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ON THE STABILITY OF A CUBIC-QUADRATIC SET-VALUED FUNCTIONAL EQUATION

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Abstract. In this paper, I prove the stability of the following set-valued functional equation

$$f(x+2y)\oplus f(x-2y)\oplus 3f(2x)\oplus f(-2x)$$
$$=4f(x+y)\oplus 4f(x-y)\oplus 10f(x)$$

by employing the direct method in the sense of Hyers and Ulam.

1. Introduction

In 1940, Ulam [18] proposed the problem concerning the stability of group homomorphisms. In 1941, Hyers [7] gave an affirmative answer to this problem for additive mappings between Banach spaces. Subsequently many mathematicians came to deal with this problem (cf. [6, 16]). Many mathematicians investigated the stability of several types of set-valued functional equations [4, 8, 9, 12, 13, 14]. The following terminologies used in this paper will be adopted from the article of Kenary et al. [9].

Throughout this paper, unless otherwise stated, let X be a real vector space and Y be a Banach space with the norm $\|\cdot\|_Y$. We denote by $C_b(Y)$, $C_c(Y)$ and $C_{cb}(Y)$ the set of all closed bounded subsets of Y, the set of all closed convex subsets of Y and the set of all closed convex bounded subsets of Y, respectively. Let A and B be two nonempty subsets of Y and $\lambda\in\mathbb{R}$. The addition and the scalar multiplication can be defined as follows

$$A+B=\{a+b|a\in A, b\in B\} \text{ and } \lambda A=\{\lambda a|a\in A\}.$$

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Furthermore, for the subsets $A, B \in C_c(Y)$, we write $A \oplus B = \overline{A+B}$, where $\overline{A+B}$ denotes the closure of $A+B$. Generally, for arbitrary $\lambda, \mu \in \mathbb{R}^+$, we can obtain that

$$\lambda A + \lambda B = \lambda(A+B) \text{ and } (\lambda+\mu)A \subset \lambda A + \mu A.$$

In particular, if A is convex, then we have $(\lambda+\mu)A = \lambda A + \mu A$. For $A, B \in C_b(Y)$, the Hausdorff distance between A and B is defined by

$$h(A,B) := \inf\{\varepsilon > 0 | A \subset B + \varepsilon\overline{S_1}, B \subset A + \varepsilon\overline{S_1}\},$$

where S_1 denotes the closed unit ball in Y, i.e., $S_1 = \{y \in Y | \|y\|_Y \leq 1\}$. Since Y is a Banach space, it is proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup [3]. Rädström [15] proved that $(C_{cb}(Y), \oplus, h)$ can be isometrically embedded in a Banach space. The following are some properties of the Hausdorff distance.

Lemma 1.1. (Castaing and Valadier [3]). *For any $A_1, A_2, B_1, B_2, C \in C_{cb}(Y)$ and $\lambda \in \mathbb{R}^+$, the following expressions hold*
(i) $h(A_1 \oplus A_2, B_1 \oplus B_2) \leq h(A_1, B_1) + h(A_2, B_2)$;
(ii) $h(\lambda A_1, \lambda B_1) = \lambda h(A_1, B_1)$;
(iii) $h(A_1 \oplus C, B_1 \oplus C) = h(A_1, B_1)$.

In particular, $h(A, C) = h(A \oplus B, B \oplus C) \leq h(A, B) + h(B, C)$ for any $A, B, C \in C_{cb}(Y)$ from (i) and (iii) in Lemma 1.1.

The main purpose of this paper is to establish the stability of the following cubic-quadratic set-valued functional equation

$$\begin{aligned} & f(x+2y) \oplus f(x-2y) \oplus 3f(2x) \oplus f(-2x) \\ & \qquad = 4f(x+y) \oplus 4f(x-y) \oplus 10f(x) \end{aligned} \tag{1}$$

by employing the direct method in the sense of P. Găvruta [6] and the fixed point method in the sense of L. Cădariu and V. Radu([1, 2]), respectively. Every solution of the set-valued functional equation (1) is called a cubic-quadratic set-valued mapping.

2. Stability of the cubic-quadratic set-valued functional equation (1)

Throughout this paper, let V and W be real vector spaces.

