n-1}-\tfrac{1}{c}\right)\left(1-g_{n-1}+\tfrac{1}{c}\right)$$

holds because the raising element r is not a zero-divisor in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ as we have seen in Remark 3.9. □

From the previous lemma, we obtain a relation between the lowering element ℓ and the elements g_i.

Proposition 3.13. *The lowering element ℓ with the $n-1$ elements $g_1, g_2, \cdots, g_{n-1}$ satisfies the following relations in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$.*

$$\begin{aligned} g_i \ell &= \ell g_{i+1}, \qquad (i \in \{1, \cdots, n-2\}) \\ g_{n-1}\ell &= \ell(g_1 + \tfrac{1}{c}) \end{aligned}$$

Proof. Note that $\ell r = (1 + g_{n-1} - \frac{1}{c})(1 - g_{n-1} + \frac{1}{c})$ commutes with all the elements $g_1, g_2, \cdots, g_{n-1}$. By Proposition 3.10, we obtain

$$g_i \ell r \;=\; \ell r g_i \;=\; \begin{cases} \ell g_{i+1} r & (i \in \{1, \cdots, n-2\}), \\ \ell (g_1 + \frac{1}{c}) r. \end{cases}$$

Since the raising element r is not a zero-divisor in the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ (Remark 3.9), we have

$$g_i \ell \;=\; \begin{cases} \ell g_{i+1} & (i \in \{1, \cdots, n-2\}), \\ \ell (g_1 + \frac{1}{c}). \end{cases}$$

□

4. Applications and Outlook

As we mentioned in Section 1, we may use the elements we introduce in Section 3 in order to study a structure of a given $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module. Here we provide simple applications and discuss what we expect to be able to do with these elements.

4.1. Applications: representation theory of $\mathbf{H}^{\mathfrak{sl}_n}(c)$

Recall that $\mathcal{G}$ denotes the commutative subalgebra of $\mathbf{H}^{\mathfrak{sl}_n}(c)$ generated by the elements $g_1, g_2, \cdots, g_{n-1}$. Hence, when we have an $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module, it is natural to consider simultaneous eigenvectors in the module M for the actions of all elements of $\mathcal{G}$.

Definition 4.1. *Let M be an $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module. An $(n-1)$-tuple $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{C}^{n-1}$ is called a **weight** of M if there exists a nonzeo vector $v \in M$ such that $g_i \cdot v = a_i v$ for all $i \in \{1, 2, \cdots, n-1\}$, and such a vector v is called a **weight vector** of weight $\mathbf{a}$.*

Given a weight vector v in an $\mathbf{H}^{\mathfrak{sl}_n}(c)$ module, we can produce several weight vectors from the given weight vector v using the auxiliary elements defined in Section 3. To be precise, we consider the actions of the intertwining elements φ_i on a module, which send weight vectors to weight vectors or zero.

Theorem 4.2. *Let M be an $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module. If a vector $v \in M$ is a weight vector of weight $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{C}^{n-1}$, then $\varphi_i \cdot v$ is either a weight vector of weight $(a_1, \cdots, a_{i+1}, a_i, \cdots, a_{n-1})$ or zero. Furthermore, $\varphi_i \cdot v = 0$ happens only if $a_i - a_{i+1} \in \{-1, 1\}$.*

Proof. Let $v \in M$ be a weight vector of weight $(a_1, a_2, \cdots, a_{n-1}) \in \mathbb{C}^{n-1}$. Then by Proposition 3.6, we directly see

$$g_j \cdot (\varphi_i \cdot v) \ = \ \varphi_i \cdot (g_{s_i(j)} \cdot v) \ = \ a_{s_i(j)} v$$

from which it follows that $\varphi_i \cdot v$ is either a weight vector of weight

$$(a_1, \cdots, a_{i+1}, a_i, \cdots, a_{n-1}) \ \in \ \mathbb{C}^{n-1}$$

or zero.

Now assume $a_i - a_{i+1} \notin \{-1, 1\}$ additionally. Then by Proposition 3.7, we obtain

$$\varphi_i \cdot (\varphi_i \cdot v) \ = \ (1 + a_i - a_{i+1})(1 - a_i + a_{i+1})v \ \neq \ 0;$$

hence, we have $\varphi_i \cdot v \neq 0$. □

We may also apply the results in Proposition 3.10 and Proposition 3.13 to the representation theory of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$. Indeed, we directly prove the following theorem using Proposition 3.10 and Proposition 3.13 in a simliar way to the proof of Theorem 4.2.

Theorem 4.3. *Let M be an $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module and let $v \in M$ be a weight vector of weight $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{C}^{n-1}$. Then $r \cdot v$ is either a weight vector of weight $(a_{n-1} - \frac{1}{c}, a_1, a_2, \cdots, a_{n-2})$ or zero, and $r \cdot v = 0$ happens only if $a_{n-1} - \frac{1}{c} \in \{-1, 1\}$. Also, $\ell \cdot v$ is either a weight vector of weight $(a_2, \cdots, a_{n-1}, a_1 + \frac{1}{c})$ or zero, and $\ell \cdot v = 0$ happens only if $a_1 \in \{-1, 1\}$.*

4.2. Outlook

Theorem 4.2 and theorem 4.3 suggest a combinatorial appoach to the representation theory of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ in terms of weights of a given $\mathbf{H}^{\mathfrak{sl}_n}(c)$-module as in [6] for the algebra $\mathbf{H}^{\mathfrak{gl}_n}(c)$. For example, we may apply Suzuki's proof of Theorem 7.2 in [5] in order to prove the fact that for a generic c, the polynomial representation of the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ admits a basis consisting of weight vectors.

Furthermore, we expect various results on the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ to be proved or explained in terms of the modified Dunkl-Cherednik elements by using the elementary representation theory or combinatorics. One of the interesting results on the algebra $\mathbf{H}^{\mathfrak{sl}_n}(c)$ is that there exists a r^{n-1}-dimensional representation when $c = \frac{r}{n}$ with $\gcd(r, n) = 1$. On the other hand, there is the notion of the (r, n)-rational parking functions in combinatorics, and it is well known that there are exactly r^{n-1} parking functions. We expect the notion of weights we defined in this paper to be a bridge between representation-theoretic values and combinatorial statistic of parking functions.

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