For a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned} Qf(x,y) :=& f(x+y) + f(x-y) - 2f(x) - 2f(y), \\ Cf(x,y) :=& \, f(x+2y) - 3f(x-y) + 3f(x) - f(x-y) - 6f(y), \\ Ef(x,y) :=& \, f(x+2y) - 4f(x-y) + 6f(x) - 4f(x-y) + f(x-2y), \\ Df(x,y) :=& \, f(x+2y) - 4f(x+y) - 4f(x-y) + f(x-2y) \\ &- 10f(x) + 3f(2x) + f(-2x) \end{aligned}$$

for all $x, y \in V$. A solution of $Qf = 0$ and $Cf = 0$ are called a quadratic mapping and a cubic mapping, respectively. Now we will show that f is a cubic-quadratic mapping if f is a solution of the functional equation $Df(x,y) = 0$ for all $x, y \in V$.

The following lemmas were proved in [11].

Lemma 2.1. (Theorem 2.2 [11]) *Suppose that the odd function $f : V \to W$ satisfies $Ef(x,y) = 0$ for all $x, y \in V$ and $f(2x) = 8f(x)$ for all $x \in V$. Then f is a cubic mapping.*

Lemma 2.2. (Theorem 2.4 [11]) *Suppose that the even function $f : V \to W$ satisfies $Ef(x,y) = 0$ for all $x, y \in V$ and $f(2x) = 4f(x)$ for all $x \in V$. Then f is a quadratic mapping.*

Theorem 2.3. *A mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$ if and only if f is a cubic-quadratic mapping.*

Proof. If a mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$, then the odd mapping f_o satisfies the equalities $f_o(2x) - 8f_o(x) = \frac{Df_o(x,0) - Df_o(-x,0)}{4} = 0$ and $Ef_o(x,y) = Df_o(x,y) = 0$ for all $x, y \in V$. Since $f(0) = \frac{Df(0,0)}{12} = 0$, the even mapping f_e satisfies the equalities $f_e(2x) - 4f_e(x) = \frac{Df(0,x)}{2} = 0$ and $Ef_e(x,y) = Df_e(x,y) = 0$ for all $x, y \in V$. It follows from Lemma 2.1 and Lemma 2.2 that f_o is a cubic mapping and f_e is a quadratic mapping.

Conversely, assume that f_1 and f_2 are mappings such that the equalities $f := f_1 + f_2$, $Qf_1(x,y) = 0$ and $Cf_2(x,y) = 0$ hold for all $x, y \in V$. Then the equalities $f_1(x) = f_1(-x)$, $f_2(x) = -f_2(-x)$, $f_1(2x) = 4f_1(x)$ and $f_2(2x) = 8f_2(x)$ hold for all $x \in V$. From the above properties, we obtain the equalities

$$\begin{aligned} Df_1(x,y) &= Qf_1(x,2y) - 4Qf_1(x,y), \\ Df_2(x,y) &= Cf_2(x,y) - Cf_2(x-y,y) \end{aligned}$$

for all $x, y \in V$, i.e.

$$Df(x,y) = Df_1(x,y) + Df_2(x,y) = 0$$

for all $x, y \in V$ as we desired. $\square$

In this section, we shall consider the stability of the set-valued functional equation (1) by employing the direct method.

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) = \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} < \infty \tag{2}$$

for all $x, y \in X$. Suppose that $f : X \to C_{cb}(Y)$ is a mapping satisfying

$$\begin{aligned} h(f(x+2y) \oplus & f(x-2y) \oplus 3f(2x) \oplus f(-2x), \\ & 4f(x+y) \oplus 4f(x-y) \oplus 10f(x)) \le \varphi(x,y) \end{aligned} \tag{3}$$

for all $x, y \in X$. Then there exists a unique cubic-quadratic mapping $F : X \to C_{cb}(Y)$ such that

$$h(f(x), F(x)) \le \sum_{n=0}^{\infty} \left(\frac{2^n + 1}{4 \cdot 8^n} \varphi(2^n x, 0) + \frac{2^n - 1}{4 \cdot 8^n} \varphi(-2^n x, 0) \right) \tag{4}$$

for all $x \in X$. In particular, F is represented by

$$F(x) = \lim_{n \to \infty} \left(\frac{2^n + 1}{2 \cdot 8^n} f(2^n x) \oplus \frac{2^n - 1}{2 \cdot 8^n} f(-2^n x) \right) \tag{5}$$

for all $x \in X$.

Proof. Setting $y = 0$ in (3), we have

$$\begin{aligned} h(16f(x), 3f(2x) \oplus f(-2x)) = & h(2f(x) \oplus 3f(2x) \oplus f(-2x), 18f(x)) \\ \le & \varphi(x, 0) \end{aligned} \tag{6}$$

for all $x \in X$. Replacing x by $2^n x$ in (6) and dividing both sides by $2 \cdot 8^{n+1}$, from the equality (ii) in Lemma 1.1, we get

$$h\left(\frac{f(2^n x)}{8^n}, \frac{3f\left(2^{n+1}x\right)}{2 \cdot 8^{n+1}} \oplus \frac{f\left(-2^{n+1}x\right)}{2 \cdot 8^{n+1}} \right) \le \frac{\varphi(2^n x, 0)}{2 \cdot 8^{n+1}}$$

for all $x \in X$ and $n \in \mathbb{N}$. By the above inequality and the inequality(i) in Lemma 1.1, we have the inequalities

$$
\begin{aligned}
h\Big(&\frac{2^n+1}{2\cdot 8^n} f(2^n x) \oplus \frac{2^n-1}{2\cdot 8^n} f(-2^n x),\\
&\quad \frac{2^{n+1}+1}{2\cdot 8^{n+1}} f(2^{n+1} x) \oplus \frac{2^{n+1}-1}{2\cdot 8^{n+1}} f(-2^{n+1} x)\Big)\\
&\le h\left(\frac{2^n+1}{2\cdot 8^n} f(2^n x), \frac{3(2^n+1)}{4\cdot 8^{n+1}} f(2^{n+1} x) \oplus \frac{2^n+1}{4\cdot 8^{n+1}} f(-2^{n+1} x)\right)\\
&\quad + h\Big(\frac{2^n-1}{2\cdot 8^n} f(-2^n x), \frac{3(2^n-1)}{4\cdot 8^{n+1}} f(-2^{n+1} x) \oplus \frac{2^n-1}{4\cdot 8^{n+1}} f(2^{n+1} x)\Big)\\
&\le \frac{2^n+1}{4\cdot 8^n} \varphi(2^n x, 0) + \frac{2^n-1}{4\cdot 8^n} \varphi(-2^n x, 0) \qquad (7)
\end{aligned}
$$

for all $x \in X$ and $n \in \mathbb{N}$. From the above inequality and the property $h(A,C) \le h(A,B) + h(B,C)$, we obtain the inequalities

$$
\begin{aligned}
h\Big(&f(x), \frac{2^n+1}{2\cdot 8^n} f(2^n x) \oplus \frac{2^n-1}{2\cdot 8^n} f(-2^n x)\Big)\\
&\le \sum_{k=0}^{n-1} h\Big(\frac{2^k+1}{2\cdot 8^k} f(2^k x) \oplus \frac{2^k-1}{2\cdot 8^k} f(-2^k x),\\
&\qquad \frac{2^{k+1}+1}{2\cdot 8^{k+1}} f(2^{k+1} x) \oplus \frac{2^{k+1}-1}{2\cdot 8^{k+1}} f(-2^{k+1} x)\Big)\\
&\le \sum_{k=0}^{n-1} \left(\frac{2^k+1}{4\cdot 8^k} \varphi\left(2^k x, 0\right) + \frac{2^k-1}{4\cdot 8^k} \varphi\left(-2^k x, 0\right)\right) \qquad (8)
\end{aligned}
$$

for all $x \in X$ and $n \in \mathbb{N}$. Now we claim that the sequence $\{\frac{2^n+1}{2\cdot 8^n} f(2^n x) \oplus \frac{2^n-1}{2\cdot 8^n} f(-2^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y),\ h)$. Indeed, for all

$m, n \in \mathbb{N}$, by (8), we can show that

$$\begin{aligned} h\bigg(&\frac{2^n+1}{2\cdot 8^n}f(2^n x)\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^n x), \\ &\qquad\qquad \frac{2^{n+m}+1}{2\cdot 8^{n+m}}f(2^{n+m}x)\oplus\frac{2^{n+m}-1}{2\cdot 8^{n+m}}f(-2^{n+m}x)\bigg) \\ &\le \sum_{k=n}^{m+n-1}\left(\frac{2^k+1}{4\cdot 8^k}\varphi\left(2^k x, 0\right)+\frac{2^k-1}{4\cdot 8^k}\varphi\left(-2^k x, 0\right)\right) \\ &\le \sum_{k=n}^{m+n-1}\frac{\varphi\left(2^k x, 0\right)+\varphi\left(-2^k x, 0\right)}{2\cdot 4^k} \end{aligned} \tag{9}$$

for all $x \in X$. From the condition (2), it follows that the last expression tends to zero as $n \to \infty$. So the sequence $\{\frac{2^n+1}{2\cdot 8^n}f(2^n x)\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^n x)\}$ is a Cauchy sequence. Therefore, from the completeness of $C_{cb}(Y)$, we can define a set-valued mapping $F : X \to C_{cb}(Y)$ represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{2^n+1}{2\cdot 8^n}f(2^n x)\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^n x)\right)$$

for all $x \in X$. Next, we show that F satisfies the set-valued functional equation (1). Replacing x and y by $\frac{x}{2n}$ and $\frac{y}{2n}$ in (3), respectively, we get

$$
\begin{aligned}
h\bigg(&\frac{2^n+1}{2\cdot 8^n}f(2^n(x+2y))\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^n(x+2y))\oplus\frac{2^n+1}{2\cdot 8^n}f(2^n(x-2y))\\
&\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^n(x-2y))\oplus\frac{3(2^n+1)}{2\cdot 8^n}f(2^{n+1}x)\\
&\oplus\frac{3(2^n-1)}{2\cdot 8^n}f(-2^{n+1}x)\oplus\frac{2^n+1}{2\cdot 8^n}f(-2^{n+1}x)\oplus\frac{2^n-1}{2\cdot 8^n}f(2^{n+1}x),\\
&\oplus\frac{4(2^n+1)}{2\cdot 8^n}f(2^n(x+y))\oplus\frac{4(2^n-1)}{2\cdot 8^n}f(-2^n(x+y))\\
&\oplus\frac{4(2^n+1)}{2\cdot 8^n}f(2^n(x-y))\oplus\frac{4(2^n-1)}{2\cdot 8^n}f(-2^n(x-y))\\
&\oplus\frac{10(2^n+1)}{2\cdot 8^n}f(2^nx)\oplus\frac{10(2^n-1)}{2\cdot 8^n}f(-2^nx)\bigg)\\
\le\ &\frac{1}{4^n}h\big(f(2^n(x+2y))\oplus f(2^n(x-2y))\oplus 3f(2^{n+1}x)\oplus f(-2^{n+1}x),\\
&\quad 4f(2^n(x+y))\oplus 4f(2^n(x-y))\oplus 10f(2^nx)\big)\\
&+\frac{1}{4^n}h\big(f(-2^n(x+2y))\oplus f(2^n(2y-x))\oplus 3f(-2^{n+1}x)\oplus f(2^{n+1}x),\\
&\quad 4f(-2^n(x+y))\oplus 4f(-2^n(x-y))\oplus 10f(-2^nx)\big)\\
\le\ &\frac{1}{4^n}\varphi\left(2^nx,2^ny\right)+\frac{1}{4^n}\varphi\left(-2^nx,-2^ny\right).
\end{aligned}
$$

Since the last expression tends to zero as $n\to\infty$, we obtain that F is a cubic-quadratic set-valued mapping. Moreover, letting $n\to\infty$ in (8), we get the desired inequality (4). To prove the uniqueness of F. Assume that F' is another cubic-quadratic set-valued mapping satisfying the inequality (4). Since the mappings F and F' are cubic-quadratic set-valued mappings, the equalities $F'(x)=\frac{2^n+1}{2\cdot 8^n}F'(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}F'(-2^nx)$

and $F(x) = \frac{2^n+1}{2\cdot 8^n}F(2^nx) \oplus \frac{2^n-1}{2\cdot 8^n}F(-2^nx)$ are obtained from the functional equation (1). Thus we can infer that

$$\begin{aligned}
&h(F(x), F'(x))\\
=&h\left(\frac{2^n+1}{2\cdot 8^n}F(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}F(-2^nx), \frac{2^n+1}{2\cdot 8^n}F'(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}F'(-2^nx)\right)\\
\le&h\left(\frac{2^n+1}{2\cdot 8^n}F(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}F(-2^nx), \frac{2^n+1}{2\cdot 8^n}f(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^nx)\right)\\
+&h\left(\frac{2^n+1}{2\cdot 8^n}f(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}f(-2^nx), \frac{2^n+1}{2\cdot 8^n}F'(2^nx)\oplus\frac{2^n-1}{2\cdot 8^n}F'(-2^nx)\right)\\
\le&\frac{2^n+1}{2\cdot 8^n}h\left(F(2^nx), f(2^nx)\right)+\frac{2^n-1}{2\cdot 8^n}h\left(F(-2^nx), f(-2^nx)\right)\\
&+\frac{2^n+1}{2\cdot 8^n}h\left(f(2^nx), F'(2^nx)\right)+\frac{2^n-1}{2\cdot 8^n}h\left(f(-2^nx), F'(-2^nx)\right)\\
\le&\frac{2^n+1}{4\cdot 8^n}\sum_{k=0}^{\infty}\left(\frac{2^k+1}{8^k}\varphi\left(2^{k+n}x,0\right)+\frac{2^k-1}{8^k}\varphi\left(-2^{k+n}x,0\right)\right)\\
&+\frac{2^n-1}{4\cdot 8^n}\sum_{k=0}^{\infty}\left(\frac{2^k+1}{8^k}\varphi\left(-2^{k+n}x,0\right)+\frac{2^k-1}{8^k}\varphi\left(2^{k+n}x,0\right)\right)\\
\le&\frac{1}{2\cdot 4^n}\sum_{k=0}^{\infty}\left(\frac{1}{4^k}\varphi\left(2^{k+n}x,0\right)+\frac{1}{4^k}\varphi\left(-2^{k+n}x,0\right)\right)\\
&+\frac{1}{2\cdot 4^n}\sum_{k=0}^{\infty}\left(\frac{1}{4^k}\varphi\left(-2^{k+n}x,0\right)+\frac{1}{4^k}\varphi\left(2^{k+n}x,0\right)\right)\\
\le&\sum_{k=n}^{\infty}\left(\frac{1}{4^k}\varphi\left(2^kx,0\right)+\frac{1}{4^k}\varphi\left(-2^kx,0\right)\right).
\end{aligned}$$

It is easy to see from the condition (2) that the last expression tends to zero as $n\to\infty$ i.e. $F(x) = F'(x)$ for all $x \in X$. This completes the proof of this theorem. □

3. Stability of the cubic-quadratic set-valued functional equation (1): The fixed point method

In this section, we will investigate the stability of the set-valued functional equation (1) by using the fixed point theorem.

We recall the following result of the fixed point theory by Margolis and Diaz.

Theorem 3.1. ([5] or [17]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:
(1) $d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;
(2) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;
(4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a positive constant $L < 1$ satisfying*

$$4L\varphi(x, y) \geq \varphi(2x, 2y) \tag{10}$$

for all $x, y \in X$. Assume that $f : X \to C_{cb}(Y)$ is the a set-valued mapping with $f(0) = \{0\}$ and satisfies the inequality (3) for all $x, y \in X$. Then there exists a unique cubic-quadratic mapping $F : X \to C_{cb}(Y)$ such that

$$h(f(x), F(x)) \leq \frac{\varphi(x, 0) + \varphi(-x, 0)}{16(1-L)} \tag{11}$$

for all $x \in X$. In particular, F is represented by (5) for all $x \in X$.

Proof. Consider the set $S = \{g | g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$ and introduce the generalized metric d on S, which is defined by

$$d(g_1, g_2) = \inf\{\mu \in (0, \infty) | h(g_1(x), g_2(x)) \leq \mu(\varphi(x, 0) + \varphi(-x, 0)), \forall x \in X\},$$

where, as usual, $\inf \emptyset = \infty$. It can be easily verified that (S, d) is a complete generalized metric space (see [10]). Now, we define an operator $T : S \to S$ by

$$Tg(x) = \frac{3g\,(2x) \oplus g\,(-2x)}{16}$$

for all $x \in X$. Let $g_1, g_2 \in S$ be given such that $d(g_1, g_2) = \varepsilon$, i.e.

$$h(g_1(x), g_2(x)) \leq \varepsilon(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$. Thus, we obtain that

$$\begin{aligned} h(Tg_1(x), Tg_2(x)) =& h\left(\frac{3g_1(2x) \oplus g_1(-2x)}{16}, \frac{3g_2(2x) \oplus g_2(-2x)}{16}\right) \\ \leq& \frac{3}{16} h\left(g_1(2x), g_2(2x)\right) + \frac{1}{16} h\left(g_1(-2x), g_2(-2x)\right) \\ \leq& \frac{\varepsilon}{4}\left(\varphi(2x,0) + \varphi(-2x,0)\right) \\ \leq& L\varepsilon(\varphi(x,0) + \varphi(-x,0)) \end{aligned}$$

for all $x \in X$. Therefore, we know that $d(Tg_1, Tg_2) \leq Ld(g_1, g_2)$ which means that T is a strictly contractive mapping with the Lipschitz constant $L < 1$. Moreover, we see that

$$\begin{aligned} h(f(x), Tf(x)) = h&\left(f(x), \frac{3f(2x) \oplus f(-2x)}{16}\right) \\ &\leq \frac{\varphi(x,0)}{16} \\ &\leq \frac{\varphi(x,0) + \varphi(-x,0)}{16} \end{aligned}$$

for all $x \in X$ from the inequality (6). It means that $d(f, Tf) \leq \frac{1}{16} < \infty$ by the definition of d. By Theorem 3.1, there exists a set-valued mapping $F : X \to C_{cb}(Y)$ satisfying the following:
(i) F is a fixed point of T, i.e., $F(x) = \frac{3F(2x) \oplus F(-2x)}{16}$ for all $x \in X$. Further, F is the unique fixed point of T in the set $\{g \in S | d(f,g) < \infty\}$, which means that there exists an $\eta \in (0, \infty)$ such that $h(f(x), F(x)) \leq \eta(\varphi(x,0) + \varphi(-x,0))$ for all $x \in X$.
(ii) $d(T^n f, F) \to 0$ as $n \to \infty$. Then we get

$$F(x) = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty}\left(\frac{2^n+1}{2 \cdot 8^n} f(2^n x) \oplus \frac{2^n-1}{2 \cdot 8^n} f(-2^n x)\right)$$

for all $x \in X$.
(iii) $d(f, F) \leq \frac{1}{1-L} d(f, Tf)$. Then we have $d(f, F) \leq \frac{1}{16(1-L)}$, which implies the inequality (11) holds.
Next, we show that F satisfies the inequality (3). Replacing x and y by

$\frac{x}{2n}$ and $\frac{y}{2n}$ in (3), respectively, we get

$$
\begin{aligned}
h\biggl(&\frac{2^n+1}{2\cdot 8^n}f(2^n(x+2y)) \oplus \frac{2^n-1}{2\cdot 8^n}f(-2^n(x+2y)) \oplus \frac{2^n+1}{2\cdot 8^n}f(2^n(x-2y))\\
&\oplus \frac{2^n-1}{2\cdot 8^n}f(-2^n(x-2y)) \oplus \frac{3(2^n+1)}{2\cdot 8^n}f(2^{n+1}x)\\
&\oplus \frac{3(2^n-1)}{2\cdot 8^n}f(-2^{n+1}x) \oplus \frac{2^n+1}{2\cdot 8^n}f(-2^{n+1}x) \oplus \frac{2^n-1}{2\cdot 8^n}f(2^{n+1}x),\\
&\frac{4(2^n+1)}{2\cdot 8^n}f(2^n(x+y)) \oplus \frac{4(2^n-1)}{2\cdot 8^n}f(-2^n(x+y))\\
&\oplus \frac{4(2^n+1)}{2\cdot 8^n}f(2^n(x-y)) \oplus \frac{4(2^n-1)}{2\cdot 8^n}f(-2^n(x-y))\\
&\oplus \frac{10(2^n+1)}{2\cdot 8^n}f(2^nx) \oplus \frac{10(2^n-1)}{2\cdot 8^n}f(-2^nx)\biggr)\\
&\le \frac{1}{4^n}\varphi\left(2^nx, 2^ny\right) + \frac{1}{4^n}\varphi\left(-2^nx, -2^ny\right)\\
&\le L^n(\varphi\left(x,y\right) + \varphi\left(-x,-y\right)).
\end{aligned}
$$

Since $L < 1$, the last expression tends to zero as $n \to \infty$. By the definition of f, we conclude that F is a cubic-quadratic set-valued mapping. If F is a cubic-quadratic set-valued mapping, then we can replacing x and y in (1) by 0 and x, respectively. So we get $F(x) = \frac{3F(2x)\oplus F(-2x)}{16}$, i.e. F is a fixed point of T. Therefore and F is a unique mapping satisfying the inequality (11) by (i). □

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